Sequences and Series
Solutions

1. It is known that

$$\begin{aligned}\frac{t_{11}+t_{13}}{t_5+t_7} &= \frac{187500}{1500}\\ \frac{ar^{10}+ar^{12}}{ar^4+ar^6} &= 125\\ \frac{ar^{10}(1+r^2)}{ar^4(1+r^2)} &= 125\\ r^6 &= 125\end{aligned}$$

Thus, $r=\pm\sqrt{5}$. Therefore, $ar^4+ar^6=25a+125a=150a=1500$ and so $a=10$. Therefore, the sequence begins $10, 10\sqrt{5}, 50$ or $10, -10\sqrt{5}, 50$.

2. Let d be the common difference in the arithmetic sequence. Since the sequence has distinct terms, we know that $d\neq 0$. Then $b-c=-d$, $c-a=2d$ and $a-b=-d$. Thus,

$$\begin{aligned}-dx^2+2dx-d &= 0\\ -d(x-1)^2 &= 0\end{aligned}$$

and since $d\neq 0$, we have $x=1$.

3. From the arithmetic sequence we have that $4=x+d$ and $y=4+d$, where d is the common difference. Therefore, $x+y=4-d+4+d=8$. From the geometric sequence we have $xr=3$ and $3r=y$, where r is the common ratio (which is not 0 since the second term is 3). Therefore, $xy=\frac{3}{r}(3r)=9$. Thus, $\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}=\frac{8}{9}$.

4. **Solution 1**

Since the product of the three numbers is non-zero, so is r, the common ratio of the geometric sequence. We let the numbers be $\frac{a}{r}$, a, and ar. Thus, $a^3=125$ and so $a=5$. Therefore, the numbers are $\frac{5}{r}$, 5, and $5r$. Let d be the common difference of the arithmetic sequence. We know that $\frac{5}{r}$ is the first term of the arithmetic sequence and 5 is the third term. Therefore, $5-\frac{5}{r}=2d$. We also know that $5r$ is the sixth term of the arithmetic sequence and therefore, $5r-5=3d$. Therefore,

$$\begin{aligned}\frac{5-\frac{5}{r}}{5r-5} &= \frac{2d}{3d}\\ 3\left(5-\frac{5}{r}\right) &= 2(5r-5)\\ 3-\frac{3}{r} &= 2r-2\\ 0 &= 2r^2-5r+3\\ 0 &= (2r-3)(r-1)\end{aligned}$$

The solution $r = 1$ gives the sequence $5, 5, 5$, but we were told that the three numbers are distinct and so we discard this solution. The solution $r = \frac{3}{2}$ gives the sequence $\frac{10}{3}, 5, \frac{15}{2}$.

Solution 2

Since the product of the three numbers is non-zero, so is r, the common ratio of the geometric sequence. We let the numbers be $\frac{a}{r}$, a, and ar. Thus, $a^3 = 125$ and so $a = 5$. We know that the middle term, 5, is the third term of an arithmetic sequence.

Let d be the common difference of this arithmetic sequence. The first term is the first term of the arithmetic sequence and therefore, it is $5 - 2d$. The third term is the sixth term of the arithmetic sequence and therefore, it is $5 + 3d$. The product of these three terms is 125 and so $(5 - 2d)5(5 + 3d) = 125$. Dividing both sides by 5 and simplifying gives $-6d^2 + 5d = 0$. Therefore, $d = 0$ or $d = \frac{5}{6}$. If $d = 0$, then the sequence is $5, 5, 5$, but we were told that the three numbers are distinct and so we discard this solution. If $d = \frac{5}{6}$, the sequence is $\frac{10}{3}, 5, \frac{15}{2}$.

5. Our sum is

$$\begin{aligned}
\sum_{k=1}^{N} \frac{k^2 + k}{2} &= \frac{\sum_{k=1}^{N} k^2 + \sum_{k=1}^{N} k}{2} \\
&= \frac{\frac{N(N+1)(2N+1)}{6} + \frac{N(N+1)}{2}}{2} \\
&= \frac{N(N+1)}{4}\left(\frac{2N+1}{3} + 1\right) \\
&= \frac{N(N+1)}{4}\left(\frac{2N+4}{3}\right) \\
&= \frac{N(N+1)(N+2)}{6}
\end{aligned}$$

Taking $N = 200$ we get

$$\begin{aligned}
\sum_{k=1}^{200} \frac{k^2 + k}{2} &= \frac{200 \cdot 201 \cdot 202}{6} \\
&= 1353400.
\end{aligned}$$

6. Represent the angles as $a - 2d$, $a - d$, a, $a + d$ and $a + 2d$. The sum of these values is 540°. Therefore, $5a = 540^\circ$ and so $a = 108^\circ$. So either $a - d = 90^\circ$ or $a - 2d = 90^\circ$. So the largest angle is either 126° or 144°.

7. We let the four positive integers be represented by k, kr, kr^2 and kr^3. Then

$$kr + kr^2 = 30 \qquad (1)$$

$$k + kr^3 = 35 \qquad (2)$$

Dividing (2) by (1) gives

$$\frac{k+kr^3}{kr+kr^2} = \frac{35}{30}$$
$$\frac{1+r^3}{r+r^2} = \frac{7}{6} \quad (k \neq 0 \text{ since } kr+kr^2=30)$$
$$6r^3-7r^2-7r+6=0$$

By inspection, we find that $r=-1$ is a solution. Using the factor theorem and long division, we arrive at

$$(r+1)(2r-3)(3r-2)=0$$

So $r=-1, \frac{2}{3}$ or $\frac{3}{2}$

Using $r=-1$, equation (2) gives $0k=35$, which is impossible. (It would also violate the condition $a<b<c<d$.)

Using $r=\frac{2}{3}$ in (1), we find $k=27$.

Using $r=\frac{3}{2}$ in (1) we find $k=8$.

Both of these value give the same list of numbers, and when arranged in increasing order they are $(a,b,c,d)=(8,12,18,27)$.

8. The sequence is arithmetic if and only if $t_1+t_3=2t_2$. There are 27 equally likely ways to pick three numbers, of which only five lead to such a sequence:

$$1,4,7$$
$$1,5,9$$
$$2,5,8$$
$$3,5,7$$
$$3,6,9$$

So the probability is $\frac{5}{27}$.

9. **Solution 1**

Since there are an odd number of integers, the average of the integers is the middle integer. Therefore, the middle integer is $\frac{500}{25}=20$. Thus, the smallest integer is 8.

Solution 2

The common difference is 1 and the number of terms is 25. Therefore, using the sum of an arithmetic sequence we get $500=\frac{25}{2}(a+(a+24))$, which simplifies to $40=2a+24$. Therefore, $a=8$,

10. The common difference is $d=2$, the first term is $a=-1994$ and so

$$-1994+2(n-1)=-1994$$

Solving for n gives $n=1995$.

11. (a) $S_1 = t_1 = 3^1 - 1 = 2$.
$S_2 = t_1 + t_2 = 3^2 - 1 = 8$ and so $t_2 = 8 - 2 = 6$.
$S_3 = t_1 + t_2 + t_3 = 3^3 - 1 = 26$ and so $t_3 = 26 - 8 = 18$.

(b)
$$\begin{aligned}\frac{t_{n+1}}{t_n} &= \frac{S_{n+1} - S_n}{S_n - S_{n-1}} \\ &= \frac{(3^{n+1} - 1) - (3^n - 1)}{(3^n - 1) - (3^{n-1} - 1)} \\ &= \frac{3^n \cdot (3-1)}{3^{n-1} \cdot (3-1)} \\ &= 3.\end{aligned}$$

12. We can see that the nth term of the sequence is $7n$. The smallest multiple of 7 that is greater than 40 is 42 and the largest multiple of 7 that is less than 28001 is 28000 (We see that $\frac{28001}{7} \approx 4000.1$ and $7 \cdot 4000 = 28000$.) So $n - 1 = \frac{28000 - (42)}{7}$ and $n = 3995$.

13. We know $f(n+1) = f(n) + \frac{1}{3}$ and so the the function evaluated at positive integers gives a sequence that is arithmetic. Its first term is 2 and its common difference is $\frac{1}{3}$. Therefore, $f(100) = 2 + 99\left(\frac{1}{3}\right) = 35$.

14. Substituting for x and y, $-p + 2q = r$ so $q - p = r - q$ and we are done!

15. If the common difference is 0, then the sequence is also a geometric sequence with a common ratio of 1. In this case, any three terms form a three-term geometric sequence.

Let's consider what happens when $d \neq 0$. For any three-term geometric sequence, x_1, x_2, x_3 we have $x_1 x_3 = (x_2)^2$. So
$$\begin{aligned}(a+4d)(a+15d) &= (a+8d)^2 \\ a^2 + 19ad + 60d^2 &= a^2 + 16ad + 64d^2 \\ 3ad &= 4d^2 \\ d &= \frac{3}{4}a \quad (\text{Since } d \neq 0)\end{aligned}$$

Thus, the general term is
$$\begin{aligned}t_k &= a + (k-1)\frac{3}{4}a \\ &= \frac{a}{4}(3k+1)\end{aligned}$$

Therefore,
$$r = \frac{t_9}{t_5} = \frac{\frac{a}{4}(3 \cdot 9 + 1)}{\frac{a}{4}(3 \cdot 5 + 1)} = \frac{7}{4}$$

We need to find an infinite number of triples (i, j, k) such that

$$\frac{t_j}{t_i} = \frac{t_k}{t_j} = \frac{7}{4}$$

which is to say that

$$\frac{3j+1}{3i+1} = \frac{3k+1}{3j+1} = \frac{7}{4}$$

Therefore, $4(3j+1) = 7(3i+1)$, which implies that $3j+1$ is a multiple of 7 and $3i+1$ is a multiple of 4.

Also, $4(3k+1) = 7(3j+1)$, which implies that $3k+1$ is a multiple of 7 and $3j+1$ is a multiple of 4. So $3j+1$ must be a multiple of 28. Let $3j+1 = 28n$ for some integer n.

We also have that $(3j+1)^2 = (3i+1)(3k+1)$ and so $(3i+1)(3k+1)$ must be a multiple of 28^2. So if we make $3i+1 = 16n$ and $3k+1 = 49n$, then we will have satisfied all the conditions.

However, we need to guarantee that i, j and k are positive integers. We note that

$$\begin{aligned} 3i+1 &= 16n = 3(5n)+n \\ 3j+1 &= 28n = 3(9n)+n \\ 3k+1 &= 49n = 3(16n)+n \end{aligned}$$

So if we choose n such that it is 1 more than a multiple of 3, then i, j and k will be integers. Therefore, let $n = 3m+1$ for some non-negative integer m and we obtain

$$\begin{aligned} i &= \frac{16(3m+1)-1}{3} = 16m+5 \\ j &= \frac{28(3m+1)-1}{3} = 28m+9 \\ k &= \frac{49(3m+1)-1}{3} = 49m+16 \end{aligned}$$

For each value of m we will obtain a three-term geometric sequence with common ratio $\frac{7}{4}$.

16. The sequence goes $5, 3, -2, -5, -3, 2, 5, 3, \ldots$. The sequence repeats in groups of 6 whose sum is 0. So the sum of 32 terms is $5+3=8$.

17.

$$\begin{aligned} t_{1998} &= \frac{1995}{1997} \times t_{1996} \\ &= \frac{1995}{1997} \times \frac{1997}{1995} \times t_{1994} \\ &= \frac{1995}{1997} \times \frac{1997}{1995} \times \frac{1995}{1993} \times \cdots \times \frac{3}{5} \times \frac{1}{3} \times t_2 \\ &= -\frac{1}{1997} \end{aligned}$$

18. The first term is $t_1 = 555 - 7 = 548$ and the common difference is -7. Therefore, the sum is $S_n = \frac{n}{2}[2(548) + (n-1)(-7)]$. Thus, the sum is negative when $1096 + (n-1)(-7) < 0$. Solving the equality $1096 - 7n + 7 = 0$ we obtain $n \approx 157.6$. We note that $S_{157} = 314$ and $S_{158} = -237$ Therefore, the smallest value of n for which S_n is negative is $n = 158$.