2003 Hypatia Contest *(Grade 11)*

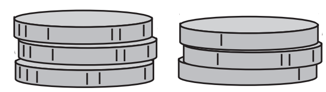
Wednesday, April 16, 2003

1. (a) Quentin has a number of square tiles, each measuring 1 cm by 1 cm. He tries to put these small square tiles together to form a larger square of side length n cm, but finds that he has 92 tiles left over. If he had increased the side length of the larger square to $(n+2)$ cm, he would have been 100 tiles short of completing the larger square. How many tiles does Quentin have?

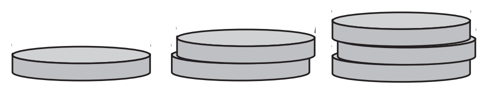
   (b) Quentin's friend Rufus arrives with a big pile of identical blocks, each in the shape of a cube. Quentin takes some of the blocks and Rufus takes the rest. Quentin uses his blocks to try to make a large cube with 8 blocks along each edge, but finds that he is 24 blocks short. Rufus, on the other hand, manages to exactly make a large cube using all of his blocks. If they use all of their blocks together, they are able to make a complete cube which has a side length that is 2 blocks longer than Rufus' cube. How many blocks are there in total?

2. Xavier and Yolanda are playing a game starting with some coins arranged in piles. Xavier always goes first, and the two players take turns removing one or more coins from any *one* pile. The player who takes the last coin wins.

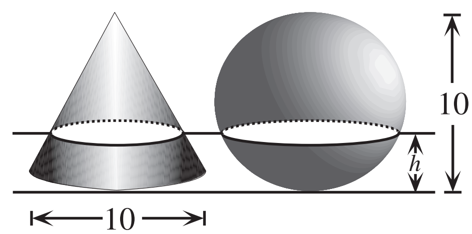
   (a) If there are two piles of coins with 3 coins in each pile, show that Yolanda can guarantee that she always wins the game.



   (b) If the game starts with piles of 1, 2 and 3 coins, explain how Yolanda can guarantee that she always wins the game.

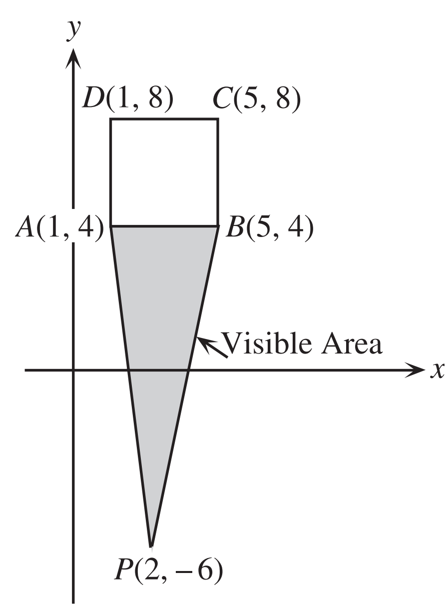


3. In the diagram, the sphere has a diameter of 10 cm. Also, the right circular cone has a height of 10 cm, and its base has a diameter of 10 cm. The sphere and cone sit on a horizontal surface. If a horizontal plane cuts both the sphere and the cone, the cross-sections will both be circles, as shown. Find the height of the horizontal plane that gives circular cross-sections of the sphere and cone of equal area.

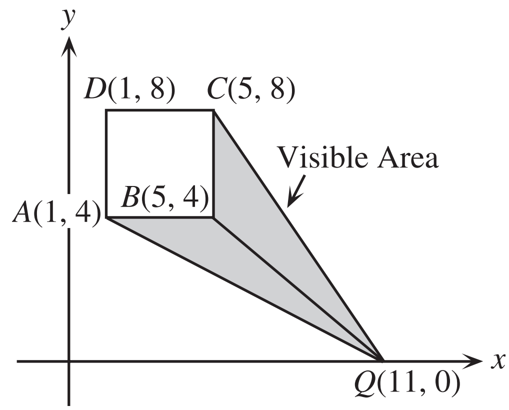


4. Square $ABCD$ has vertices $A(1,4)$, $B(5,4)$, $C(5,8)$, and $D(1,8)$. From a point P outside the square, a vertex of the square is said to be *visible* if it can be connected to P by a straight line that does not pass through the square. Thus, from any point P outside the square, either two or three of the vertices of the square are visible. The *visible area of P* is the area of the one triangle or the sum of the areas of the two triangles formed by joining P to the two or three visible vertices of the square.

   (a) Show that the *visible area* of $P(2,-6)$ is 20 square units.



(b) Show that the visible area of $Q(11, 0)$ is also 20 square units.



(c) The set of points P for which the visible area equals 20 square units is called the *20/20 set*, and is a polygon. Determine the perimeter of the 20/20 set.

Extensions (Attempt these only when you have completed as much as possible of the four main problems.)

Extension to Problem 1:
As in Question 1(a), Quentin tries to make a large square out of square tiles and has 92 tiles left over. In an attempt to make a second square, he increases the side length of this first square by *an unknown number of tiles* and finds that he is 100 tiles short of completing the square. How many different numbers of tiles is it possible for Quentin to have?

Extension to Problem 2:
If the game starts with piles of 2, 4 and 5 coins, which player wins if both players always make their best possible move? Explain the winning strategy.

Extension to Problem *3*:
A sphere of diameter d and a right circular cone with a base of diameter d stand on a horizontal surface. In this case, the height of the cone is equal to the *radius* of the sphere. Show that, for any horizontal plane that cuts both the cone and the sphere, the *sum* of the areas of the circular cross-sections is always the same.

Extension to Problem 4:
From any point P outside a unit cube, 4, 6 or 7 vertices are visible in the same sense as in the case of the square. Connecting point P to each of these vertices gives 1, 2 or 3 square-based pyramids, which make up the *visible volume* of P. The *20/20 set* is the set of all points P for which the visible volume is 20, and is a polyhedron. What is the surface area of this 20/20 set?