Centre for Education
in Mathematics and Computing

Euclid eWorkshop # 1

Logarithms and Exponents

TOOLKIT

If a, b, x, and y are real numbers and n is a nonzero integer then the rules for exponents are:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \qquad a^0 = 1 \text{ if } a \neq 0 \qquad a^{-x} = \frac{1}{a^x} \text{ if } a \neq 0$$

$$a^x a^y = a^{x+y} \qquad \frac{a^x}{a^y} = a^{x-y} \text{ if } a \neq 0 \qquad (a^x)^y = a^{xy}$$

$$a^x \cdot b^x = (ab)^x \qquad \frac{a^x}{b^x} = (\frac{a}{b})^x \text{ if } b \neq 0$$

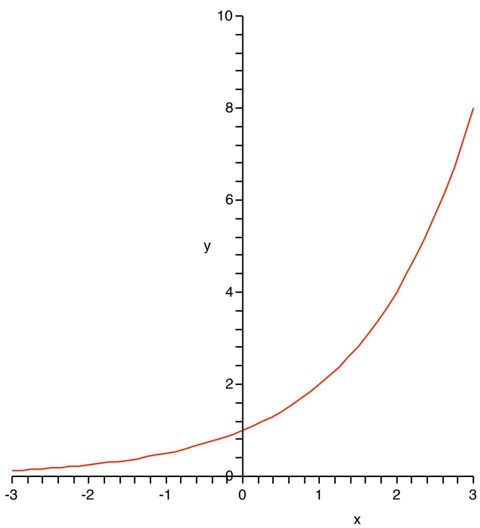
Also, 0^0 is not defined if it is encountered using any of the above formulae.
If a, x, and y are nonzero real numbers then:

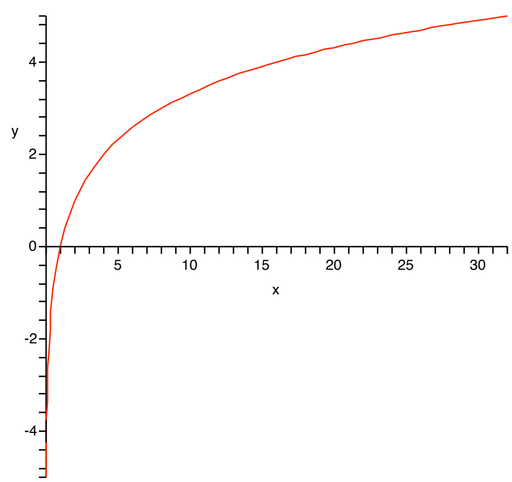
$$\log_a(xy) = \log_a x + \log_a y \qquad \log_a(\frac{x}{y}) = \log_a x - \log_a y$$

$$\log_a(x^y) = y \log_a x \qquad \log_a(a^x) = a^{\log_a x} = x \qquad \log_a 1 = 0$$

$$\log_a x = \frac{1}{\log_x a} \qquad \frac{\log_a x}{\log_a y} = \log_y x$$

If $f(x) = a^x$ then $f^{-1}(x) = \log_a x$. You should be able to sketch the graphs of both these functions. The graphs are shown for $a = 2$ below.





SAMPLE PROBLEMS

1. Calculate the ratio $\frac{x}{y}$ if $2\log_5(x-3y) = \log_5(2x) + \log_5(2y)$

Solution

First we note that if the 3 logarithmic terms are to be defined in the original equation then their arguments must be positive. So $x > 0$, $y > 0$, and $x > 3y$. Now

$$\begin{aligned} 2\log_5(x-3y) &= \log_5(2x) + \log_5(2y) \\ \log_5(x-3y)^2 &= \log_5(4xy). \end{aligned}$$

Now since the log function takes on each value its range only once, this implies

$$\begin{aligned} (x-3y)^2 &= 4xy \\ x^2 - 6xy + 9y^2 &= 4xy \\ x^2 - 10xy + 9y^2 &= 0 \\ (x-y)(x-9y) &= 0 \end{aligned}$$

So $\frac{x}{y} = 1$ or 9. But $\frac{x}{y} > 3$ from our restrictions so that $\frac{x}{y} = 9$.

2. If m and k are integers, find all solutions to the equation $9(7^k + 7^{k+2}) = 5^{m+3} + 5^m$.

Solution

We factor both sides of this equation to arrive at:

$$\begin{aligned} 9(1+7^2)7^k &= 5^m(1+5^3) \\ 3^2 \cdot 2 \cdot 5^2 \cdot 7^k &= 5^m \cdot 2 \cdot 3^2 \cdot 7 \end{aligned}$$

Now since both sides of this equation are integers and have unique factorizations it follows that $m = 2$ and $k = 1$ is the only solution.

3. Determine the points of intersection of the curves $y = \log_{10}(x-2)$ and $y = 1 - \log_{10}(x+1)$.

Solution

Again the arguments of the log functions, $x-2$ and $x+1$ must be positive which implies that $x > 2$. Now

$$\begin{aligned}
\log_{10}(x-2) &= 1-\log_{10}(x+1)\\
\log_{10}(x-2)+\log_{10}(x+1) &= 1\\
\log_{10}[(x-2)(x+1)] &= 1\\
(x-2)(x+1) &= 10\\
x^2-x-2 &= 10\\
x^2-x-12 &= 0\\
(x-4)(x+3) &= 0
\end{aligned}$$

So $x = 4$ or -3, but $x > 2$ from our restrictions so $x = 4$. The point of intersection is $(4, \log_{10} 2)$ or $(4, 1 - \log_{10} 5)$. Since $\log_{10} 2 + \log_{10} 5 = 1$, these are equivalent answers.

4. Solve for x if $\log_2(9 - 2^x) = 3 - x$.

Solution

Once again the argument of the log must be positive, implying that $9 > 2^x$.

$$\begin{aligned}
\log_2(9-2^x) &= 3-x\\
(9-2^x) &= 2^{3-x} = \frac{8}{2^x}
\end{aligned}$$

Substituting $y = 2^x$ we have:

$$\begin{aligned}
9-y &= \frac{8}{y}\\
y^2-9y+8 &= 0
\end{aligned}$$

Thus, $y = 1$ or $y = 8$. Substituting back in, we see that $x = 0$ or $x = 3$. Both of these satisfy the restriction.

5. The graph of $y = m^x$ passes through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

Solution

From the given information, $m^2 = 5$ and $n = m^5$. Thus

$$\begin{aligned}
m &= \pm\sqrt{5}\\
n &= (\pm\sqrt{5})^5\\
mn &= (\sqrt{5})^6 = 125.
\end{aligned}$$

PROBLEM SET

1. Determine x such that $\log_x 2 + \log_x 4 + \log_x 8 = 1$.

2. Determine the values of x such that $12^{2x+1} = 2^{3x+7} \cdot 3^{3x-4}$.

3. What is the sum of the following series?

$$\log_{10} \frac{3}{2} + \log_{10} \frac{4}{3} + \log_{10} \frac{5}{4} + \cdots\cdots \log_{10} \frac{200}{199}.$$

4. If $x^3y^5 = 2^{11} \cdot 3^{13}$ and $\dfrac{x}{y^2} = \dfrac{1}{27}$, solve for x and y.

5. If $\log_8 3 = k$ then express $\log_8 18$ in terms of k.

6. Solve the equations for the point of intersection of the graphs of $y = \log_2(2x)$ and $y = \log_4 x$.

7. The points $A(x_1, y_1)$ and $B(x_2, y_2)$ lie on the graph of $y = \log_a x$. Through the midpoint of AB a horizontal line is drawn to cut the curve at $C(x_3, y_3)$. Show that $(x_3)^2 = x_1x_2$.

8. The function $y = ax^r$ passes through the points $(2, 1)$ and $(32, 4)$. Calculate the value of r.

9. If $2^{x+3} + 2^x = 3^{y+2} - 3^y$ and x and y are integers, determine the values of x and y.

10. If $f(x) = 2^{4x-2}$, calculate, in simplest form, $f(x) \cdot f(1 - x)$.

11. Find all values of x such that $\log_5(x - 2) + \log_5(x - 6) = 2$.

12. Prove that a, b, and c are 3 numbers that form a geometric sequence if and only if $\log_x a$, $\log_x b$ and $\log_x c$ form an arithmetic sequence.