Canadian
Mathematics
Competition

An activity of The Centre for Education
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University of Waterloo, Waterloo, Ontario

2004 Solutions
Pascal Contest (Grade 9)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
Awards

2004 Pascal Contest Solutions

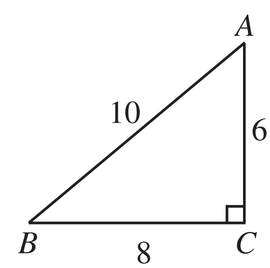
1. Using the proper order of operations,
$$5\times(10-6)\div 2=5\times 4\div 2=20\div 2=10$$
Answer: (A)

2. *Solution 1*
Since the average of 2, x and 12 is 8,
$$\frac{2+x+12}{3}=8$$
$$14+x=24$$
$$x=10$$

Solution 2
Since the average of three numbers is 8, their sum must be 24. Since the sum of 2, x and 12 is 24, then x must be 10.
Answer: (E)

3. The lowest common denominator of three fractions is the least common multiple of the three denominators. The least common multiple of 9, 4 and 18 is 36.
Answer: (D)

4. The area of triangle ABC is $\frac{1}{2}(BC)(AC)=4(AC)$, since the triangle is right-angled at C.
By the Pythagorean Theorem, $AC^2+BC^2=AB^2$ or $AC^2+8^2=10^2$ or $AC^2=36$, which tells us that $AC=6$.
Therefore, the area of the triangle is 24.



Answer: (E)

5. Calculating,
$$\frac{5-\sqrt{4}}{5+\sqrt{4}}=\frac{5-2}{5+2}=\frac{3}{7}$$
Answer: (A)

6. Calculating,
$$4^1+3^2-2^3+1^4=4+9-8+1=6$$
Answer: (C)

7. Substituting $x=-3$,

$$3x^2+2x=3(-3)^2+2(-3)=3(9)-6=21$$

Answer: (D)

8. *Solution 1*

Since 18% of 42 is equal to 27% of *x*, then

$$\frac{18}{100}(42)=\frac{27}{100}x$$
$$18(42)=27x$$
$$x=28$$

Solution 2

Since 18% is two-thirds of 27%, and 18% of 42 is equal to 27% of *x*, then *x* must be two-thirds of 42. Thus, *x* equals 28.

Answer: (A)

9. A cube has six faces, each of which is a square.

If a cube has a surface area of 96 cm^2, then each of its faces has an area of one-sixth of this total, or 16 cm^2.

Since each face has an area of 16 cm^2, then the edge length of the cube is 4 cm.

Since the edge length of the cube is 4 cm, its volume is $4^3\text{ cm}^3=64\text{ cm}^3$.

Answer: (B)

10. *Solution 1*

Since $y=3x-5$ and $z=3x+3$, then $z-y=(3x+3)-(3x-5)=8$, ie. z is 8 more than y. Since $y=1$, then $z=9$.

Solution 2

Since $y=3x-5$ and $y=1$, then $3x-5=1$ or $x=2$.

Since $x=2$, then $z=3(2)+3=9$.

Answer: (E)

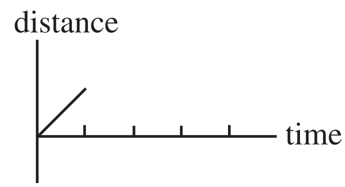
11. In the diagram, the square is divided into four rectangles, each of which has been divided in half to form two identical triangles. So in each of the four rectangles, the area of the shaded triangle equals the area of the unshaded triangle. Thus, the area of the shaded region is $\frac{1}{2}$ of the overall area of the square, or $\frac{1}{2}$ of 16 square units, or 8 square units.

Answer: (B)

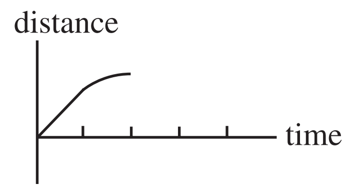
12. From the two given balances, 3 ▲'s balance 5 ●'s, and 1 ▲ balances 2 ■'s and 1 ●. Tripling the quantities on the second balance implies that 3 ▲'s will balance 6 ■'s and 3 ●'s. Therefore, 5 ●'s will balance 6 ■'s and 3 ●'s, and removing 3 ●'s from each side implies that 2 ●'s will balance 6 ■'s, or 1 ● will balance 3 ■'s.

Answer: (C)

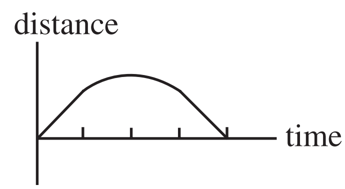
13. Suppose the length of one side of the park is 1. When Nadia is one quarter of the way around the park (ie. the top corner), she is at a distance of 1 from *S*.



From the quarter-way point to the half-way point, her distance from *S* is steadily increasing. When she is half-way around the park, her distance from *S* will be $\sqrt{2}$ (since we can form a right-angled triangle with two sides of length 1 and the hypotenuse joining *S* to the opposite corner).



As she completes her circuit of the park, the graph will be completed as follows.



Therefore, the graph must be the one from (C), since this is the only one which satisfies this condition. (The graph is indeed slightly rounded in the middle.)

Answer: (C)

14. *Solution 1*

The first figure has 8 unshaded squares. The second figure has 12 unshaded squares. The third figure has 16 unshaded squares. So the number of unshaded squares increases by 4 with each new figure. So the number of unshaded squares in the tenth figure should be $8+4(9)=44$ (we add 4 nine times to get from the first figure to the second, from the second to the third, and so on).

Solution 2

The first figure is a 3 by 3 square with a 1 by 1 square shaded in. The second figure is a 4 by 4 square with a 2 by 2 square shaded in. The third figure is a 5 by 5 square with a 3 by 3 square shaded in. Therefore, the tenth figure should be a 12 by 12 square with a 10 by 10 square shaded in. So the number of unshaded squares is $12^2-10^2=44$.

Answer: (D)

15. Since each child has at least 2 brothers, then each boy has at least 2 brothers. So there have to be at least 3 boys.
Since each child has at least 1 sister, then each girl has at least 1 sister. So there have to be at least 2 girls.
Therefore, the Pascal family has at least 5 children.

Answer: (C)

16. We try positive integer values for a to see the resulting value of b is a positive integer.
If $a=1$, then $a^2=1$, so $3b=32$, so b is not a positive integer.
If $a=2$, then $a^2=4$, so $3b=29$, so b is not a positive integer.
If $a=3$, then $a^2=9$, so $3b=24$, so $b=8$.
If $a=4$, then $a^2=16$, so $3b=17$, so b is not a positive integer.
If $a=5$, then $a^2=25$, so $3b=8$, so b is not a positive integer.
If a is at least 6, then a^2 is at least 36, so $3b$ is negative, and b cannot be a positive integer.
Therefore, there is only one possible pair of values for a and b, so $ab=3(8)=24$.

Answer: (B)

17. Since $0.\overline{12}$ has a period of length 2 and $0.\overline{123}$ has a period of length 3, we must expand each of the three given decimals to six places:

$$0.\overline{1}=0.111111...$$
$$0.\overline{12}=0.121212...$$
$$0.\overline{123}=0.123123...$$

When we add these three numbers as decimals, we get

$$0.\overline{1}+0.\overline{12}+0.\overline{123}=0.355446...$$

so the answer must be $0.\overline{1}+0.\overline{12}+0.\overline{123}=0.\overline{355446}$.

Answer: (D)

18. Using the definition of the symbol,

$$\begin{aligned}(x-1)(-5)-(2)(3)&=9\\ -5x+5-6&=9\\ -5x&=10\\ x&=-2\end{aligned}$$

Answer: (C)

19. Every week, a branch that is at least two weeks old produces a new branch. Therefore, after a given week, the total number of branches is equal to the number of branches at the beginning of the week plus the number of branches that were not new during the previous week (ie. the number of "old" branches). So we make a chart:

Week #	Branches at beginning	Old branches at beginning	Branches at end
1	0	0	1
2	1	0	1
3	1	1	2
4	2	1	3
5	3	2	5
6	5	3	8
7	8	5	13
8	13	8	21

Therefore, there are 21 branches at the end of the eighth week.

Answer: (A)

20. We start by making a chart in which we determine the position after the next turn given any current position:

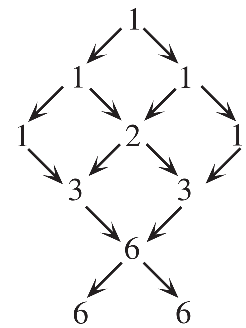
Current Position	Position after next turn
1	2
2	4
3	6
4	1
5	3
6	5
7	7

So for the arrow to point to the 6 after the 21st turn, it must have pointed to the 3 after the 20th turn (from the third row of the chart). For it to point to the 3 after the 20th turn, it must have pointed to the 5 after the 19th turn, and to the 6 after the 18th turn.
This pattern now continues in a cycle, from which we can conclude the arrow pointed to the 6 after the 15th turn, the 12th turn, the 9th turn, the 6th turn and the 3rd turn.
Since it pointed to the 6 after the 3rd turn, it pointed to the 3 after the 2nd turn and the 5 after the 1st turn.

Answer: (C)

21. First we notice that any path following the arrows from the top P to one of the two bottom L's actually does spell the word "PASCAL", so we need to count the total number of paths from the top to the bottom.
We will proceed by counting the number of paths that reach each of the letters in the diagram. To do this, we see that the number of paths reaching a letter in the diagram is the sum of the number of paths reaching all of the letters which directly lead to the desired letter.
So we can fill in the number of paths leading to each letter

So in total, there are 12 paths through the diagram – 6 which lead to the left "L" and 6 which lead to the right "L".



Answer: (C)

22. Let d be the original depth of the water.
Then the total volume of water in the container initially is $20 \times 20 \times d = 400d$.
The volume of the gold cube is $15 \times 15 \times 15 = 3375$.
After the cube has been added to the water, the total volume inside the container that is filled is $20 \times 20 \times 15 = 6000$, since the container with base 20 cm by 20 cm has been filled to a depth of 15 cm.
Therefore, $400d + 3375 = 6000$ or $400d = 2625$ or $d = 6.5625$. So d is closest to 6.56 cm.

Answer: (A)

23. We start by labelling the two quarters Q_1 and Q_2, the two dimes D_1 and D_2, and the two nickels N_1 and N_2. We then make a chart where the labels on the left tell us which coin is chosen first and the labels on the top tell us which coin is chosen second. Inside the chart, we put a Y if the combination will pay the toll and an N if the combination will not pay the toll. (We put X's on the diagonal, since we cannot choose a coin first and then the same coin the second time.)

	Q_1	Q_2	D_1	D_2	N_1	N_2
Q_1	X	Y	Y	Y	Y	Y
Q_2	Y	X	Y	Y	Y	Y
D_1	Y	Y	X	N	N	N
D_2	Y	Y	N	X	N	N
N_1	Y	Y	N	N	X	N
N_2	Y	Y	N	N	N	X

For example, if Q_1 is chosen first and then N_1, the driver has chosen 30 cents, so can pay the toll. If D_2 is chosen first and then D_1, then 20 cents has been chosen, and he cannot pay the toll.

From the chart, there are 30 possible combinations that can be chosen (since the X's on the main diagonal indicate that these combinations are not possible), with 18 of them enough to pay the toll, and 12 not enough to pay the toll.

Therefore, the probability is $\frac{18}{30} = \frac{3}{5}$.

Answer: (A)

24. In analyzing this sequence of fractions, we start by observing that this large sequence is itself made up of smaller sequences. Each of these smaller sequences is of the form

$$\frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}, \ldots, \frac{3}{n-2}, \frac{2}{n-1}, \frac{1}{n}$$

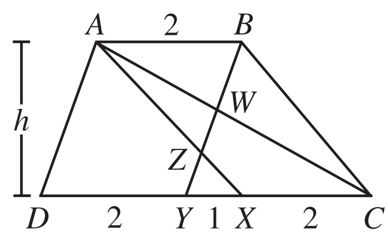
with the denominators increasing from 1 to n and the numerators decreasing from n to 1.
We observe that there is 1 term in the first of these smaller sequences, 2 terms in the second of these, and so on. This can be seen in the following grouping:

$$\left(\tfrac{1}{1}\right), \left(\tfrac{2}{1}, \tfrac{1}{2}\right), \left(\tfrac{3}{1}, \tfrac{2}{2}, \tfrac{1}{3}\right), \left(\tfrac{4}{1}, \tfrac{3}{2}, \tfrac{2}{3}, \tfrac{1}{4}\right), \ldots, \left(\tfrac{9}{1}, \tfrac{8}{2}, \tfrac{7}{3}, \tfrac{6}{4}, \tfrac{5}{5}, \tfrac{4}{6}, \tfrac{3}{7}, \tfrac{2}{8}, \tfrac{1}{9}\right), \ldots$$

If we take any fraction in any of these smaller sequences, the sum of the numerator and denominator is 1 greater than the number of terms in this smaller sequence. For example, if we take the first occurrence of $\frac{3}{7}$, it would occur in the sequence with 9 terms.
This implies that the fifth occurrence of a fraction equivalent to $\frac{3}{7}$, namely $\frac{5\times 3}{5\times 7} = \frac{15}{35}$, would occur in the sequence with 49 terms, and would be the 35^{th} term in that sequence.
Since the smaller sequences before this particular sequence have 1, 2, 3, …, 48 terms, so the term $\frac{15}{35}$ is term number $(1+2+\cdots+47+48)+35 = \frac{1}{2}(48)(49)+35 = 1176+35 = 1211$.

Answer: (E)

25. Let the height of trapezoid $ABCD$ be h.
Then its total area is $\frac{1}{2}(AB+CD)h = \frac{7}{2}h$.
Since $AB = 2$ and AX is parallel to BC, then $XC = 2$.
Since $AB = 2$ and BY is parallel to AD, then $DY = 2$.
Since $CD = 5$, $XC = 2$ and $DY = 2$, then $YX = 1$.
Now we want to determine the area of ΔAZW, so we will determine the areas of ΔAZB and ΔAWB and subtract them.



First, we calculate the area of ΔAZB. Since AB is parallel to CD, $\angle ZAB = \angle ZXY$ and $\angle ZBA = \angle ZYX$, so ΔAZB is similar to ΔXZY. Since the ratio of AB to XY is 2 to 1, then the ratio of the heights of these two triangles will also be 2 to 1, since they are similar. But the sum of their heights must be the height of the trapezoid, h, so the height of ΔAZB is $\frac{2}{3}h$. Therefore, the area of ΔAZB is $\frac{1}{2}(2)\left(\frac{2}{3}h\right) = \frac{2}{3}h$.

Next, we calculate the area of ΔAWB. Since AB is parallel to CD, $\angle WAB = \angle WCY$ and $\angle WBA = \angle WYC$, so ΔAWB is similar to ΔCWY. Since the ratio of AB to CY is 2 to 3, then the ratio of the heights of these two triangles will also be 2 to 3, since they are similar. But the sum of their heights must be the height of the trapezoid, h, so the height of ΔAWB is $\frac{2}{5}h$. Therefore, the area of ΔAWB is $\frac{1}{2}(2)\left(\frac{2}{5}h\right) = \frac{2}{5}h$.

Therefore, the area of ΔAZW is the difference between the areas of ΔAZB and ΔAWB, or $\frac{2}{3}h - \frac{2}{5}h = \frac{4}{15}h$. Thus, the ratio of the area of ΔAZW to the area of the whole trapezoid is $\frac{4}{15}h : \frac{7}{2}h = \frac{4}{15} : \frac{7}{2} = 4(2) : 7(15) = 8 : 105$

Answer: (B)