Centre for Education
in Mathematics and Computing

# *Euclid eWorkshop # 3*

## *Analytic Geometry*

## TOOLKIT

As well as the material on parabolas in the previous workshop, some useful formulae include:

| **Description** | **Formula** |
|---|---|
| 1. The standard form for a line with slope $\frac{-A}{B}$ and intercepts $\frac{-C}{A}$ and $\frac{-C}{B}$ | $Ax + By + C = 0$ |
| 2. The equation of the line with slope $m$ through $(x_0,y_0)$ | $(y - y_0) = m(x - x_0)$ |
| 3. The equation of the line with intercepts at $(a, 0)$ and $(0, b)$ | $\frac{x}{a} + \frac{y}{b} = 1$ |
| 4. The formula for the midpoint $M$ of $A(x_1,y_1)$ and $B(x_2,y_2)$ | $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ |
| 5. The distance $D$ between points $A(x_1,y_1)$ and $B(x_2,y_2)$ | $D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ |
| 6. The distance $D$ between the line $Ax + By + C = 0$ and the point $(x_0,y_0)$ | $D = \frac{\mid Ax_0 + By_0 + C \mid}{\sqrt{A^2 + B^2}}$ |
| 7. The area of the triangle $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ | $\frac{1}{2} \mid x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3 \mid$ |
| 8. The equation of the circle centre with $(h,k)$ and radius $r$ | $(x - h)^2 + (y - k)^2 = r^2$ |

## SAMPLE PROBLEMS

1. If the line $2x - 3y - 6 = 0$ is reflected in the line $y = -x$, find the equation of the image line.

### Solution

Recall that after reflection in a line the image of any point is equally distant from the line but on the opposite side of the line. Thus the line segment joining any point to its image is perpendicular to the reflecting line and has its midpoint on the reflecting line.

The line $2x - 3y - 6 = 0$ has intercepts of $(3{,}0)$ and $(0{,}-2)$. Since the image of a line after reflection is another line we reflect these two points and then find the equation of the line through their image points. The image of the point $(3{,}0)$ upon reflection in the line $y = -x$ is $(0{,}-3)$. This result follows since the segment from $(3{,}0)$ to $(0{,}-3)$ has slope 1, which makes the line perpendicular to $y = -x$, and midpoint $\left(\frac{3}{2}, \frac{-3}{2}\right)$, which is on $y = -x$. Similarly the image of the point $(0{,}-2)$ reflected in the line $y = -x$ is $(2{,}0)$. Therefore the required line is $\frac{x}{2} + \frac{y}{-3} = 1$ (using #3 from the toolkit) or $3x - 2y - 6 = 0$.

2. If $A(3{,}5)$ and $B(11{,}11)$ are fixed points, find the point(s) $P$ on the $x$-axis such that the area of the triangle $ABP$ equals 30.

### Solution 1

The length $AB = \sqrt{8^2 + 6^2} = 10$ and the equation of $AB$ is $(y - 11) = \frac{3}{4}(x - 11)$ or $3x - 4y + 11 = 0$. Now if we think of $AB$ as the base of the triangle then the distance from $P(a{,}0)$ to the line $AB$ must be 6, the height of the triangle. Thus

$$\frac{|\ 3a - 4 \cdot 0 + 11\ |}{\sqrt{3^2 + (-4)^2}} = 6$$
$$3a + 11 = \pm 30$$
$$a = \frac{-41}{3} \text{ or } \frac{19}{3}$$

The points are $(\frac{19}{3}, 0)$ and $(\frac{-41}{3}, 0)$.

### Solution 2

Let $P = (p, 0)$. Then using the area formula in the toolkit

$$|\ \Delta ABP\ | = \frac{1}{2}\ |\ x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3\ |$$
$$= \frac{1}{2}\ |\ 33 + 0 + 5p - 55 - 11p - 0\ |$$
$$|\ -22 - 6p\ | = 60$$
$$p = \frac{19}{3} \text{ or } \frac{-41}{3}$$

The points are $(\frac{19}{3}, 0)$ and $(\frac{-41}{3}, 0)$.

3. Given the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x + 2 = 0$, find the length of their common chord.

**Solution**

The first circle has centre (0,0) and radius 2. The equation for the second circle can be written as

$$(x-3)^2 + y^2 = 7$$

thus it has centre (3,0) and radius $\sqrt{7}$. Since the line joining the centres is horizontal, the common chord is vertical. We can find the intersection points by solving

$$\begin{aligned} x^2 + y^2 - 4 &= x^2 + y^2 - 6x + 2 \\ 6x &= 6 \\ x &= 1 \end{aligned}$$

Substituting this value back into the equation for either circle gives intersection points $(1, \pm\sqrt{3})$. Thus the length of the common chord is $2\sqrt{3}$.

4. A line has slope $-2$ and is a distance of 2 units from the origin. What is the area of the triangle formed by this line and the axes?

**Solution**

Let the $x$ intercept be $k$. Since the line has slope -2, the $y$ intercept is $2k$, and the equation of the line is $2x + y - 2k = 0$. Therefore the formula for the distance from a line to a point shows that the distance from this line to (0,0) is $\left|\frac{2k}{\sqrt{5}}\right|$, which must equal 2. So $k = \pm\sqrt{5}$ and the area of the required triangle is $\frac{1}{2} \cdot k \cdot 2k = k^2 = (\sqrt{5})^2 = 5$.

## PROBLEM SET

1. A vertical line divides the triangle with vertices $O(0,0)$, $C(9,0)$ and $D(8,4)$ into two regions of equal area. Find the equation of the line.

2. Find all values of $c$ such that the line $y = x + c$ is tangent to the circle $x^2 + y^2 = 8$.

3. Find all values of $k$ so that the circle with equation $x^2+y^2 = k^2$ will intersect the circle $(x-5)^2+(y+12)^2 = 49$ in exactly one point.

4. A circle intersects the axes at $A(0{,}10)$, $O(0{,}0)$ and $B(8{,}0)$. A line through $P(2{,}-3)$ cuts the circle in half. What is the $y$ intercept of the line?

5. If triangle $ABC$ has vertices $A(0{,}0)$, $B(3{,}3)$ and $C(-4{,}4)$, determine the equation of the bisector of $\angle CAB$.

6. What are the length and the slope of the tangent(s) from the origin to the circle $(x - 3)^2 + (y - 4)^2 = 4$?

7. Find the equation of the set of points equidistant from $C(0, 3)$ and $D(6, 0)$.

8. In quadrilateral $KWAD$, $K$ is at the origin, $D$ is on the positive $x$-axis and $A$ and $W$ are in the first quadrant. The midpoints of $KW$ and $AD$ are $M$ and $N$ respectively. If $MN = \dfrac{1}{2}(AW + DK)$ prove that $WA$ is parallel to $KD$.

9. The point $A$ is on the line $4x + 3y - 48 = 0$ and the point $B$ is on the line $x + 3y + 10 = 0$. If the midpoint of $AB$ is (4,2), find the co-ordinates of $A$ and $B$.