



# Canadian Mathematics Competition

An activity of The Centre for Education  
in Mathematics and Computing,  
University of Waterloo, Waterloo, Ontario

## Euclid Contest

for

**The CENTRE for EDUCATION in MATHEMATICS and COMPUTING  
Awards**

**Wednesday, April 14, 2004**

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iAnywhere Solutions


**Time:**  $2\frac{1}{2}$  hours

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**Calculators are permitted**, provided they are non-programmable and without graphic displays.

Do not open this booklet until instructed to do so. The paper consists of 10 questions, each worth 10 marks. Parts of each question can be of two types. **SHORT ANSWER** parts are worth 2 marks each (questions 1-2) or 3 marks each (questions 3-7). **FULL SOLUTION** parts are worth the remainder of the 10 marks for the question.


### Instructions for SHORT ANSWER parts:


1. **SHORT ANSWER** parts are indicated like this: .
2. **Enter the answer in the appropriate box in the answer booklet.** For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

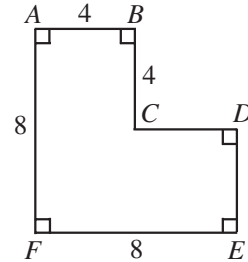
### Instructions for FULL SOLUTION parts:


1. **FULL SOLUTION** parts are indicated like this: .
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be supplied by your supervising teacher. Insert these pages into your answer booklet.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.

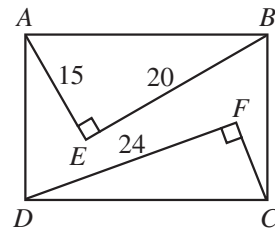
**NOTE:** At the completion of the contest, insert the information sheet inside the answer booklet.


- NOTE:
- Please read the instructions on the front cover of this booklet.
  - Place all answers in the answer booklet provided.
  - For questions marked “”, full marks will be given for a correct answer placed in the appropriate box in the answer booklet. **Marks may be given for work shown if an incorrect answer is given.** Students are strongly encouraged to show their work.
  - It is expected that all calculations and answers will be expressed as exact numbers such as  $4\pi$ ,  $2 + \sqrt{7}$ , etc., except where otherwise indicated.

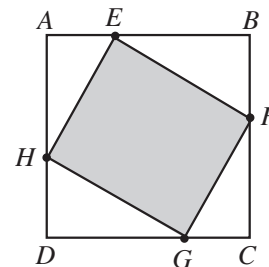
1.  (a) In the diagram, what is the area of figure  $ABCDEF$ ?





-  (b) In the diagram,  $ABCD$  is a rectangle with  $AE = 15$ ,  $EB = 20$  and  $DF = 24$ . What is the length of  $CF$ ?

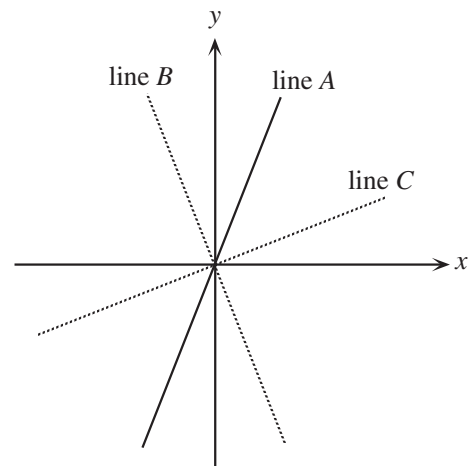


-  (c) In the diagram,  $ABCD$  is a square of side length 6. Points  $E$ ,  $F$ ,  $G$ , and  $H$  are on  $AB$ ,  $BC$ ,  $CD$ , and  $DA$ , respectively, so that the ratios  $AE:EB$ ,  $BF:FC$ ,  $CG:GD$ , and  $DH:HA$  are all equal to 1:2. What is the area of  $EFGH$ ?



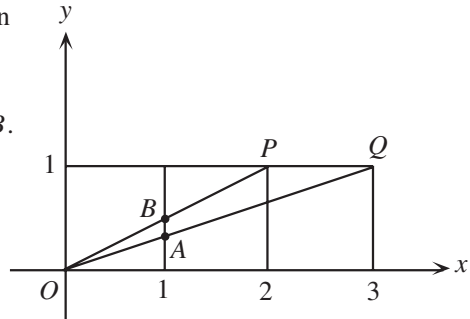
2.  (a) A horizontal line has the same  $y$ -intercept as the line  $3x - y = 6$ . What is the equation of this horizontal line?


-  (b) In the diagram, line  $A$  has equation  $y = 2x$ . Line  $B$  is obtained by reflecting line  $A$  in the  $y$ -axis. Line  $C$  is perpendicular to line  $B$ . What is the slope of line  $C$ ?






- (c) Three squares, each of side length 1, are drawn side by side in the first quadrant, as shown. Lines are drawn from the origin to  $P$  and  $Q$ . Determine, with explanation, the length of  $AB$ .



3.  (a) In an arithmetic sequence with five terms, the sum of the first two terms is 2 and the sum of the last two terms is  $-18$ . What is the third term of this sequence? (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7, 9, 11 is an arithmetic sequence with five terms.)




- (b) If  $x - y = 4\sqrt{2}$  and  $xy = 56$ , determine the two possible values of  $x + y$ .

4.  (a) Two fair dice, each having six faces numbered 1 to 6, are thrown. What is the probability that the *product* of the two numbers on the top faces is divisible by 5?

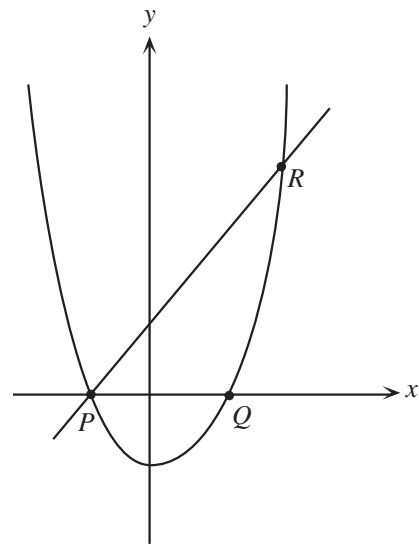



- (b) If  $f(x) = x^2 - x + 2$ ,  $g(x) = ax + b$ , and  $f(g(x)) = 9x^2 - 3x + 2$ , determine all possible ordered pairs  $(a, b)$  which satisfy this relationship.

5.  (a) If  $16^x = 2^{x+5} - 2^{x+4}$ , determine the value of  $x$ .



- (b) In the diagram, the parabola with equation  $y = x^2 + tx - 2$  intersects the  $x$ -axis at points  $P$  and  $Q$ . Also, the line with equation  $y = 3x + 3$  intersects the parabola at points  $P$  and  $R$ . Determine the value of  $t$  and the area of triangle  $PQR$ .



6.  (a) Lori has a loonie, three quarters, three dimes, three nickels, and five pennies. She wishes to purchase a toy helicopter for exactly \$1.34. What is the maximum number of coins that she can use to make the purchase? (In Canada, a loonie is worth \$1.00, a quarter is worth \$0.25, a dime is worth \$0.10, a nickel is worth \$0.05, and a penny is worth \$0.01.)




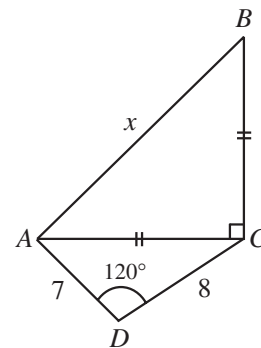
- (b) Digital images consist of a very large number of equally spaced dots called *pixels*. The *resolution* of an image is the number of pixels/cm in each of the horizontal and vertical directions.

Thus, an image with dimensions 10 cm by 15 cm and a resolution of 75 pixels/cm has a total of  $(10 \times 75) \times (15 \times 75) = 843\,750$  pixels.

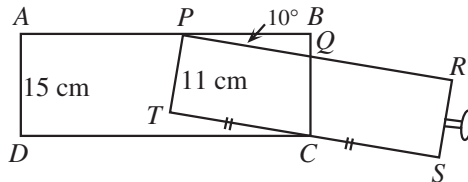
If each of these dimensions was increased by  $n\%$  and the resolution was decreased by  $n\%$ , the image would have 345 600 pixels.


Determine the value of  $n$ .

7.  (a) In the diagram,  $AC = BC$ ,  $AD = 7$ ,  $DC = 8$ , and  $\angle ADC = 120^\circ$ . What is the value of  $x$ ?



- (b) In the diagram, a drawer,  $PRST$ , that is 11 cm high is in a long slot,  $ABCD$ , which is 15 cm high. The drawer is pulled out so that the midpoint of its base rests at  $C$ . The drawer is tilted so that the top back edge of the drawer touches the top of the slot. If the angle between the drawer and the slot is  $10^\circ$ , determine the length of the drawer, to the nearest tenth of a centimetre.



8.  (a) If  $T = x^2 + \frac{1}{x^2}$ , determine the values of  $b$  and  $c$  so that  $x^6 + \frac{1}{x^6} = T^3 + bT + c$  for all non-zero real numbers  $x$ .

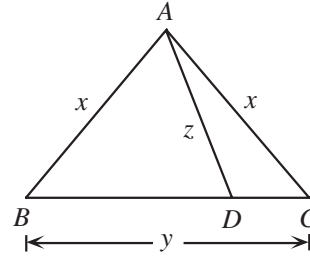



- (b) If  $x$  is a real number satisfying  $x^3 + \frac{1}{x^3} = 2\sqrt{5}$ , determine the exact value of  $x^2 + \frac{1}{x^2}$ .

9.  A *Kirk triplet* is a triple  $(x, y, z)$  of integers such that:

- i)  $x > z$ ,
- ii)  $z$  is a prime number, and
- iii) there is a triangle  $ABC$  with  $AB = AC = x$ ,  $BC = y$ , and a point  $D$  on  $BC$  such that  $AD = z$  and  $\angle ADB = 60^\circ$ .

- (a) Find the Kirk triplet with  $x = 7$  and  $z = 5$ .
- (b) Determine all other Kirk triplets with  $z = 5$ .
- (c) Determine the Kirk triplet for which  $\cos(\angle ABC)$  is as close to 0.99 as possible.



10.  A *Skolem sequence* of order  $n$  is a sequence  $(s_1, s_2, \dots, s_{2n})$  of  $2n$  integers satisfying the conditions:

- i) for every  $k$  in  $\{1, 2, 3, \dots, n\}$ , there exist exactly two elements  $s_i$  and  $s_j$  with  $s_i = s_j = k$ , and
- ii) if  $s_i = s_j = k$  with  $i < j$ , then  $j - i = k$ .

For example,  $(4, 2, 3, 2, 4, 3, 1, 1)$  is a Skolem sequence of order 4.

- (a) List all Skolem sequences of order 4.
- (b) Determine, with justification, all Skolem sequences of order 9 which satisfy all of the following three conditions:
  - I)  $s_3 = 1$ ,
  - II)  $s_{18} = 8$ , and
  - III) between any two equal even integers, there is exactly one odd integer.
- (c) Prove that there is no Skolem sequence of order  $n$ , if  $n$  is of the form  $4k + 2$  or  $4k + 3$ , where  $k$  is a non-negative integer.

## PUBLICATIONS

Students and parents who enjoy solving problems for fun and recreation may find the following publications of interest. They are an excellent resource for enrichment, problem solving and contest preparation.

### Copies of Previous Canadian Mathematics Competitions

Copies of previous contests and solutions are available at no cost in both English and French at <http://www.cemc.uwaterloo.ca>

### Problems Problems Problems Books

Each volume is a collection of problems (multiple choice and full solution), grouped into 9 or more topics. Questions are selected from previous Canadian Mathematics Competition contests, and full solutions are provided for all questions. The price is \$15. **(Available in English only.)**

#### Volume 1

- over 300 problems and full solutions
- 10 topics
- for students in Grades 9, 10, & 11
- French version of Volume 1 is available

#### Volume 3

- over 235 problems and full solutions
- 12 topics
- for senior high school students

#### Volume 5

- over 200 problems and full solutions
- 9 topics (different from Volume 3)
- for senior high school students

#### Volume 7

- over 300 problems and full solutions
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- for students in Grades 9 and 10

#### Volume 9

- over 300 problems and full solutions
- 11 topics
- for students in Grades 7 and 8

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#### Volume 4

- over 325 problems and full solutions
- 12 topics
- for students in Grades 7, 8, & 9

#### Volume 6

- over 300 problems and full solutions
- 11 topics
- for students in Grades 7, 8, & 9

#### Volume 8

- over 200 problems and full solutions
- 10 topics
- for students in Grades 11 and 12

Orders should be addressed to: Canadian Mathematics Competition  
Faculty of Mathematics, Room 5181  
University of Waterloo  
Waterloo, ON N2L 3G1

Include your name, address (with postal code), and telephone number.

Cheques or money orders in Canadian funds should be made payable to "Centre for Education in Mathematics and Computing". In Canada, add \$3.00 for the first item ordered for shipping and handling, plus \$1.00 for each subsequent item. No Provincial Sales Tax is required, but 7% GST must be added. Orders *outside of Canada ONLY*, add \$10.00 for the first item ordered for shipping and handling, plus \$2.00 for each subsequent item. **Prices for these publications will remain in effect until September 1, 2004.**

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