Centre for Education
in Mathematics and Computing

Euclid eWorkshop # 2

Solutions

SOLUTIONS

1. Subtract one equation from the other and factor the resulting expression.

$$\begin{aligned} xy + y - 8 - 8x &= 0 \\ x(y-8) + y - 8 &= 0 \\ (x+1)(y-8) &= 0 \end{aligned}$$

   There are solutions when $x = -1$ and when $y = 8$. If $x = -1$ then $y = -9$. If $y = 8$ then $x = 4 \pm 2\sqrt{2}$. The solutions are $(-1, -9)$ and $(4 \pm 2\sqrt{2}, 8)$.

2. We are asked for the x value of the midpoint of zeros, which is the x value of the vertex. The equation is written in vertex form already, having an x value of 1.

   Alternately Soluion: Find the intercepts:

$$\begin{aligned} (x-1)^2 - 4 &= 0 \\ (x-1)^2 &= 4 \\ x &= 1 \pm 2 \end{aligned}$$

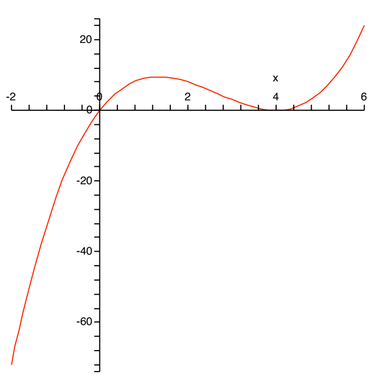
   Thus $x = 3$ or -1. Thus $a = \dfrac{-1+3}{2} = 1$.

3. (a) Consider $a = 0$ and $a = 1$, and find the intersection point of the resulting equations, $y = x^2$ and $y = x^2 + 2x + 1$. Then $0 = 2x + 1$ and the intersection point is $(-\frac{1}{2}, \frac{1}{4})$. Now substitute this point into the general equation to show that this point is on all the parabolas, since

$$\begin{aligned} y &= x^2 + 2ax + a \\ &= \frac{1}{4} + 2a \cdot \frac{-1}{2} + a \\ &= \frac{1}{4} \end{aligned}$$

   (b) Now $y = x^2 + 2ax + a = (x+a)^2 + a - a^2$ so the vertex is at $(-a, a - a^2)$. If we represent the coordinates of the vertex by (p,q) we have $p = -a$ and $q = a - a^2$ or $q = -p^2 - p$, the required parabola.

4. (a) .



(b) From the graph $x \geq 0$.

5. Factoring both equations we arrive at:

$$p(1+r+r^2) = 26 \quad (1)$$
$$p^2r(1+r+r^2) = 156 \quad (2)$$

Dividing (2) by (1) gives $pr = 6$. Substituting this relation back into (1) we get

$$\begin{aligned} \frac{6}{r} + 6 + 6r &= 26 \\ 6 - 20r + 6r^2 &= 0 \\ 3r^2 - 10r + 3 &= 0 \\ (3r-1)(r-3) &= 0 \end{aligned}$$

Hence $(r, p) = (3, 2)$ or $(\frac{1}{3}, 18)$.

6. We assume, on the contrary, that the coefficients are in geometric sequence. Then $\dfrac{b}{a} = \dfrac{c}{b}$ or $b^2 = ac$. But now the discriminant $b^2 - 4ac = -3b^2 < 0$ so that the roots are not real. Thus we have a contradiction of the condition set out in the statement of the problem and our assumption is false.

7. Let r and s be the integer roots. The equation can be written as

$$\begin{aligned} a(x-r)(x-s) &= a(x^2 - (r+s)x + rs) \\ &= ax^2 - a(r+s)x + ars \\ &= ax^2 + bx + c \end{aligned}$$

with $b = -a(r+s)$ and $c = ars$. Since a, b, c are in arithmetic sequence, we have

$$\begin{aligned} c - b &= b - a \\ a + c - 2b &= 0 \\ a + ars + 2a(r+s) &= 0 \\ 1 + rs + 2(r+s) &= 0 \quad \text{we can divide by } a \text{ since } a \neq 0 \\ (r+2)(s+2) &= 3 \end{aligned}$$

Since there are only 2 integer factorings of 3 we have $\{r, s\} = \{1, -1\}$ or $\{-3, -5\}$.

8. **Solution 1**

Multiplying out and collecting terms results in $x^4 - 6x^3 + 8x^2 + 2x - 1 = 0$. We look for a factoring with integer coefficients, using the fact that the first and last coefficients are 1. So

$$x^4 - 6x^3 + 8x^2 + 2x - 1 = (x^2 + ax + 1)(x^2 + bx - 1)$$

where a and b are undetermined coefficients. However multiplication now gives $a + b = -6$ and $-a + b = 2$ and $ab = 8$. Since all 3 equations are satisfied by $a = -4$ and $b = -2$, we have factored the original expression as

$$x^4 - 6x^3 + 8x^2 + 2x - 1 = (x^2 - 4x + 1)(x^2 - 2x - 1)$$

Factoring these two quadratics gives roots of $x = 2 \pm \sqrt{3}$ and $x = 1 \pm \sqrt{2}$.

Solution 2

We observe that the original equation is of the form $f(f(x)) = x$ where $f(x) = x^2 - 3x + 1$. Now if we can find x such that $f(x) = x$ then $f(f(x)) = x$. So we solve $f(x) = x^2 - 3x + 1 = x$ which gives the first factor $x^2 - 4x + 1$ above. With polynomial division, we can then determine that

$$x^4 - 6x^3 + 8x^2 + 2x - 1 = (x^2 - 4x + 1)(x^2 - 2x - 1)$$

and continue as in Solution 1.

9. The vertex has $x = 2$ and $y = -16$ so $A = (2, -16)$. When $y = 0$ we get intercepts at -2 and 6. The larger value is 6, so $B = (6, 0)$. Therefore we want the line through $(2, -16)$ and $(6, 0)$ which is $4x - y - 24 = 0$.

10. **Solution 1**

Multiplying gives

$$\begin{aligned} x^2 - (b+c)x + bc &= a^2 - (b+c)a + bc \\ 0 &= x^2 - (b+c)x + a(b+c-a) \\ x &= \frac{b+c \pm \sqrt{(b+c)^2 - 4a(b+c-a)}}{2} \\ &= \frac{b+c \pm \sqrt{(b+c-2a)^2}}{2} \\ &= a \quad \text{OR} \quad b+c-a \end{aligned}$$

Solution 2 Observe that $x = a$ is one solution. Rearrange as above to get $x^2 - (b+c)x + a(b+c-a) = 0$. Using the sum/product of roots, the other solution is $x = b + c - a$.

11. Since $x = -2$ is a solution of $x^3 - 7x - 6$, thus $x + 2$ is a factor. Factor as

$$\begin{aligned} x^3 - 7x - 6 &= (x+2)(x^2 - 2x - 3) \\ &= (x+2)(x+1)(x-3) \end{aligned}$$

so the roots are -2, -1 and 3.

12. Let the roots be r and s. By the sum and product rule,

$$\begin{aligned} r + s &= \frac{-4(a-2)}{4} \\ &= 2 - a \\ rs &= \frac{-8a^2 + 14a + 31}{4} \\ &= -2a^2 + \frac{7}{2}a + \frac{31}{4} \end{aligned}$$

Then

$$\begin{aligned} r^2 + s^2 &= (r+s)^2 - 2rs \\ &= (2-a)^2 - 2(-2a^2 + \frac{7}{2}a + \frac{31}{4}) \\ &= 4 - 4a + a^2 + 4a^2 - 7a - \frac{31}{2} \\ &= 5a^2 - 11a - \frac{23}{2}. \end{aligned}$$

It appears that the minimum value should be at the vertex of the parabola $f(a) = 5a^2 - 11a - \dfrac{23}{2}$, that is at $a = \dfrac{11}{10}$ (found by completing the square). But we have ignored the condition that the roots are real. The discriminant of the original equation is

$$\begin{aligned}
B^2 - 4AC &= [4(a-2)]^2 - 4(4)(-8a^2 + 14a + 31) \\
&= 16(a^2 - 4a + 4) + 128a^2 - 224a - 496 \\
&= 144a^2 - 288a - 432 \\
&= 144(a^2 - 2a - 3) \\
&= 144(a-3)(a+1).
\end{aligned}$$

Thus we have real roots only when $a \geq 3$ or $a \leq -1$. Therefore $a = \dfrac{11}{10}$ cannot be our final answer, since the roots are not real for this value. However $f(a) = 5a^2 - 11a - \dfrac{23}{2}$ is a parabola opening up and is symmetrical about its axis of symmetry $a = \dfrac{11}{10}$. So we move to the nearest value of a to the axis of symmetry that gives real roots, which is $a = 3$.

13. Let $g(2) = k$. Since f and g are inverse functions, thus $f(k) = 2$. We need to solve

$$\begin{aligned}
\frac{3k-7}{k+1} &= 2 \\
3k - 7 &= 2(k+1) \\
k &= 9
\end{aligned}$$

Thus $g(2) = 9$.

14. Write

$$\begin{aligned}
y &= -2x^2 - 4ax + k \\
&= -2(x^2 + 2ax + \frac{k}{2}) \\
&= -2(x+a)^2 + k + 2a^2
\end{aligned}$$

The vertex is at $(-a, k + 2a^2)$ or $(-2, 7)$ and we can solve for $a = 2$ and $k = -1$.

15. Using sum and product of roots we have the 4 equations:

$$a + b = -c \qquad ab = d$$

$$c + d = -a \qquad cd = b.$$

$$\begin{aligned}
\text{Therefore} \quad -(c+d) + cd &= -c \\
cd - d &= 0 \\
d(c-1) &= 0
\end{aligned}$$

But none of a, b, c or d are zero, so $c = 1$. Then we get $d = b$, $a = 1$ and$d = b = -2$. Thus $a + b + c + d = -2$.

16. The most common way to do this problem uses calculus. However we make the substitution $z = x - 4$. To get y in terms of z, try

$$\begin{aligned} y &= x^2 - 2x - 3 \\ &= (x-4)^2 + 6x - 19 \\ &= (x-4)^2 + 6(x-4) + 5 \\ &= z^2 + 6z + 5 \end{aligned}$$

The value we want to minimize is then $\dfrac{y-4}{(x-4)^2} = \dfrac{z^2+6z+1}{z^2} = 1 + \dfrac{6}{z} + \dfrac{1}{z^2}$. If we now let $u = \dfrac{1}{z}$, we have the up-opening parabola $1 + 6u + u^2$ which has its minimum at $u = -3$ with minimum value of -8. Note that since x can assume any real value except 4, z and u will assume all real values except zero. Thus the minimum value of this expression is -8.