Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest (Grade 12)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

Tuesday, April 16, 2002

C.M.C. Sponsors: University of Waterloo; Deloitte & Touche Chartered Accountants

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Time: $2\frac{1}{2}$ hours

Calculators are permitted, provided they are non-programmable and without graphic displays.

Do not open this booklet until instructed to do so. The paper consists of 10 questions, each worth 10 marks. Parts of each question can be of two types. **SHORT ANSWER** parts are worth 2 marks each (questions 1-2) or 3 marks each (questions 3-7). **FULL SOLUTION** parts are worth the remainder of the 10 marks for the question.

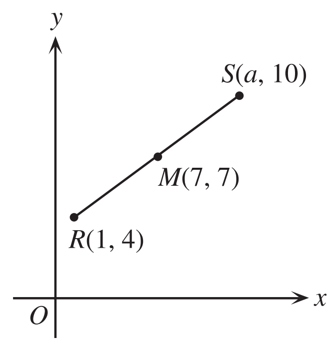
Instructions for SHORT ANSWER parts:
1. **SHORT ANSWER** parts are indicated like this: .
2. **Enter the answer in the appropriate box in the answer booklet.** For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

Instructions for FULL SOLUTION parts:
1. **FULL SOLUTION** parts are indicated like this: .
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be supplied by your supervising teacher. Insert these pages into your answer booklet.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.

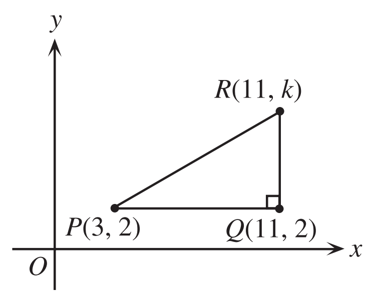
NOTE: At the completion of the contest, insert the information sheet inside the answer booklet.

NOTE: 1. Please read the instructions on the front cover of this booklet.
2. Place all answers in the answer booklet provided.
3. For questions marked "lightbulb", full marks will be given for a correct answer placed in the appropriate box in the answer booklet. **Marks may be given for work shown**. Students are strongly encouraged to show their work.
4. It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc., except where otherwise indicated.

1. (a) If $M(7,7)$ is the midpoint of the line segment which joins $R(1,4)$ and $S(a,10)$, what is the value of a?



(b) In the diagram, points $P(3,2)$, $Q(11,2)$ and $R(11,k)$ form a triangle with area 24, where $k>0$. What is the value of k?



(c) Lines are *concurrent* if they each pass through the same point. The lines $y=2x+3$, $y=8x+15$, and $y=5x+b$ are concurrent. What is the value of b?

2. (a) The quadratic equation $x^2-3x+c=0$ has $x=4$ as one of its roots. What is its second root?

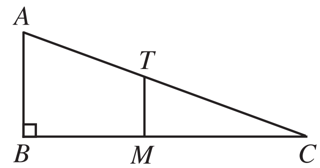
(b) The rational expression $\dfrac{2x^2+1}{x^2-3}$ may be written as $2+\dfrac{A}{x^2-3}$, where A is an integer. What is the value of A?

(c) The parabola $y=x^2-4x+3$ is translated 5 units to the right. In this new position, the equation of the parabola is $y=x^2-14x+d$. Determine the value of d.

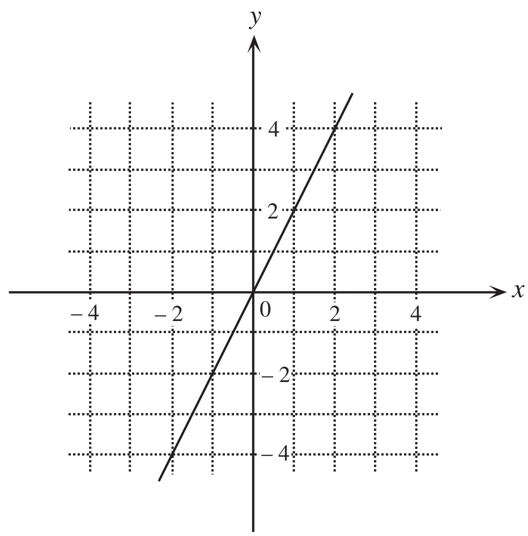
3. (a) Three bins are labelled A, B and C, and each bin contains four balls numbered 1, 2, 3, and 4. The balls in each bin are mixed, and then a student chooses one ball at random from each of the bins. If a, b and c are the numbers on the balls chosen from bins A, B and C, respectively, the student wins a toy helicopter when $a=b+c$. There are 64 ways to choose the three balls. What is the probability that the student wins the prize?

(b) Three positive integers a, ar and ar^2 form an increasing sequence. If the product of the three integers in this sequence is 216, determine all sequences satisfying the given conditions.

4. (a) In the diagram, triangle ABC is right-angled at B. MT is the perpendicular bisector of BC with M on BC and T on AC. If $AT = AB$, what is the size of $\angle ACB$?



(b) The graph of $y = f(x)$, where $f(x) = 2x$, is given on the grid below.



(i) On the grid in the answer booklet, draw and label the graphs of the inverse function $y = f^{-1}(x)$ and the reciprocal function $y = \dfrac{1}{f(x)}$.

(ii) State the coordinates of the points where $f^{-1}(x) = \dfrac{1}{f(x)}$.

(iii) Determine the numerical value of $f^{-1}\left(\dfrac{1}{f\left(\frac{1}{2}\right)}\right)$.

5. (a) What are all values of x such that

$$\log_5(x+3) + \log_5(x-1) = 1?$$

(b) A chef aboard a luxury liner wants to cook a goose. The time t in hours to cook a goose at 180°C depends on the mass of the goose m in kilograms according to the formula

$$t = am^b$$

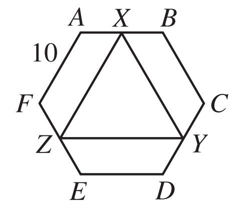
where a and b are constants. The table below gives the times observed to cook a goose at 180°C.

Mass, m (kg)	Time, t (h)
3.00	2.75
6.00	3.75

(i) Using the data in the table, determine both a and b to two decimal places.

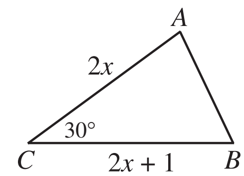
(ii) Suppose that the chef wants to cook a goose with a mass of 8.00 kg at 180°C. How long will it take until his goose is cooked?

6. (a) In the diagram, $ABCDEF$ is a regular hexagon with a side length of 10. If X, Y and Z are the midpoints of AB, CD and EF, respectively, what is the length of XZ?

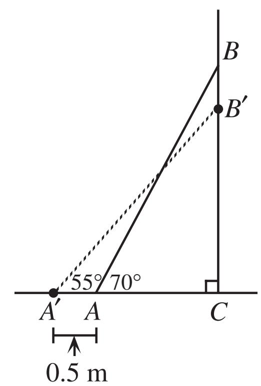


(b) A circle passes through the origin and the points of intersection of the parabolas $y = x^2 - 3$ and $y = -x^2 - 2x + 9$. Determine the coordinates of the centre of this circle.

7. (a) In the diagram, $AC = 2x$, $BC = 2x + 1$ and $\angle ACB = 30°$. If the area of ΔABC is 18, what is the value of x?



(b) A ladder, AB, is positioned so that its bottom sits on horizontal ground and its top rests against a vertical wall, as shown. In this initial position, the ladder makes an angle of 70° with the horizontal. The bottom of the ladder is then pushed 0.5 m away from the wall, moving the ladder to position $A'B'$. In this new position, the ladder makes an angle of 55° with the horizontal. Calculate, to the nearest centimetre, the distance that the ladder slides down the wall (that is, the length of BB').

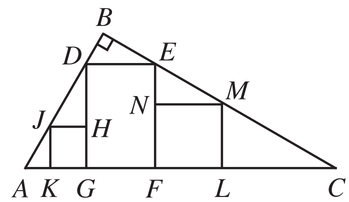


8. (a) In a soccer league with 5 teams, each team plays 20 games (that is, 5 games with each of the other 4 teams). For each team, every game ends in a win (W), a loss (L), or a tie (T). The numbers of wins, losses and ties for each team at the end of the season are shown in the table. Determine the values of x, y and z.

Team	W	L	T
A	2	15	3
B	7	9	4
C	6	12	2
D	10	8	2
E	x	y	z

(b) Prove that it is not possible to create a sequence of 4 numbers a, b, c, d, such that the sum of any two consecutive terms is positive, and the sum of any three consecutive terms is negative.

9. (a) In triangle ABC, $\angle ABC = 90°$. Rectangle $DEFG$ is inscribed in ΔABC, as shown. Squares $JKGH$ and $MLFN$ are inscribed in ΔAGD and ΔCFE, respectively. If the side length of $JHGK$ is v, the side length of $MLFN$ is w, and $DG = u$, prove that $u = v + w$.



(b) Three thin metal rods of lengths 9, 12 and 15 are welded together to form a right-angled triangle, which is held in a horizontal position. A solid sphere of radius 5 rests in the triangle so that it is tangent to each of the three sides. Assuming that the thickness of the rods can be neglected, how high above the plane of the triangle is the top of the sphere?

10. A triangle is called *Heronian* if each of its side lengths is an integer and its area is also an integer. A triangle is called *Pythagorean* if it is right-angled and each of its side lengths is an integer.

(a) Show that every Pythagorean triangle is Heronian.

(b) Show that every odd integer greater than 1 is a side length of some Pythagorean triangle.

(c) Find a Heronian triangle which has all side lengths different, and no side length divisible by 3, 5, 7 or 11.

PUBLICATIONS

Students and parents who enjoy solving problems for fun and recreation may find the following publications of interest. They are an excellent resource for enrichment, problem solving and contest preparation.

COPIES OF PREVIOUS CONTESTS (WITH FULL SOLUTIONS)

Copies of previous contests, together with solutions, are available in a variety of packages, as described below. Please order by package number, and note that the number defines the competition and the number of papers included. Each package has two numbers. Numbers prefixed with E are English language supplies – Numbers prefixed with F are French language supplies. Each package is considered as one title.

(For copies of 1999 - 2001 contests and solutions, please see our website (http://www.cemc.uwaterloo.ca)).

Included is one copy of any one contest, together with solutions, for each of 1998, 1999, and 2000. Recommended for individuals.

E 213, F 213	Gauss Contest (Grades 7, 8)	\$10.00
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University of Waterloo
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