35th Anniversary 1963 – 1998

Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

1998 Solutions
Fermat Contest (Grade 11)

for the
NATIONAL BANK OF CANADA
Awards

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PART A:

1. The value of $\frac{1+2+3+4+5}{2+4+6+8+10}$ is

(**A**) $\frac{1}{3}$ (**B**) 2.5 (**C**) $\frac{1}{2}$ (**D**) $\frac{11}{26}$ (**E**) $\frac{3}{8}$

Solution 1

$$\frac{1+2+3+4+5}{2+4+6+8+10} = \frac{15}{30} = \frac{1}{2}$$

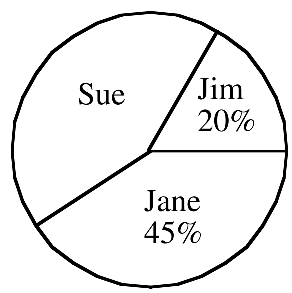
Solution 2

$$\frac{1+2+3+4+5}{2+4+6+8+10} = \frac{(1+2+3+4+5)}{2(1+2+3+4+5)} = \frac{1}{2}$$

ANSWER: (C)

2. The pie chart shows a percentage breakdown of 1000 votes in a student election. How many votes did Sue receive?

(**A**) 550 (**B**) 350 (**C**) 330
(**D**) 450 (**E**) 935

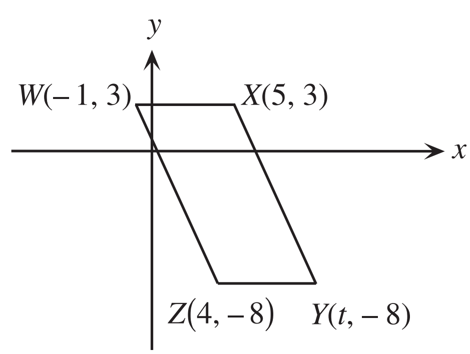


Solution

Sue received $100-(20+45)=35$ percent of the total number of votes. Since there was a total of 1000 votes, Sue received $0.35(1000)=350$ votes. ANSWER: (B)

3. If $WXYZ$ is a parallelogram, then t equals

(**A**) 8 (**B**) 9 (**C**) 10
(**D**) 11 (**E**) 12



Solution

Since $WXYZ$ is a parallelogram, opposite sides are equal in length. The length of WX is $5-(-1)=6$. Since WX=ZY, then $t-4=6$ or $t=10$. ANSWER: (C)

4. The product of two positive integers p and q is 100. What is the largest possible value of $p+q$?
(**A**) 52 (**B**) 101 (**C**) 20 (**D**) 29 (**E**) 25

Solution
The pairs of positive integers whose product is 100 are: 1 and 100, 2 and 50, 4 and 25, 5 and 20, 10 and 10. The pair with the largest sum is 1 and 100. The sum is 101.

ANSWER: (B)

5. If $\otimes$ is a new operation defined as $p \otimes q = p^2 - 2q$, what is the value of $7 \otimes 3$?

Solution
Using the definition of the new operation $\otimes$,
$$\begin{aligned} 7 \otimes 3 &= 7^2 - 2(3) \\ &= 49 - 6 \\ &= 43 \end{aligned}$$

ANSWER: (A)

6. The value of $\frac{1}{3}$ of 6^{30} is
(**A**) 6^{10} (**B**) 2^{30} (**C**) 2^{10} (**D**) 2×6^{29} (**E**) 2×6^{10}

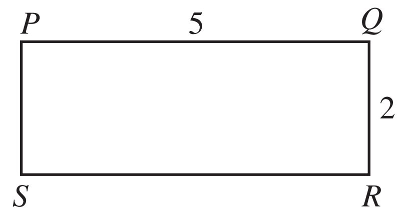
Solution
$$\begin{aligned} \frac{1}{3} \times 6^{30} &= \frac{1}{3} \times 6 \times 6^{29} \\ &= 2 \times 6^{29} \end{aligned}$$

ANSWER: (D)

7. The average (mean) of a list of 10 numbers is 0. If 72 and -12 are added to the list, the new average will be
(**A**) 30 (**B**) 6 (**C**) 0 (**D**) 60 (**E**) 5

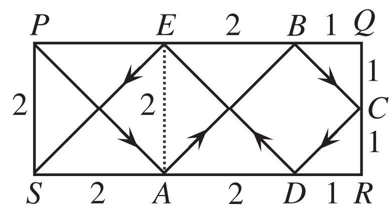
Solution
If the average (mean) of a list of 10 numbers is 0, then the sum of the numbers is 10(0) = 0. When 72 and –12 are added to the list, the sum of these 12 numbers is $0 + 72 - 12 = 60$.
Thus, the average of the 12 numbers is $60 \div 12 = 5$. ANSWER: (E)

8. On a rectangular table 5 units long and 2 units wide, a ball is rolled from point P at an angle of $45°$ to PQ and bounces off SR. The ball continues to bounce off the sides at $45°$ until it reaches S. How many bounces of the ball are required?



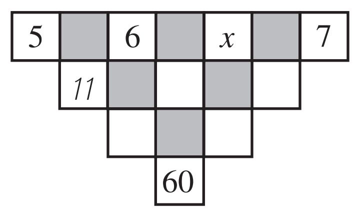
(**A**) 9 (**B**) 8 (**C**) 7
(**D**) 5 (**E**) 4

Solution
Since the ball bounces off the sides of the rectangular table at 45°, right-angled isosceles triangles are created as shown. The ball begins at point P then bounces at points A,B,C,D,and E before reaching S, for a total of 5 bounces.



ANSWER: (D)

9. The number in an unshaded square is obtained by adding the numbers connected to it from the row above. (The '*11*' is one such number.) The value of x must be



(**A**) 4 (**B**) 6 (**C**) 9
(**D**) 15 (**E**) 10

Solution
The three entries in row two, from left to right, are 11, $6+x$, and $x+7$. The two entries in row three, from left to right, are $11+(6+x)=17+x$ and $(6+x)+(x+7)=2x+13$. The single entry in row four is $(17+x)+(2x+13)=3x+30$.

Thus, $3x+30=60$
$$3x=30$$
$$x=10$$

ANSWER: (E)

10. Four points are on a line segment, as shown.
If $AB:BC=1:2$ and $BC:CD=8:5$, then $AB:BD$ equals

(**A**) 4:13 (**B**) 1:13 (**C**) 1:7
(**D**) 3:13 (**E**) 4:17

Solution
In order to compare the given ratios, we must rewrite the ratio $AB:BC=1:2$ as $AB:BC=4:8$. Now both ratios express BC as 8 units and we can write $AB:BC:CD=4:8:5$.

Thus, $AB : BD = 4:(8+5)$
$= 4:13$

ANSWER: (A)

PART B:

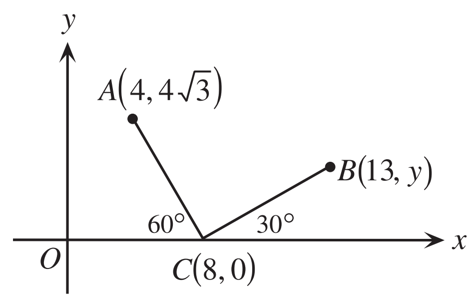
11. The number of solutions (x, y) of the equation $3x + y = 100$, where x and y are positive integers, is
(**A**) 33 (**B**) 35 (**C**) 100 (**D**) 101 (**E**) 97

Solution

Rewrite the given equation as $x = \frac{100 - y}{3}$. Since x must be an integer, $100 - y$ must be divisible by 3. Since both x and y must be positive integers, the only possible values of y are 1, 4, 7, 10, 13, ... , 94, and 97. Thus, there are 33 possible values for y and 33 solutions (x, y) that meet the given conditions.

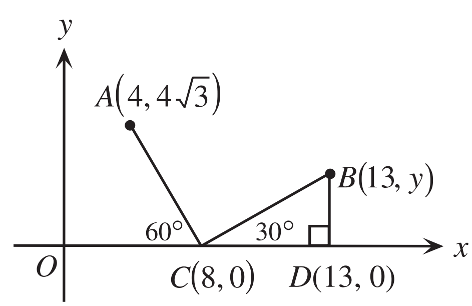
ANSWER: (A)

12. In the diagram, the value of y is
(**A**) $\frac{13}{2\sqrt{3}}$ (**B**) $\frac{5}{\sqrt{3}}$ (**C**) 2
(**D**) 12 (**E**) $\frac{\sqrt{3}}{5}$



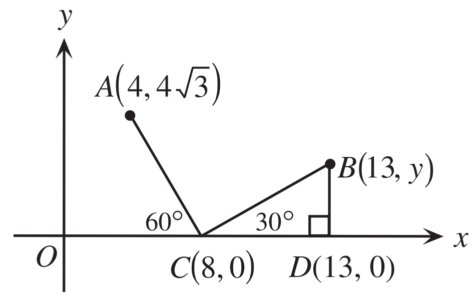
Solution 1

Label point D (13,0), where ΔBDC is a right-angled triangle. The slope of AC is $\frac{4\sqrt{3} - 0}{4 - 8} = -\sqrt{3}$. Since $\angle ACB$ is a right angle, AC is perpendicular to CB and the slope of CB is $\frac{1}{\sqrt{3}}$. The length of CD is $13 - 8 = 5$ and the length of DB is $\frac{1}{\sqrt{3}}(5) = \frac{5}{\sqrt{3}}$. Thus, $y = \frac{5}{\sqrt{3}}$.



Solution 2

Label point D (13,0), where ΔBDC is a right-angled triangle. The length of BD is y. The length of CD is $13-8=5$. Since ΔCBD is a 30°-60°-90° triangle, its sides are in the ratio $1:\sqrt{3}:2$.



Thus, $\frac{BD}{CD}=\frac{1}{\sqrt{3}}$

$$\frac{y}{5}=\frac{1}{\sqrt{3}}$$

$$y=\frac{5}{\sqrt{3}}$$

ANSWER: (B)

13. Three-digit integers are formed using only the digits 1 and/or 2. The sum of all such integers formed is

(**A**) 1332 (**B**) 333 (**C**) 999 (**D**) 666 (**E**) 1665

Solution

The only three-digit integers that can be formed are 111, 112, 121, 122, 211, 212, 221, 222. The sum of these integers is 1332. ANSWER: (A)

14. Three straight lines, l_1, l_2 and l_3, have slopes $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$, respectively. All three lines have the same y-intercept. If the sum of the x-intercepts of the three lines is 36, then the y-intercept is

(**A**) $\frac{-13}{12}$ (**B**) $\frac{-12}{13}$ (**C**) -4 (**D**) 4 (**E**) -9

Solution

Let b represent the common y-intercept of the three lines. The first line, l_1, has equation $y=\frac{1}{2}x+b$.

Set $y=0$ in this equation to find the x-intercept of the first line.

$$0=\frac{1}{2}x+b$$

$$-\frac{1}{2}x=b$$

$$x=-2b$$

Similarly the second line, l_2, has equation $y=\frac{1}{3}x+b$ and x - intercept $-3b$. The third line, l_3, has equation $y=\frac{1}{4}x+b$ and x - intercept $-4b$.

We know that $-2b-3b-4b=36$

$$-9b=36$$

$$b=-4$$

Thus, the common y-intercept of the three lines is -4. ANSWER: (C)

15. If $-2 \le x \le 5$, $-3 \le y \le 7$, $4 \le z \le 8$, and $w = xy - z$, then the smallest value w may have is
(A) -14 **(B)** -18 **(C)** -19 **(D)** -22 **(E)** -23

Solution
We obtain the smallest value of $w = xy - z$ by finding the smallest value of the product xy and then subtracting the largest value of z.
Since both x and y can take on positive or negative values, the smallest product xy will be negative with one of x and y positive and the other negative. The smallest such product xy is $(5)(-3) = -15$.
Thus, the smallest possible value of w is $-15 - 8 = -23$. ANSWER: (E)

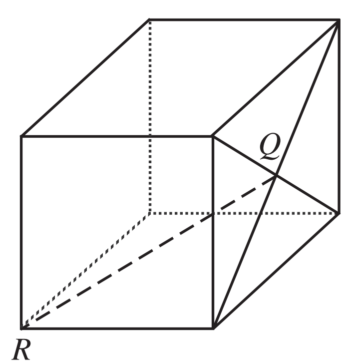
16. If $N = \left(7^{p+4}\right)\left(5^{q}\right)\left(2^{3}\right)$ is a perfect cube, where p and q are positive integers, the smallest possible value of $p + q$ is
(A) 5 **(B)** 2 **(C)** 8 **(D)** 6 **(E)** 12

Solution
In order for N to be a perfect cube, each prime factor of N must have an exponent that is divisible by 3. Since p and q must be positive integers, the smallest value of p is 2 and the smallest value of q is 3. Thus, the smallest value of $p + q$ is 5. ANSWER: (A)

17. Using only digits 1, 2, 3, 4, and 5, a sequence is created as follows: one 1, two 2's, three 3's, four 4's, five 5's, six 1's, seven 2's, and so on.
The sequence appears as: 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1, 2, 2, ...
The 100th digit in the sequence is
(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

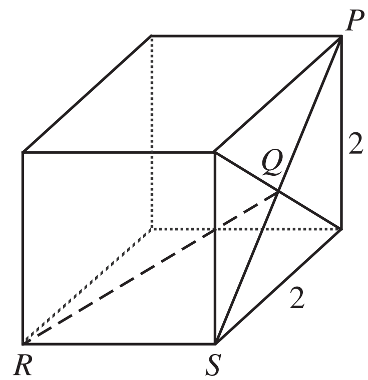
Solution
The total number of digits in n groups of the sequence is given by $1 + 2 + 3 + \ldots + n$. In order to determine the group containing the 100th digit in the sequence, we must find the positive integer n such that $1 + 2 + 3 + \ldots + (n-1) < 100$ and $1 + 2 + 3 + \ldots + n > 100$. By examining a few of these sums we find that $1 + 2 + 3 + \ldots + 13 = 91$ and $1 + 2 + 3 + \ldots + 13 + 14 = 105$. Thus the 100th digit in the sequence is in the 14th group. The 100th digit is a 4. ANSWER: (D)

18. Q is the point of intersection of the diagonals of one face of a cube whose edges have length 2 units. The length of QR is
(A) 2 **(B)** $\sqrt{8}$ **(C)** $\sqrt{5}$
(D) **(E)** $\sqrt{6}$



Solution

Label points P and S as shown. Since each face of the cube is a square with sides of length 2, use the Pythagorean Theorem to find the length of diagonal PS.



$$PS^2 = 2^2 + 2^2$$
$$= 8$$
$$PS = 2\sqrt{2}$$

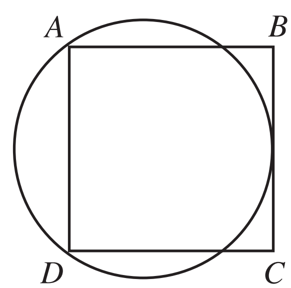
Then QS has length $\sqrt{2}$, as Q is the midpoint of diagonal PS.

Because we are working with a cube, $\angle QSR = 90^\circ$ and ΔQRS is a right-angled triangle. Use the Pythagorean Theorem in ΔQRS to get

$$QR^2 = 2^2 + \left(\sqrt{2}\right)^2$$
$$= 6$$
$$QR = \sqrt{6}$$

ANSWER: (E)

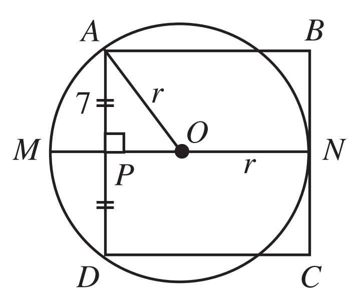
19. Square $ABCD$ has sides of length 14. A circle is drawn through A and D so that it is tangent to BC, as shown. What is the radius of the circle?



(**A**) 8.5 (**B**) 8.75 (**C**) 9
(**D**) 9.25 (**E**) 9.5

Solution

Let r represent the length of the radius and let O represent the centre of the circle. Draw diameter MN that bisects chord AD perpendicularly at P. Join OA.



ΔOAP is a right-angled triangle with $\angle APO = 90^\circ$. The length of AP is 7, since it is half of a side of the square. The length of OA is r, and the length of PO is $PN - ON = 14 - r$.

Using the Pythagorean Theorem we get

$$r^2 = 7^2 + (14 - r)^2$$
$$r^2 = 49 + 196 - 28r + r^2$$
$$28r = 245$$
$$r = 8.75$$

Thus, the radius of the circle is 8.75.

ANSWER: (B)

20. A deck of 100 cards is numbered from 1 to 100. Each card has the same number printed on both sides. One side of each card is red and the other side is yellow. Barsby places all the cards, red side up, on a table. He first turns over every card that has a number divisible by 2. He then examines all the cards, and turns over every card that has a number divisible by 3. How many cards have the red side up when Barsby is finished?

(**A**) 83 (**B**) 17 (**C**) 66 (**D**) 50 (**E**) 49

Solution

Initially, all 100 cards have the red side up. After Barsby's first pass only the 50 odd-numbered cards have the red side up, since he has just turned all the even-numbered cards from red to yellow.

During Barsby's second pass he turns over all cards whose number is divisible by 3. On this pass Barsby will turn any odd-numbered card divisible by 3 from red to yellow. Between 1 and 100, there are 17 odd numbers that are divisible by 3, namely 3, 9, 15, 21, ... , 93, and 99. Also on this pass, Barsby will turn any even-numbered card divisible by 3 from yellow to red. Between 1 and 100, there are 16 even numbers that are divisible by 3, namely 6, 12, 18, 24, ... , 90, and 96.

When Barsby is finished, the cards that have the red side up are the 50 odd-numbered cards from the first pass, minus the 17 odd-numbered cards divisible by 3 from the second pass, plus the 16 even-numbered cards divisible by 3, also from the second pass.

Thus, $50-17+16=49$ cards have the red side up. ANSWER: (E)

PART C:

21. The numbers $123\,456\,789$ and $999\,999\,999$ are multiplied. How many of the digits in the final result are 9's?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 17

Solution

Rewrite the product as follows:

$$\begin{aligned}(123\ 456\ 789)(999\ 999\ 999) &= (123\ 456\ 789)\left(10^9-1\right)\\ &= (123\ 456\ 789)\times 10^9-(123\ 456\ 789)\end{aligned}$$

When 123 456 789 is subtracted from $(123\ 456\ 789)\times 10^9$ the result is 123 456 788 876 543 211. None of the digits are 9's. ANSWER: (A)

22. There are four unequal, positive integers a, b, c, and N such that $N=5a+3b+5c$. It is also true that $N=4a+5b+4c$ and N is between 131 and 150. What is the value of $a+b+c$?

(**A**) 13 (**B**) 17 (**C**) 22 (**D**) 33 (**E**) 36

Solution

We are told that $N = 5a + 3b + 5c$ (1) and $N = 4a + 5b + 4c$ (2). Multiply equation (1) by 4 to get $4N = 20a + 12b + 20c$ (3). Similarly, multiply equation (2) by 5 to get $5N = 20a + 25b + 20c$ (4). Subtract equation (3) from equation (4) to get $N = 13b$.

Since N and b are both positive integers with $131 < N < 150$, N must be a multiple of 13. The only possible value for N is 143, when $b = 11$.
Substitute $N = 143$ and $b = 11$ into equation (1) to get

$$143 = 5a + 3(11) + 5c$$
$$110 = 5a + 5c$$
$$22 = a + c$$

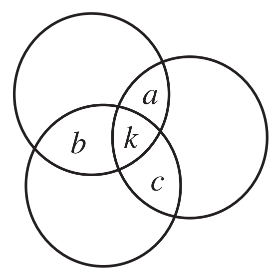
Thus, the value of a+b+c is 22+11=33. ANSWER: (D)

23. Three rugs have a combined area of 200 m^2. By overlapping the rugs to cover a floor area of 140 m^2, the area which is covered by exactly two layers of rug is 24 m^2. What area of floor is covered by three layers of rug?

(**A**) 12 m^2 (**B**) 18 m^2 (**C**) 24 m^2 (**D**) 36 m^2 (**E**) 42 m^2

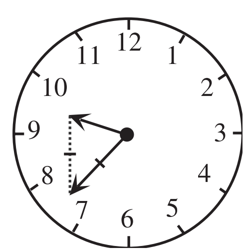
Solution

Draw the rugs in the following manner, where $a + b + c$ represents the amount of floor covered by exactly two rugs and k represents the amount of floor covered by exactly three rugs. We are told that $a + b + c = 24$ (1).



Since the total amount of floor covered when the rugs do not overlap is 200 m^2 and the total covered when they do overlap is 140 m^2, then 60 m^2 of rug is wasted on double or triple layers. Thus, $a + b + c + 2k = 60$ (2). Subtract equation (1) from equation (2) to get $2k = 36$ and solve for $k = 18$. Thus, the area of floor covered by exactly three layers of rug is 18 m^2. ANSWER: (B)

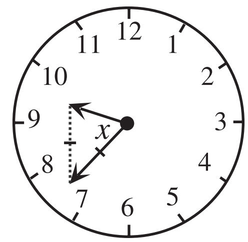
24. At some time between 9:30 and 10 o'clock the triangle determined by the minute hand and the hour hand is an isosceles triangle (see diagram). If the two equal angles in this triangle are each twice as large as the third angle, what is the time?



(**A**) 9:35 (**B**) 9:36 (**C**) 9:37
(**D**) 9:38 (**E**) 9:39

Solution

Let x represent the angle, in degrees, between the hour and the minute hands. We are told that the triangle in the diagram is isosceles, with the two equal angles each twice as large as the third angle.



Thus, $x + x + \frac{1}{2}x = 180$

$$\frac{5}{2}x = 180$$

$$x = 72$$

For each minute that passes, the minute hand moves through an angle of $360^\circ \div 60 = 6^\circ$, and the hour hand moves through an angle of $\left(360^\circ \div 12\right) \div 60 = \frac{1}{2}^\circ$.

At 9:00 there is an angle of 270° between the hour and the minute hands. At the time shown in the diagram there is an angle of 72° between the hour and the minute hands. Since the minute hand gains $5\frac{1}{2}^\circ$ on the hour hand every minute, it takes $\frac{270 - 72}{5\frac{1}{2}} = 36$ minutes from 9:00 for the hour and minute hands to reach the given position. Thus, the time is 9:36. ANSWER: (B)

25. For each value of x, $f(x)$ is defined to be the minimum value of the three numbers $2x + 2$, $\frac{1}{2}x + 1$ and $-\frac{3}{4}x + 7$. What is the maximum value of $f(x)$?

(**A**) $\frac{2}{3}$ (**B**) 2 (**C**) $\frac{17}{5}$ (**D**) $\frac{62}{11}$ (**E**) 7

Solution

The three numbers $2x + 2$, $\frac{1}{2}x + 1$ and $\frac{-3}{4}x + 7$ can be viewed as the y-coordinates of points lying on the lines $y = 2x + 2$ (1), $y = \frac{1}{2}x + 1$ (2), and $y = \frac{-3}{4}x + 7$ (3), respectively.

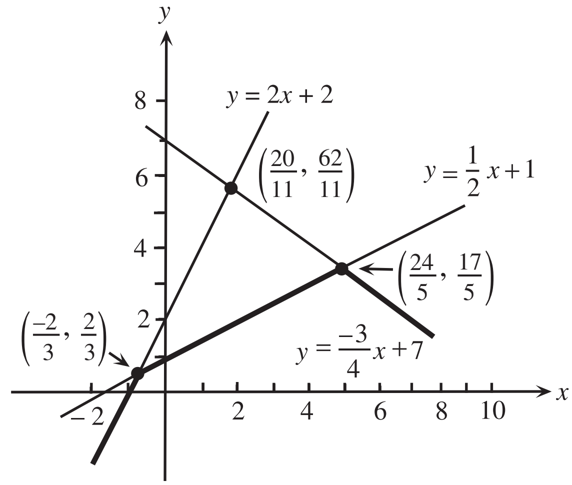
Draw all three lines on the same set of axes and find the points of intersection.

Subtract equation (2) from equation (1) to get

$$0 = \frac{3}{2}x + 1$$

$$\frac{-3}{2}x = 1$$

$$x = \frac{-2}{3}$$



Substitute $x = \frac{-2}{3}$ into equation (1) and get

$$y = 2\left(\frac{-2}{3}\right) + 2$$

$$y = \frac{2}{3}$$

Thus, the point of intersection of the lines $y = 2x + 2$ and $y = \frac{1}{2}x + 1$ is $\left(\frac{-2}{3}, \frac{2}{3}\right)$.

Similarly, we find $\left(\frac{20}{11}, \frac{62}{11}\right)$ as the point of intersection of the lines $y = 2x + 2$ and $y = \frac{-3}{4}x + 7$, and $\left(\frac{24}{5}, \frac{17}{5}\right)$ as the point of intersection of the lines $y = \frac{1}{2}x + 1$ and $y = \frac{-3}{4}x + 7$.

The minimum value of the three numbers $2x + 2$, $\frac{1}{2}x + 1$, and $\frac{-3}{4}x + 7$ is shown in the diagram as the smallest of the *y*-coordinates of points on the three lines for a given value of *x*. The maximum of these *y*-coordinates is $\frac{17}{5}$.

ANSWER: (C)