Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

1999 Solutions
Fermat Contest (Grade 11)

for the
NATIONAL BANK OF CANADA
Awards

Part A

1. The value of $\left(\sqrt{25}-\sqrt{9}\right)^2$ is

(**A**) 26 (**B**) 16 (**C**) 34 (**D**) 8 (**E**) 4

Solution

$\left(\sqrt{25}-\sqrt{9}\right)^2=(5-3)^2=4$ ANSWER: (E)

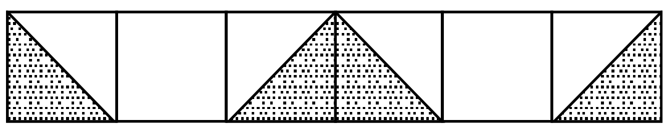
2. Today is Wednesday. What day of the week will it be 100 days from now?

(**A**) Monday (**B**) Tuesday (**C**) Thursday (**D**) Friday (**E**) Saturday

Solution

Since there are 7 days in a week it will be Wednesday in 98 days.
In 100 days it will be Friday. ANSWER: (D)

3. Six squares are drawn and shaded as shown. What fraction of the total area is shaded?



(**A**) $\frac{1}{2}$ (**B**) $\frac{1}{3}$ (**C**) $\frac{1}{4}$ (**D**) $\frac{2}{5}$ (**E**) $\frac{2}{3}$

Solution

Out of a possible six squares, there is the equivalent of two shaded squares.
Thus $\frac{1}{3}$rd of the figure is shaded. ANSWER: (B)

4. Turning a screwdriver $90°$ will drive a screw 3 mm deeper into a piece of wood. How many complete revolutions are needed to drive the screw 36 mm into the wood?

(**A**) 3 (**B**) 4 (**C**) 6 (**D**) 9 (**E**) 12

Solution

One complete revolution of the screw driver, $360°$, will drive it 12 mm deeper into the wood.
In order for the screw to go 36 mm into the wood it will take three revolutions.

ANSWER: (A)

5. A value of x such that $(5-3x)^5=-1$ is

(**A**) $\frac{4}{3}$ (**B**) 0 (**C**) $\frac{10}{3}$ (**D**) $\frac{5}{3}$ (**E**) 2

Solution
Since $(-1)^5 = -1$, $5 - 3x = -1$ ***or*** $x = 2$. ANWSER: (E)

6. The number which is 6 less than twice the square of 4 is

(**A**) -26 (**B**) 10 (**C**) 26 (**D**) 38 (**E**) 58

Solution
$2(4)^2 - 6 = 26$ ANSWER: (C)

7. The Partridge family pays each of their five children a weekly allowance. The average allowance for each of the three younger children is \$8. The two older children each receive an average allowance of \$13. The total amount of allowance money paid per week is

(**A**) \$50 (**B**) \$52.50 (**C**) \$105 (**D**) \$21 (**E**) \$55

Solutiion
The total paid out was, $3 \times \$8 + 2 \times \$13 = \$50$. ANSWER: (A)

8. The time on a digital clock is 5:55. How many minutes will pass before the clock next shows a time with all digits identical?

(**A**) 71 (**B**) 72 (**C**) 255 (**D**) 316 (**E**) 436

Solution
The digits on the clock will next be identical at 11:11. This represents a time difference of 316 minutes. (Notice that times like 6:66, 7:77 etc. are not possible.) ANSWER: (D)

9. In an election, Harold received 60% of the votes and Jacquie received all the rest. If Harold won by 24 votes, how many people voted?

(**A**) 40 (**B**) 60 (**C**) 72 (**D**) 100 (**E**) 120

Solution
If Harold received 60% of the votes this implies that Jacquie received 40% of the total number of votes. The difference between them, 20%, represents 24 votes.
Therefore, the total number of votes cast was $5 \times 24 = 120$. ANSWER: (E)

10. If x and y are each chosen from the set $\{1, 2, 3, 5, 10\}$, the largest possible value of $\frac{x}{y} + \frac{y}{x}$ is

(**A**) 2 (**B**) $12\frac{1}{2}$ (**C**) $10\frac{1}{10}$ (**D**) $2\frac{1}{2}$ (**E**) 20

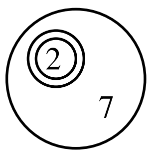
Solution
The best strategy is to choose the largest value and the smallest so that, $\frac{x}{y} > 1$, is as large as possible.

When we consider the reciprocal, $\frac{y}{x}$, this will always produce a number less than 1 and will be of little consequence in our final consideration. The best choices, then, are $x = 10$ and $y = 1$ and $\frac{x}{y} + \frac{y}{x}$ becomes $\frac{10}{1} + \frac{1}{10} = 10\frac{1}{10}$.

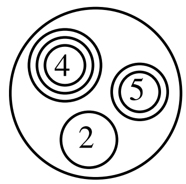
ANSWER: (C)

Part B

11. In *Circle Land*, the numbers 207 and 4520 are shown in the following way:

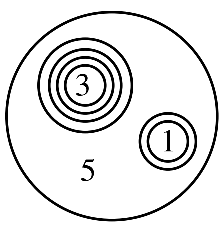


207



4520

In *Circle Land*, what number does the following diagram represent?



(**A**) 30 105 (**B**) 30 150 (**C**) 3105 (**D**) 3015 (**E**) 315

Solution 1

$= 3 \times 10^4 = 30\,000$

$= 1 \times 10^2 = 100$

5 $= 5 \times 10^0 = 5$

The required number is $30\,000 + 100 + 5 = 30\,105$.

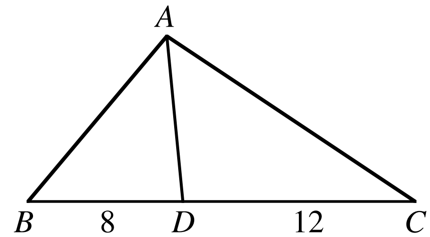
Solution 2

Since there are four circles around the ‘3’ this corresponds to $3 \times 10^4 = 30\,000$.

The ‘5’ corresponds to a 5 in the units digit which leads to $30\,105$ as the only correct possibility.

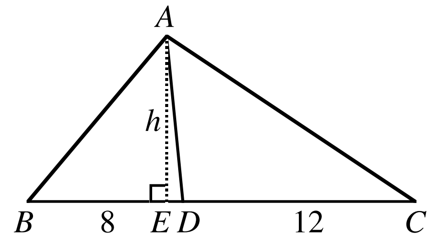
ANSWER: (A)

12. The area of ΔABC is 60 square units. If $BD = 8$ units and $DC = 12$ units, the area (in square units) of ΔABD is



(**A**) 24 (**B**) 40 (**C**) 48
(**D**) 36 (**E**) 6

Solution
From A, draw a line perpendicular to BC to meet BC at E. Thus the line segment AE which is labelled as h is the height of ΔABD and ΔABC. Since the heights of the two triangles are equal, their areas are then proportionate to their bases. If the area of ΔABC is 60, then the area of ΔABD is $\frac{8}{20} \times 60 = 24$.



ANSWER: (A)

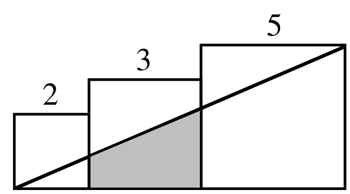
13. Crippin wrote four tests each with a maximum possible mark of 100. The average mark he obtained on these tests was 88. What is the lowest score he could have achieved on one of these tests?

(**A**) 88 (**B**) 22 (**C**) 52 (**D**) 0 (**E**) 50

Solution
If the average score of four tests was 88, this implies that a total of 4×88 or 352 marks were obtained. The lowest mark would be obtained if Crippin had three marks of 100 and one mark of 52.

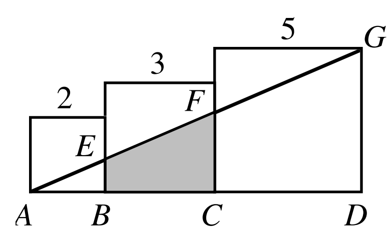
ANSWER: (C)

14. Three squares have dimensions as indicated in the diagram. What is the area of the shaded quadrilateral?



(**A**) $\frac{21}{4}$ (**B**) $\frac{9}{2}$ (**C**) 5
(**D**) $\frac{15}{4}$ (**E**) $\frac{25}{4}$

Solution 1
In the first solution, we use similar triangles. We start by labelling the diagram as shown. The objective in this question is to calculate the lengths EB and FC which will allow us to calculate the area of ΔAEB and ΔAFC. We first note that ΔAFC and ΔAGD are similar and that,



$$\frac{AC}{AD} = \frac{FC}{GD} = \frac{5}{10} = \frac{1}{2}.$$

Therefore, $FC = \frac{1}{2} GD = \frac{1}{2}(5) = \frac{5}{2}$.

Using the same reasoning, ΔAEB and ΔAFC are also similar triangles meaning that, $\frac{EB}{FC} = \frac{2}{5}$.

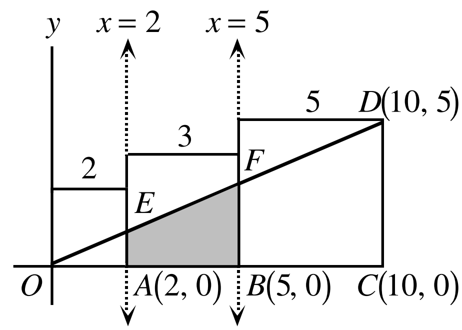
Thus, $EB = \frac{2}{5}\left(\frac{5}{2}\right) = 1$.

We find the required area to be

$$\begin{aligned} \text{area } \Delta AFC - \text{area } \Delta AEB &= \tfrac{1}{2}(5)\left(\tfrac{5}{2}\right) - \tfrac{1}{2}(2)(1) \\ &= \tfrac{21}{4}. \end{aligned}$$

Solution 2

We start by putting the information on a coordinate axes and labelling as shown. The line containing OD has equation $y = \frac{1}{2}x$ while $x = 2$ and $x = 5$ contains AE and BF. Solving the systems $y = \frac{1}{2}x$, $x = 2$ and $y = \frac{1}{2}x$, $x = 5$ gives the coordinates of E to be $(2, 1)$ and F to be $\left(5, \frac{5}{2}\right)$. This makes $AE = 1$ and $BF = \frac{5}{2}$ which now leads to exactly the same answer as in solution 1.



ANSWER: (A)

15. If $(a + b + c + d + e + f + g + h + i)^2$ is expanded and simplified, how many different terms are in the final answer?

(**A**) 36 (**B**) 9 (**C**) 45 (**D**) 81 (**E**) 72

Solution

$$\underbrace{(a + b + c + d + e + f + g + h + i)}_{\text{Bracket 1}}\underbrace{(a + b + c + d + e + f + g + h + i)}_{\text{Bracket 2}}$$

If we wish to determine how many different terms can be produced we begin by multiplying the 'a' in bracket 1 by each term in bracket 2. This calculation gives 9 different terms. We continue this process by now multiplying the 'b' in bracket 1 by the elements from b to i in bracket 2 to give 8 different terms. We continue this process until we finally multiply the 'i' in the first bracket by the 'i' in the second bracket. Altogether we have, $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$ different terms.

ANSWER: (C)

16. If $px + 2y = 7$ and $3x + qy = 5$ represent the same straight line, then p equals

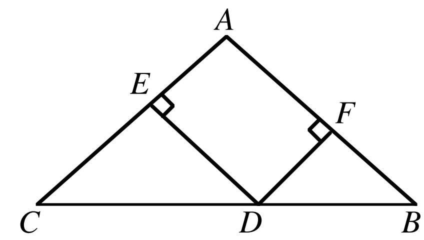
(**A**) 5 (**B**) 7 (**C**) 21 (**D**) $\frac{21}{5}$ (**E**) $\frac{10}{7}$

Solution

If we multiply the equation of the first line by 5 and the second by 7 we obtain, $5px + 10y = 35$ and $21x + 7qy = 35$. Comparing coefficients gives, $5p = 21$ or $p = \frac{21}{5}$.

ANSWER: (D)

17. In ΔABC, $AC = AB = 25$ and $BC = 40$. D is a point chosen on BC. From D, perpendiculars are drawn to meet AC at E and AB at F. $DE + DF$ equals

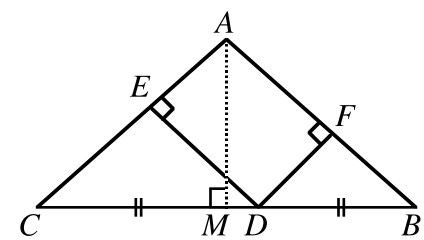


(**A**) 12 (**B**) 35 (**C**) 24

(**D**) 25 (**E**) $\frac{35}{2}\sqrt{2}$

Solution

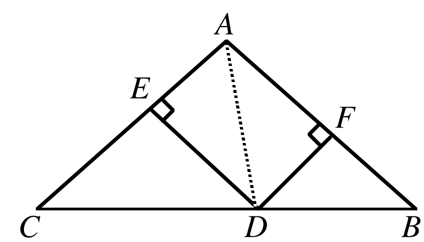
We start by drawing a line from A that is perpendicular to the base CB. Since ΔABC is isosceles, M is the midpoint of CB thus making $CM = MB = 20$. Using pythagoras in ΔACM we find AM to be $\sqrt{25^2 - 20^2} = 15$.



Join A to D. The area of ΔABC is $\frac{1}{2}(40)(15) = 300$ but it is also, $\frac{1}{2}(ED)(25) + \frac{1}{2}(DF)(25)$

$$= \frac{25}{2}(ED + DF).$$

Therefore, $ED + DF = \frac{2}{25}(300) = 24$.



ANSWER: (C)

18. The number of solutions (P, Q) of the equation $\frac{P}{Q} - \frac{Q}{P} = \frac{P+Q}{PQ}$, where P and Q are integers from 1 to 9 inclusive, is

(**A**) 1 (**B**) 8 (**C**) 16 (**D**) 72 (**E**) 81

Solution

If we simplify the rational expression on the left side of the equation and then factor the resulting numerator as a difference of squares we obtain,

$$\frac{(P-Q)(P+Q)}{PQ}.$$

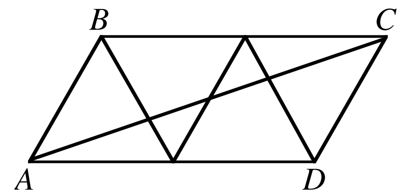
The equation can now be written as,

$$\frac{(P-Q)(P+Q)}{PQ} = \frac{P+Q}{PQ} \quad \text{or} \quad P - Q = 1 \quad (PQ \neq 0 \text{ and } P + Q \neq 0).$$

The only integers that satisfy this are: $(2,1)$, $(3,2)$, $(4,3)$, ..., $(9,8)$.

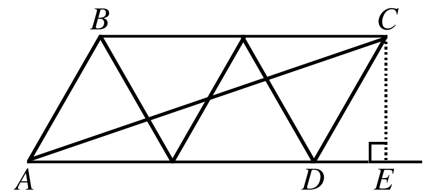
Thus there are 8 possibilities. ANSWER: (B)

19. Parallelogram $ABCD$ is made up of four equilateral triangles of side length 1. The length of diagonal AC is



(**A**) $\sqrt{5}$ (**B**) $\sqrt{7}$ (**C**) 3
(**D**) $\sqrt{3}$ (**E**) $\sqrt{10}$

Solution
From C, we draw a line perpendicular to AD extended so that they meet at point E as shown in the diagram.
This construction makes ΔCDE a $30° - 60° - 90°$ triangle with $\angle CDE = 60°$ and $CD = 1$. Thus $CE = \frac{\sqrt{3}}{2}$ and $DE = \frac{1}{2}$. Using pythagoras in ΔACE, we have $AE = \frac{5}{2}$ and $CE = \frac{\sqrt{3}}{2}$, $AC = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \sqrt{7}$.



ANSWER: (B)

20. If $a_1 = \dfrac{1}{1-x}$, $a_2 = \dfrac{1}{1-a_1}$, and $a_n = \dfrac{1}{1-a_{n-1}}$, for $n \geq 2$, $x \neq 1$ and $x \neq 0$, then a_{107} is

(**A**) $\dfrac{1}{1-x}$ (**B**) x (**C**) $-x$ (**D**) $\dfrac{x-1}{x}$ (**E**) $\dfrac{1}{x}$

Solution

$$a_1 = \frac{1}{1-x}$$

$$a_2 = \frac{1}{1-\frac{1}{1-x}} = \frac{(1-x)(1)}{(1-x)\left(1-\frac{1}{1-x}\right)} = \frac{1-x}{1-x-1} = \frac{1-x}{-x} = \frac{x-1}{x}$$

$$a_3 = \frac{1}{1-\frac{x-1}{x}} = \frac{x(1)}{x\left(1-\frac{x-1}{x}\right)} = \frac{x}{x-(x-1)} = x$$

$$a_4 = \frac{1}{1-x}$$

Since $a_1 = a_4$, we conclude $a_1 = a_4 = a_7 = ... = a_{3n-2} = a_{106}$.
Also, $a_2 = a_5 = a_8 = ... = a_{3n-1} = a_{107}$ for $n = 36$.
Since $a_2 = \dfrac{x-1}{x}$ then $a_{107} = \dfrac{x-1}{x}$.

ANSWER: (D)

Part C

21. How many integers can be expressed as a sum of three distinct numbers if chosen from the set $\{4, 7, 10, 13, ..., 46\}$?

(**A**) 45 (**B**) 37 (**C**) 36 (**D**) 43 (**E**) 42

Solution

Since each number is of the form $1+3n$, $n=1, 2, 3, ..., 15$, the sum of the three numbers will be of the form $3+3k+3l+3m$ where k, l and m are chosen from $\{1, 2, 3, ..., 15\}$. So the question is equivalent to the easier question of, 'How many distinct integers can be formed by adding three numbers from, $\{1, 2, 3, ..., 15\}$?'

The smallest is $1+2+3=6$ and the largest is $13+14+15=42$.

It is clearly possible to get every sum between 6 and 42 by:

(a) increasing the sum by one replacing a number with one that is 1 larger or,

(b) decreasing the sum by one by decreasing one of the addends by 1.

Thus all the integers from 6 to 42 inclusive can be formed.

This is the same as asking, 'How many integers are there between 1 and 37 inclusive?' The answer, of course, is 37. ANSWER: (B)

22. If $x^2+ax+48=(x+y)(x+z)$ and $x^2-8x+c=(x+m)(x+n)$, where y, z, m, and n are integers between -50 and 50, then the maximum value of ac is

(**A**) 343 (**B**) 126 (**C**) 52 234 (**D**) 784 (**E**) 98 441

Solution

For the equation, $x^2+ax+48=(x+y)(x+z)$ we consider the possible factorizations of 48 which produce different values for a. The factorizations and possible values for a are listed in the table that follows:

Possible Factorizations of 48	Possible Values for a
1×48, -1×-48	49, -49
2×24, -2×-24	26, -26
3×16, -3×-16	19, -19
4×12, -4×-12	16, -16
6×8, -6×-8 14, -14	

For the equation, $x^2-8x+c=(x+m)(x+n)$, we list some of its possible factorizations and the related possible values of c.

Possible Factorizations	Related Values of c
$(x-49)(x+41)$	$-49\times41=-2009$
$(x-48)(x+40)$	$-48\times40=-1920$
$\vdots$	$\vdots$

$(x-9)(x+1)$	$-9\times 1=-9$
$(x-8)(x+0)$	0

From these tables, we can see that the maximum value of ac is $-49\times -2009=98\,441$.

ANSWER: (E)

23. The sum of all values of x that satisfy the equation $\left(x^2-5x+5\right)^{x^2+4x-60}=1$ is

(**A**) -4 (**B**) 3 (**C**) 1 (**D**) 5 (**E**) 6

Solution

We consider the solution in three cases.

Case 1 It is possible for the base to be 1.

Therefore, $x^2-5x+5=1$

$$x^2-5x+4=0$$

$$(x-1)(x-4)=0$$

Therefore $x=1$ or $x=4$.

Both these values are acceptable for *x*.

Case 2 It is possible that the exponent be 0.

Therefore, $x^2+4x-60=0$

$$(x+10)(x-6)=0$$

$$x=-10 \text{ or } x=6$$

Note: It is easy to verify that neither $x=-10$ nor $x=6$ is a zero of x^2-5x+5, so that the indeterminate form $0°$ does not occur.

Case 3 It is possible that the base is –1 and the exponent is even.

Therefore, $x^2-5x+5=-1$ but $x^2+4x-60$ must also be even.

$$x^2-5x+5=-1$$

$$x^2-5x+6=0$$

$$(x-2)(x-3)=0$$

$$x=2 \text{ or } x=3$$

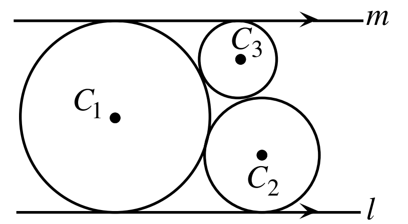
If $x=2$, then $x^2-4x-60$ is even, so $x=2$ is a solution.

If $x=3$, then $x^2-4x-60$ is odd, so $x=3$ is ***not*** a solution.

Therefore the sum of the solutions is $1+4-10+6+2=3$.

ANSWER: (B)

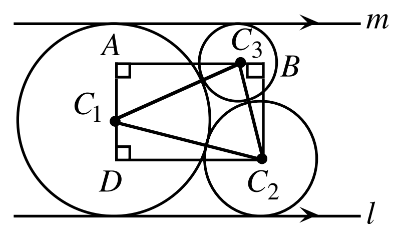
24. Two circles C_1 and C_2 touch each other externally and the line l is a common tangent. The line m is parallel to l and touches the two circles C_1 and C_3. The three circles are mutually tangent. If the radius of C_2 is 9 and the radius of C_3 is 4, what is the radius of C_1?



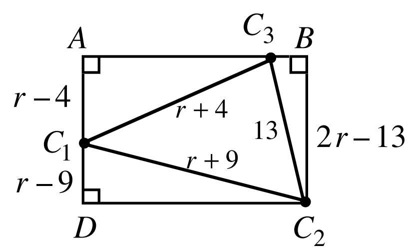
(**A**) 10.4 (**B**) 11 (**C**) $8\sqrt{2}$
(**D**) 12 (**E**) $7\sqrt{3}$

Solution

We start by joining the centres of the circles to form $\Delta C_1C_2C_3$. (The lines joining the centres pass through the corresponding points of tangency.)
Secondly, we construct the rectangle ABC_2D as shown in the diagram. If the radius of the circle with centre C_1 is r we see that: $C_1C_2 = r+9$, $C_1C_3 = r+4$ and $C_2C_3 = 13$.



We now label lengths on the rectangle in the way noted in the diagram.



To understand this labelling, look for example at C_1D. The radius of the large circle is r and the radius of the circle with centre C_2 is 9. The length C_1D is then $r-9$.
This same kind of reasoning can be applied to both C_1A and BC_2.

Using Pythagoras we can now derive the following:

In ΔAC_3C_1, $\quad C_3A^2 = (r+4)^2 - (r-4)^2$
$= 16r$.

Therefore $C_3A = 4\sqrt{r}$.

In ΔDC_1C_2, $\quad (DC_2)^2 = (r+9)^2 - (r-9)^2$
$= 36r$.

Therefore $DC_2 = 6\sqrt{r}$.

In ΔBC_3C_2, $\quad (C_3B)^2 = 13^2 - (2r-13)^2$
$= -4r^2 + 52r$.

Therefore $C_3B = \sqrt{-4r^2 + 52r}$.

In a rectangle opposite sides are equal, so:

$DC_2 = C_3A + C_3B$

or, $6\sqrt{r} = 4\sqrt{r} + \sqrt{-4r^2 + 52r}$

$2\sqrt{r} = \sqrt{-4r^2 + 52r}$.

Squaring gives, $4r = -4r^2 + 52r$

$4r^2 - 48r = 0$

$4r(r - 12) = 0$

Therefore $r = 0$ or $r = 12$.

Since $r > 0$, $r = 12$. ANSWER: (D)

25. Given that n is an integer, for how many values of n is $\dfrac{2n^2 - 10n - 4}{n^2 - 4n + 3}$ an integer?

(**A**) 9 (**B**) 7 (**C**) 6 (**D**) 4 (**E**) 5

Solution

We start by dividing $n^2 - 4n + 3$ into $2n^2 - 10n - 4$.

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\ 2 \\ n^2 - 4n + 3 & 2n^2 - 10n - 4 \\ & \underline{2n^2 - 8n\ \ + 6} \\ & \quad\ \ -2n - 10 \end{array}$$

This allows us to write the original expression in the following way,

$$\frac{2n^2 - 10n - 4}{n^2 - 4n + 3} = 2 + \frac{-2n - 10}{n^2 - 4n + 3} = 2 - \frac{2n + 10}{n^2 - 4n + 3}.$$

The original question comes down to the consideration of $\dfrac{2n + 10}{n^2 - 4n + 3}$ and when this expression is an integer. This rational expression can only assume integer values when, $2n + 10 \ge n^2 - 4n + 3$ (the numerator must be greater than the denominator) and when $2n + 10 = 0$.

Or, $n^2 - 6n - 7 \le 0$ and $n = -5$

or, $(n - 7)(n + 1) \le 0$

$-1 \le n \le 7$.

This means that we only have to consider values of n, $-1 \le n \le 7$, $n \in Z$ and $n = -5$. Also note that since $n^2 - 4n + 3 = (n - 1)(n - 3)$ we can remove $n = 1$ and $n = 3$ from consideration. We construct a table and check each value.

n	-5	-1	0	2	4	5	6	7
$\dfrac{2n + 10}{(n - 3)(n - 1)}$	0	$+1$	$\frac{10}{3}$	-14	6	$\frac{5}{2}$	$\frac{22}{15}$	1

From this table we can see that there are just four acceptable values of n that produce an integer.

Note also that $\frac{2n+10}{n^2-4n+3}$ would also be an integer if $2n+10=0$ and $n^2-4n+3 \neq 0$. Thus $n=-5$ is a fifth value since the denominator $\neq 0$. ANSWER: (E)