Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2004 Solutions
Fryer Contest (Grade 9)

1. (a) Since $\xrightarrow{R}$ indicates that we take a reciprocal, then $3 \xrightarrow{R} \frac{1}{3}$.
Since $\xrightarrow{A}$ indicates to take a reciprocal, then $\frac{1}{3} \xrightarrow{A} \frac{1}{3}+1=\frac{4}{3}$.
Continuing, $\frac{4}{3} \xrightarrow{R} \frac{3}{4}$, $\frac{3}{4} \xrightarrow{A} \frac{3}{4}+1=\frac{7}{4}$ and $\frac{7}{4} \xrightarrow{R} \frac{4}{7}$.
In summary, $3 \xrightarrow{R} \frac{1}{3} \xrightarrow{A} \frac{4}{3} \xrightarrow{R} \frac{3}{4} \xrightarrow{A} \frac{7}{4} \xrightarrow{R} \frac{4}{7}$.

(b) As in (a),

$$x \xrightarrow{R} \frac{1}{x}$$
$$\frac{1}{x} \xrightarrow{A} \frac{1}{x}+1=\frac{1}{x}+\frac{x}{x}=\frac{1+x}{x}$$
$$\frac{1+x}{x} \xrightarrow{R} \frac{x}{1+x}$$
$$\frac{x}{1+x} \xrightarrow{A} \frac{x}{1+x}+1=\frac{x}{1+x}+\frac{1+x}{1+x}=\frac{1+2x}{1+x}$$
$$\frac{1+2x}{1+x} \xrightarrow{R} \frac{1+x}{1+2x}$$

In summary,

$$x \xrightarrow{R} \frac{1}{x} \xrightarrow{A} \frac{1+x}{x} \xrightarrow{R} \frac{x}{1+x} \xrightarrow{A} \frac{1+2x}{1+x} \xrightarrow{R} \frac{1+x}{1+2x}$$

(Since we started by taking the reciprocal of x, we are assuming that x is not equal to 0.)

(c) *Solution 1*
By inspection, we observe that if $\frac{1+x}{1+2x}$ equals $\frac{14}{27}$, then $x+1=14$ or $x=13$. We check to see that the value $x=13$ makes the denominator equal to 27, which it does.
(We note that we have to be careful to *check* that our answer works here, because it will not work for instance if $\frac{1+y}{1+3y}=\frac{2}{5}$, where the solution is $y=3$.)

Solution 2
From (b), if the operations begin with x, the final result is $\frac{1+x}{1+2x}$.
Since we want the final result to be $\frac{14}{27}$, then we set $\frac{1+x}{1+2x}=\frac{14}{27}$ and solve for x to determine the original input:

$$\frac{1+x}{1+2x}=\frac{14}{27}$$
$$27(1+x)=14(1+2x)$$
$$27+27x=14+28x$$
$$13=x$$

Therefore, the input should be 13 to obtain a final result of $\frac{14}{27}$. (We can check this by actually doing the 5 operations starting with 13.)

Solution 3

We start with $\frac{14}{27}$ and work backwards through the operations, using the "inverse" of each of the steps.

Since the fifth operation was taking a reciprocal to get $\frac{14}{27}$, then the number at the previous step must have been $\frac{27}{14}$.

Since the fourth operation was adding 1 to get $\frac{27}{14}$, then the number at the previous step must have been $\frac{27}{14} - 1 = \frac{13}{14}$.

Since the third operation was taking a reciprocal to get $\frac{13}{14}$, then the number at the previous step must have been $\frac{14}{13}$.

Since the second operation was adding 1 to get $\frac{14}{13}$, then the number at the previous step must have been $\frac{14}{13} - 1 = \frac{1}{13}$.

Since the first operation was taking a reciprocal to get $\frac{1}{13}$, then the original input must have been 13.

2. (a) Since at least one of each type of prize is given out, then these four prizes account for $\$5 + \$25 + \$125 + \$625 = \$780$.
Since there are five prizes given out which total \$905, then the fifth prize must have a value of $\$905 - \$780 = \$125$.
Thus, the Fryer Foundation gives out one \$5 prize, one \$25 prize, two \$125 prizes, and one \$625 prize.

(b) As in (a), giving out one of each type of prize accounts for \$780.
The fifth prize could be a \$5 prize for a total of $\$780 + \$5 = \$785$.
The fifth prize could be a \$25 prize for a total of $\$780 + \$25 = \$805$.
The fifth prize could be a \$625 prize for a total of $\$780 + \$625 = \$1405$.
(We already added an extra \$125 prize in (a).)

(c) *Solution 1*
Since at least one of each type of prize is given out, this accounts for \$780. So we must figure out how to distribute the remaining $\$880 - \$780 = \$100$ using at most 5 of each type of prize. We cannot use any \$125 or \$625 prizes, since these are each greater than the remaining amount.
We could use four additional \$25 prizes to make up the \$100.
Could we use fewer than four \$25 prizes? If we use three additional \$25 prizes, this accounts for \$75, which leaves \$25 remaining in \$5 prizes, which can be done by using five additional \$5 prizes.
Could we use fewer than three \$25 prizes? If so, then we would need to make at least \$50 with \$5 prizes, for which we need at least ten such prizes. But we can use at most six \$5 prizes in total, so this is impossible.

Therefore, the two ways of giving out $880 in prizes under the given conditions are:

i) one $625 prize, one $125 prize, five $25 prizes, one $5 prize

ii) one $625 prize, one $125 prize, four $25 prizes, six $5 prizes

We can check by addition that each of these totals $880.

Solution 2

We know that the possible total values using at least one of each type of prize and exactly five prizes are $785, $805, $905 and $1405.

We try starting with $785 and $805 to get to $880. (Since $905 and $1405 are already larger than $880, we do not need to try these.)

Starting with $785, we need to give out an additional $95 to get to $880. Using three $25 prizes accounts for $75, leaving $20 to be split among four $5 prizes. (Using fewer than three $25 prizes will mean we need more than six $5 prizes in total.) So in this way, we need one $625 prize, one $125 prize, four $25 prizes, and six $5 prizes (since there were already two included in the $785).

Starting with $805, we need to give out an additional $75 to get to $880. Using three $25 prizes will accomplish this, for a total of one $625 prize, one $125 prize, five $25 prizes, and one $5 prize. We could also use two $25 prizes and five $5 prizes to make up the $75, for a total of one $625 prize, one $125 prize, four $25 prizes, and six $5 prizes (which is the same as we obtained above starting with $785). If we use fewer than two additional $25 prizes, we would need too many $5 prizes.

Therefore, the two ways of giving out $880 in prizes under the given conditions are:

i) one $625 prize, one $125 prize, five $25 prizes, one $5 prize

ii) one $625 prize, one $125 prize, four $25 prizes, six $5 prizes

We can check by addition that each of these totals $880.

3. (a) If Bob places a 3, then the total of the two numbers so far is 8, so Avril should place a 7 to bring the total up to 15.

Since Bob can place a 3 in any the eight empty circles, Avril should place a 7 in the circle directly opposite the one in which Bob places the 3. This allows Avril to win on her next turn.

(b) As in (a), Bob can place any of the numbers 1, 2, 3, 4, 6, 7, 8, 9 in any of the eight empty circles. On her next turn, Avril should place a disc in the circle directly opposite the one in which Bob put his number. What number should Avril use? Avril should place the number that brings the total up to 15, as shown below:

Bob's First Turn	Total so far	Avril's Second Turn
1	6	9
2	7	8
3	8	7
4	9	6
6	11	4
7	12	3
8	13	2
9	14	1

Since each of these possibilities is available to Avril on her second turn (since 5 is not in the list and none is equal to Bob's number), then she can always win on her second turn.

(c) Bob can place any of the numbers 1, 2, 4, 5, 7, 8 in any of the six empty circles. We can pair these numbers up so that the sum of the two numbers in the pair plus 6 is equal to 15:

1 and 8; 2 and 7; 4 and 5

So when Bob uses one of these numbers, Avril can use the other number from the pair, place it directly opposite the one that Bob entered, and the total of the three numbers on this line through the centre will be 15, so Avril will win the game.

4. (a) The 10th triangular number is

$$1+2+3+4+5+6+7+8+9+10=55$$

The 24th triangular number is $1+2+3+\cdots+23+24$. We could add this up by hand or on a calculator to obtain 300. Instead, we could notice that if we pair up the numbers starting with the first and last, then the second and second last, and so on, we obtain:

$$(1+24)+(2+23)+(3+22)+\cdots+(11+14)+(12+13)$$

that is, 12 pairs, each adding to 25, giving a total of $12\times 25=300$. (This pairing is the basis for a general method of finding a formula for $1+2+\cdots+n$.)

(b) *Solution 1*

Let the three consecutive triangle numbers be

$1+2+3+\cdots+(n-1)+n \qquad (1)$,

$1+2+3+\cdots+(n-1)+n+(n+1) \qquad (2)$, and

$1+2+3+\cdots+(n-1)+n+(n+1)+(n+2) \qquad (3)$.

We add these three numbers up, and bring the last term (the $(n+2)$) from (3) to (1):

$$
\begin{aligned}
&\left[1+2+3+\cdots+(n-1)+n+(n+2)\right] \\
&\qquad +\left[1+2+3+\cdots+(n-1)+n+(n+1)\right] \\
&\qquad +\left[1+2+3+\cdots+(n-1)+n+(n+1)\right] \\
&=\left[1+2+3+\cdots+(n-1)+n+(n+1)\right]+1 \qquad \text{(writing } n+2 \text{ as } (n+1)+1\text{)} \\
&\qquad +\left[1+2+3+\cdots+(n-1)+n+(n+1)\right] \\
&\qquad +\left[1+2+3+\cdots+(n-1)+n+(n+1)\right] \\
&=3\left[1+2+3+\cdots+(n-1)+n+(n+1)\right]+1
\end{aligned}
$$

which is 1 more than three times the middle of these three numbers.

Solution 2

This solution uses the formula $1+2+3+\cdots+(n-1)+n=\frac{n(n+1)}{2}$ (which we can obtain using a similar pairing argument that we saw in (a)).

So the three consecutive triangle numbers are

$1+2+3+\cdots+(n-1)+n=\frac{n(n+1)}{2}$,

$1+2+3+\cdots+(n-1)+n+(n+1)=\frac{(n+1)(n+2)}{2}$, and

$1+2+3+\cdots+(n-1)+n+(n+1)+(n+2)=\frac{(n+2)(n+3)}{2}$

Adding these three, we obtain

$$
\begin{aligned}
&\frac{n(n+1)}{2}+\frac{(n+1)(n+2)}{2}+\frac{(n+2)(n+3)}{2} \\
&=\frac{n^2+n}{2}+\frac{n^2+3n+2}{2}+\frac{n^2+5n+6}{2} \\
&=\frac{3n^2+9n+8}{2} \\
&=\frac{3n^2+9n+6}{2}+1 \\
&=3\left(\frac{n^2+3n+2}{2}\right)+1
\end{aligned}
$$

which is 1 more then three times the middle number of the three consecutive triangular numbers.

(c) *Solution 1*

We are told that the 3rd, 6th and 8th triangular numbers are in arithmetic sequence, and that the 8th, 12th and 15th triangular numbers are in arithmetic sequence. This seems to suggest a pattern, since we add 3 to 3 to 6 and then 2 to 6 to get 8, followed by 4 to 8 to get 12 and 3 to 12 to get 15.

This suggests that the 15th, 20th and 24th (adding 5 to 15 and then adding 4) triangular numbers are in arithmetic sequence. We know that the 15th triangular number is 120 (given) and that the 24th triangular number is 300 (from (a)). We can check using a calculator that the 20th triangular number is 210. This suggests that the pattern continues. (But this doesn't prove that the pattern works!)

Continuing the pattern we get the following groups of numbers:

24th, 30th and 35th
35th, 42nd and 48th
48th, 56th and 63rd

Let's try the third set:

$$1+2+\cdots+47+48=(1+48)+(2+47)+\cdots+(24+25)=24\times 49=1176$$

so this is still too small, since we want all three to be bigger than 2004.

Continuing the pattern:

63rd, 72nd, 80th,
80th, 90th and 99th

Let's try this set. Using our pairing technique from above:

$$1+2+\cdots+79+80=40\times 81=3240$$
$$1+2+\cdots+89+90=45\times 91=4095$$
$$1+2+\cdots+98+99=(1+2+\cdots+99+100)-100=50\times 101-100=4950$$

Checking these, $4950-4095=855=4095-3240$, so the 80th, 90th and 99th triangular numbers are in arithmetic sequence.

Solution 2

In the two given examples, the difference between the positions of the first two numbers (ie. the 3rd and 6th in the first example) is one more than the difference between the positions of the second two numbers.

Let's try this pattern and see if we can continue this.

Let's look at the nth triangular number, the $(n+12)$th triangular number (12 positions further along) and the $(n+23)$th triangular number (11 positions further along). Can we find a value of n which makes these in arithmetic sequence? (There was no special reason to choose $(n+12)$; we could try larger or smaller numbers to see if they work.) If these are in arithmetic sequence, then

$$\begin{aligned}
[1+2+\cdots+(n+12)]-[1+2+\cdots+n] &= [1+2+\cdots+(n+23)]-[1+2+\cdots+(n+12)] \\
(n+1)+(n+2)+\cdots+(n+12) &= (n+13)+(n+14)+\cdots+(n+23) \\
12n+(1+2+\cdots+12) &= 11n+(13+14+\cdots+23) \\
n &= (13+14+\cdots+23)-(1+2+\cdots+11)-12 \\
n &= 11(12)-12 \\
n &= 120
\end{aligned}$$

So the 120th, 132nd and 143rd triangular numbers are in arithmetic progression.
We could calculate these numbers using the pairing idea from (a) to check our answer:

$1+2+\cdots+119+120 = 60\times 121 = 7260$

$1+2+\cdots+131+132 = 66\times 133 = 8778$

$1+2+\cdots+142+143 = (1+2+\cdots+143+144)-144 = 72\times 145-144 = 10296$

Checking these, $8778-7260 = 1518 = 10296-8778$, so the 120th, 132nd and 143rd triangular numbers are in arithmetic sequence.