Canadian
Mathematics
Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2006 Cayley Contest

(Grade 10)

Wednesday, February 22, 2006

Solutions

1. Calculating, $\frac{1}{2} + \left(\frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$.

Answer: (E)

2. Calculating, $(\sqrt{100} - \sqrt{36})^2 = (10 - 6)^2 = 4^2 = 16$.

Answer: (A)

3. We determine the value of this expression, by calculating each difference first, so

$$43 - 41 + 39 - 37 + 35 - 33 + 31 - 29 = 2 + 2 + 2 + 2 = 8$$

Answer: (A)

4. When $a = -3$ and $b = 2$, we have $a(b - 3) = (-3)(2 - 3) = (-3)(-1) = 3$.

Answer: (C)

5. To determine the number by which we multiply the third term to obtain the fourth term, we can divide the second term by the first term. Doing this, we get $\dfrac{0.02}{0.001} = 20$.
(We can check that this is also equal to the third term divided by the second term.)
Therefore, the fourth term is $20(0.4) = 8$.

Answer: (B)

6. The base of $\triangle ABC$ (that is, BC) has length 20.
Since the area of $\triangle ABC$ is 240, then $240 = \frac{1}{2}bh = \frac{1}{2}(20)h = 10h$, so $h = 24$.
Since the height of $\triangle ABC$ (from base BC) is 24, then the y-coordinate of A is 24.

Answer: (D)

7. *Solution 1*
Rewriting $\frac{3}{2}$ as $\frac{6}{4}$, we see that $\dfrac{6}{x+1} = \dfrac{6}{4}$ and so comparing denominators, $x + 1 = 4$ or $x = 3$.

Solution 2
Since $\dfrac{6}{x+1} = \dfrac{3}{2}$, then cross-multiplying, we obtain $2(6) = 3(x + 1)$ or $12 = 3x + 3$ or $3x = 9$ or $x = 3$.

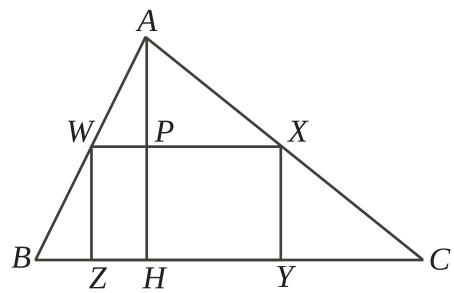
Solution 3
Inverting both sides of the equation, we get $\dfrac{x+1}{6} = \dfrac{2}{3}$ or $x + 1 = 6 \times \frac{2}{3} = 4$.
Thus, $x = 3$.

Answer: (C)

8. *Solution 1*
Since $WXYZ$ is a rectangle, then $\angle ZWX = \angle WXY = 90^\circ$.
Therefore, $\angle AWX = 180^\circ - \angle ZWX - \angle BWZ = 180^\circ - 90^\circ - 22^\circ = 68^\circ$.
Also, $\angle AXW = 180^\circ - \angle WXY - \angle CXY = 180^\circ - 90^\circ - 65^\circ = 25^\circ$.
So, looking at $\triangle AWX$, $\angle BAC = \angle WAX = 180^\circ - \angle AWX - \angle AXW = 180^\circ - 68^\circ - 25^\circ = 87^\circ$.

Solution 2
Since $WXYZ$ is a rectangle, then $\angle WZY = \angle XYZ = 90^\circ$, so $\triangle WZB$ and $\triangle XYC$ are both right-angled.
Thus, $\angle WBZ = 90^\circ - \angle BWZ = 90^\circ - 22^\circ = 68^\circ$ and $\angle XCY = 90^\circ - \angle CXY = 90^\circ - 65^\circ = 25^\circ$.
Looking at $\triangle ABC$, $\angle BAC = 180^\circ - \angle ABC - \angle ACB = 180^\circ - 68^\circ - 25^\circ = 87^\circ$.

Solution 3
Draw a perpendicular from A to H on BC.



Then WZ, AH and XY are all parallel since each is perpendicular to BC, so $\angle BAH = \angle BWZ = 22^\circ$ and $\angle CAH = \angle CXY = \angle 65^\circ$.
Therefore, $\angle BAC = 22^\circ + 65^\circ = 87^\circ$.

Answer: (A)

9. Since the perimeter of the triangle is 36, then $7 + (x + 4) + (2x + 1) = 36$ or $3x + 12 = 36$ or $3x = 24$ or $x = 8$.
Thus, the lengths of the three sides of the triangle are 7, $8+4 = 12$ and $2(8)+1 = 17$, of which the longest is 17.

Answer: (C)

10. From the given information, the total amount of marks obtained by the class is $20(80) + 8(90) + 2(100) = 2520$.
Therefore, the class average is $\dfrac{2520}{30} = 84$.

Answer: (B)

11. *Solution 1*
Since the side lengths of $\triangle DEF$ are 50% larger than the side lengths of $\triangle ABC$, then these new side lengths are $\frac{3}{2}(6) = 9$, $\frac{3}{2}(8) = 12$, and $\frac{3}{2}(10) = 15$.
We know that $\triangle DEF$ is right-angled, and this right angle must occur between the sides of length 9 and 12 (since it is opposite the longest side).
Therefore, the area of $\triangle DEF$ is $\frac{1}{2}(9)(12) = 54$.

Solution 2
The area of $\triangle ABC$ is $\frac{1}{2}(6)(8) = 24$.
Since the side lengths of $\triangle ABC$ are each multiplied by $\frac{3}{2}$ to get the side lengths in $\triangle DEF$, then the area of $\triangle DEF$ is $\left(\frac{3}{2}\right)^2$ times that of $\triangle ABC$.
Therefore, the area of $\triangle DEF$ is $\left(\frac{3}{2}\right)^2(24) = \frac{9}{4}(24) = 54$.

Answer: (E)

12. *Solution 1*
Since Jim drives from 7:45 p.m. to 9:30 p.m., then Jim drives for 1 hour and 45 minutes or $1\frac{3}{4}$ hours or $\frac{7}{4}$ hours.
Since Jim drives 84 km in $\frac{7}{4}$ hours at a constant speed, then this speed is $\dfrac{84}{\frac{7}{4}} = 84 \times \dfrac{4}{7} = 48$ km/h.

Solution 2
Since Jim drives from 7:45 p.m. to 9:30 p.m., then Jim drives for 1 hour and 45 minutes, which is the same as 7 quarters of an hour.
Since he drives 84 km in 7 quarters of an hour, he drives 12 km in 1 quarter of an hour, or 48 km in one hour, so his speed is 48 km/h.

ANSWER: (E)

13. Since $x + 1 = y - 8$ and $x = 2y$, then $2y + 1 = y - 8$ or $y = -9$.
Thus, $x = 2(-9) = -18$, and so $x + y = (-9) + (-18) = -27$.

ANSWER: (D)

14. We check the value of each of the five possibilities when $x = -3$:

$$(-3)^2 - 3 = 6 \qquad (-3-3)^2 = 36 \qquad (-3)^2 = 9 \qquad (-3+3)^2 = 0 \qquad (-3)^2 + 3 = 12$$

So the smallest value is that of $(x+3)^2$.

ANSWER: (D)

15. *Solution 1*
Since the units digit of the product $39P \times Q3$ comes from multiplying $P \times 3$, and this units digit is a 1, then P must be the digit 7.
Therefore, $397 \times Q3 = 32\,951$ so $Q3 = \dfrac{32\,951}{397} = 83$, so $Q = 8$.
Thus, $P + Q = 15$.

Solution 2
Since the units digit of the product $39P \times Q3$ comes from multiplying $P \times 3$, and this units digit is a 1, then P must be the digit 7.
Therefore, the product becomes

$$\begin{array}{r} 3\,9\,7 \\ \times \quad Q\,3 \\ \hline 1\,1\,9\,1 \\ \square \\ \hline 3\,2\,9\,5\,1 \end{array}$$

Therefore, the product $Q \times 7$ must have a units digit of 6 (where the $\square$ is) in order to get the tens digit of 5 in the product. Since $Q \times 7$ has a units digit of 6, then $Q = 8$.
Thus, $P + Q = 15$.

ANSWER: (C)

16. Let c cents be the cost of downloading 1 song in 2005.
Then the cost of downloading 1 song in 2004 was $c + 32$ cents.
The total cost in 2005 was $360c$ and the total cost in 2004 was $200(c + 32)$.
Thus, $360c = 200(c + 32)$ or $160c = 6400$ or $c = 40$ cents, and so the total cost in 2005 was $360(40) = 14400$ cents, or \$144.00.

ANSWER: (A)

17. Since w is a positive integer, then $w \neq 0$, so $w^3 = 9w$ implies $w^2 = 9$ (we can divide by w since it is non-zero).
Since $w^2 = 9$, then $w = 3$ (because we know $w > 0$).
Thus, $w^5 = 3^5 = 243$.

ANSWER: (B)

18. Suppose the side lengths of the triangle are a, b, and c, with c the hypotenuse.
Then $c^2 = a^2 + b^2$ by the Pythagorean Theorem.
We are told that $a^2 + b^2 + c^2 = 1800$.
Since $a^2 + b^2 = c^2$, then $c^2 + c^2 = 1800$ or $2c^2 = 1800$ or $c^2 = 900$ or $c = 30$ (since the side lengths are positive).
So the hypotenuse has length 30.

Answer: (D)

19. *Solution 1*
Of the original 200 candies, since 90% (or 180 candies) are black, then 10% (or 20 candies) are gold.
Since Yehudi eats only black candies, the number of gold candies does not change.
After Yehudi has eaten some of the black candies, the 20 gold candies then represent 20% of the total number of candies, so there must be 100 candies in total remaining.
Thus, Yehudi ate $200 - 100 = 100$ black candies.

Solution 2
Of the original 200 candies, 90% or 180 candies are black.
Suppose that Yehudi eats b black candies.
This will leave $200 - b$ candies in total, of which $180 - b$ are black.
Since 80% of the remaining candies are black, then $\dfrac{180-b}{200-b} = \dfrac{4}{5}$ or $5(180 - b) = 4(200 - b)$ or $900 - 5b = 800 - 4b$ or $b = 100$.
Thus, Yehudi ate 100 black candies.

Answer: (D)

20. *Solution 1*
The y-intercept of the line $y = -\frac{3}{4}x + 9$ is $y = 9$, so Q has coordinates $(0, 9)$.
To determine the x-intercept, we set $y = 0$, and so obtain $0 = -\frac{3}{4}x + 9$ or $\frac{3}{4}x = 9$ or $x = 12$.
Thus, P has coordinates $(12, 0)$.
Therefore, the area of $\triangle POQ$ is $\frac{1}{2}(12)(9) = 54$, since $\triangle POQ$ is right-angled at O.
Since we would like the area of $\triangle TOP$ to be $\frac{1}{3}$ that of $\triangle POQ$, then the area of $\triangle TOP$ should be 18.
If T has coordinates (r, s), then $\triangle TOP$ has base TO of length 12 and height s, so $\frac{1}{2}(12)(s) = 18$ or $6s = 18$ or $s = 3$.
Since T lies on the line, then $s = -\frac{3}{4}r + 9$ or $3 = -\frac{3}{4}r + 9$ or $\frac{3}{4}r = 6$ or $r = 8$.
Thus, $r + s = 8 + 3 = 11$.

Solution 2
The y-intercept of the line $y = -\frac{3}{4}x + 9$ is $y = 9$, so Q has coordinates $(0, 9)$.
To determine the x-intercept, we set $y = 0$, and so obtain $0 = -\frac{3}{4}x + 9$ or $\frac{3}{4}x = 9$ or $x = 12$.
Thus, P has coordinates $(12, 0)$.
The areas of triangles are proportional to their heights if the bases are equal.
Since the area of $\triangle POQ$ is 3 times the area of $\triangle TOP$, then the height of $\triangle TOP$ is $\frac{1}{3}$ that of $\triangle POQ$.
Thus, T is $\frac{1}{3}$ along PQ from P towards Q.
Since P has coordinates $(12, 0)$ and Q has coordinates $(0, 9)$, then T has coordinates $\left(\frac{2}{3}(12), \frac{1}{3}(9)\right) = (8, 3)$.
Thus, $r + s = 8 + 3 = 11$.

Answer: (C)

21. *Solution 1*
We know $p + \dfrac{1}{q + \dfrac{1}{r}} = \dfrac{25}{19} = 1 + \dfrac{6}{19} = 1 + \dfrac{1}{\frac{19}{6}} = 1 + \dfrac{1}{3 + \dfrac{1}{6}}$.
Therefore, comparing the two fractions, $p = 1$, $q = 3$ and $r = 6$.

Solution 2
Since p, q and r are positive integers, then $q + \dfrac{1}{r}$ is at least 1, so $\dfrac{1}{q + \dfrac{1}{r}}$ is between 0 and 1.

Since $p + \dfrac{1}{q + \dfrac{1}{r}}$ is equal to $\dfrac{25}{19}$ which is between 1 and 2, then p must be equal to 1.
Therefore, $\dfrac{1}{q + \dfrac{1}{r}} = \dfrac{25}{19} - 1 = \dfrac{6}{19}$ or $q + \dfrac{1}{r} = \dfrac{19}{6}$.
Since r is a positive integer, then $\dfrac{1}{r}$ is between 0 and 1, so since $\dfrac{19}{6}$ is between 3 and 4, then $q = 3$.
(We are not asked to determine what the value of r, but we can check that $r = 6$.)

Answer: (C)

22. Suppose that n is a multiplicatively perfect positive integer.
If n can be written as $n = a \times b$ with a and b different positive integers, neither equal to 1, then n cannot have any other proper divisors other than 1, since $n = 1 \times a \times b$, so any other proper divisors would make this product too large.
So n can only have 2 proper divisors other than 1.
This tells us that n cannot have more than two distinct prime divisors (otherwise, it would certainly have three proper divisors larger than one).
If n has 2 distinct prime divisors, p and q, then neither can occur in the prime factorization of n more than once, otherwise p^2 or q^2 would be another proper divisor of n. So if n has 2 distinct prime divisors, then $n = pq$ (which indeed has only three proper divisors: 1, p and q).
The integers between 2 and 30 which are of the form $n = pq$ are 6, 10, 14, 15, 21, 22, and 26, or a total of 7 integers.
If n has only 1 prime factor, p, then $n = p^3$ in order to have exactly 3 proper divisors (namely, 1, p and p^2). Higher powers of p will have more than 3 proper divisors and lower powers of p will have fewer than 3 proper divisors. (For example, p^2 has 2 proper divisors and p^5 has 5 proper divisors.)
The integers between 2 and 30 which are of the form $n = p^3$ are 8 and 27, or 2 integers in total.
Therefore, there are 9 multiplicatively perfect numbers between 2 and 30.

Answer: (A)

23. First, we do some experimentation.
Since Celine moves small boxes faster and Quincy moves large boxes faster, then suppose Celine moves all 16 small boxes (taking 32 minutes) and Quincy moves all 10 large boxes (taking 50 minutes). Thus, they would finish in 50 minutes.
We could transfer two large boxes from Quincy to Celine, who now moves 16 small and 2 large boxes, taking 44 minutes. Quincy would then move 8 large boxes, taking 40 minutes. So they would finish in 44 minutes. (So (E) is not the answer.)
If we transfer one small box from Celine to Quincy, then Quincy moves 8 large boxes and 1 small box, taking 43 minutes, and Celine moves 15 small and 2 large boxes, taking 42 minutes.

So they would finish in 43 minutes. (So (D) is not the answer.)

Why is 43 minutes the smallest possible total time?
Suppose that it took them at most 42 minutes to finish the job. Then the total amount of working time would be at most 84 minutes.
Suppose that Celine moves x small boxes and y large boxes, which would take $2x+6y$ minutes.
Then Quincy moves $16-x$ small boxes and $10-y$ large boxes, which would take
$3(16-x)+5(10-y)=98-3x-5y$ minutes.
Since the total working time is at most 84 minutes, then $(2x+6y)+(98-3x-5y)\leq 84$ or $14\leq x-y$.
Since $0\leq x\leq 16$ and $0\leq y\leq 10$, then the possible pairs of x and y which satisfy $14\leq x-y$ are $(16,0)$, $(16,1)$, $(16,2)$, $(15,0)$, $(15,1)$, $(14,0)$, which produce working times as follows:

x	y	Celine Small	Celine Large	Celine Time	Quincy Small	Quincy Large	Quincy Time
16	0	16	0	32	0	10	50
16	1	16	1	38	0	9	45
16	2	16	2	44	0	8	40
15	0	15	0	30	1	10	53
15	1	15	1	36	1	9	48
14	0	14	0	28	2	10	56

In each of these cases, while the total working time is no more than 84 minutes, it takes longer than 43 minutes for the two of them to finish.
Therefore, it is impossible for them to finish in 42 minutes or less, so the earliest possible finishing time is 9:43 a.m.

Answer: (C)

24. Let us consider some small values for n.
If $n=1$, Anne wins by taking the first toothpick.
If $n=2$, Anne must take 1 toothpick (since she cannot take 3 or 4), so Brenda takes 1 toothpick and wins.
If $n=3$ or $n=4$, Anne can win by taking all of the toothpicks.
When $n=5$, if Anne takes 3 toothpicks to start, then Brenda is left with 2 toothpicks, and so cannot win, since Anne could not win when starting with 2 toothpicks. So Anne wins.
When $n=6$, if Anne takes 4 toothpicks to start, then Anne will win in the same way as in the $n=5$ case. So Anne wins.
When $n=7$, if Anne takes 1, 3 or 4 toothpicks, then 6, 4 or 3 toothpicks will be left for Brenda. But choosing from 6, 4 or 3 toothpicks is a winning position for the first person choosing (Brenda in this case). So $n=7$ is a losing position for Anne, since Brenda can use Anne's winning strategy in each case.
In general, Brenda will have a winning strategy for n if $n-1$, $n-3$ and $n-4$ all give Anne a winning strategy (since if Anne starts with n toothpicks an chooses 1, 3 or 4, then Brenda will start choosing from $n-1$, $n-3$ or $n-4$ toothpicks).
Anne will have a winning strategy otherwise (since if Anne would lose starting with an initial position of $n-1$, $n-3$ or $n-4$, then Anne should win starting with an initial position of n because she can leave Brenda with whichever of $n-1$, $n-3$ or $n-4$ is a losing position for the first person choosing).
We construct lists of the winning positions for each, adding a new n to Brenda's list if $n-1$, $n-3$ and $n-4$ are all in Anne's list, and to Anne's list otherwise:

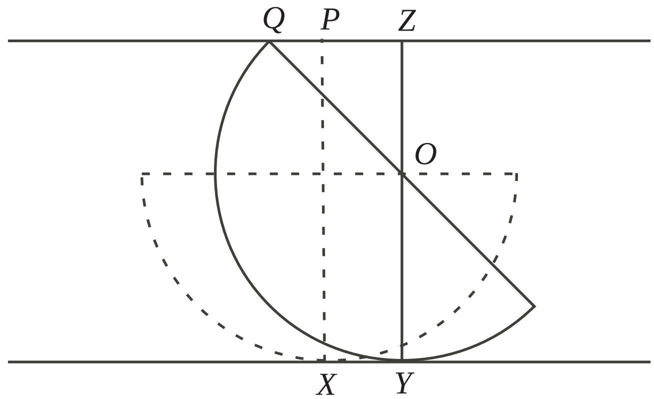
A's winning positions: 1, 3, 4, 5, 6, 8, 10, 11, 12, 13, 15, 17, 18, 19, 20, 22, 24, 25, 26, 27, 29, 31, 32, 33, 34

B's winning positions: 2, 7, 9, 14, 16, 21, 23, 28, 30, 35

So of the given possibilities, Brenda has a winning strategy when $n = 35$.

Answer: (E)

25. In its initial position, suppose the semi-circle touches the bottom line at X, with point P directly above X on the top line.
Consider when the semi-circle rocks to the right.



Suppose now the semi-circle touches the bottom line at Y (with O the point on the top of the semi-circle directly above Y, and Z the point on the top line directly above Y) and touches the top line at Q. Note that $XY = PZ$.
Q is one of the desired points where the semi-circle touches the line above. Because the diagram is symmetrical, the other point will be the mirror image of Q in line XP. Thus, the required distance is 2 times the length of PQ.
Now $PQ = QZ - PZ = QZ - XY$.
Since the semi-circle is tangent to the bottom line, YO is perpendicular to the bottom line, and O lies on a diameter, then O is the centre of the circle, so $OY = OQ = 8$ cm, since both are radii (or since the centre always lies on a line parallel to the bottom line and a distance of the radius away).
Also, $OZ = 4$ cm, since the distance between the two lines is 12 cm
By the Pythagorean Theorem (since $\angle QZO = 90^\circ$), then

$$QZ^2 = QO^2 - ZO^2 = 8^2 - 4^2 = 64 - 16 = 48$$

so $QZ = 4\sqrt{3}$ cm.
Also, since $QZ : ZO = \sqrt{3} : 1$, then $\angle QOZ = 60^\circ$.
Thus, the angle from QO to the horizontal is 30°, so the semi-circle has rocked through an angle of 30°, ie. has rocked through $\frac{1}{12}$ of a full revolution (if it was a full circle).
Thus, the distance of Y from X is $\frac{1}{12}$ of the circumference of what would be the full circle of radius 8, or $XY = \frac{1}{12}(2\pi(8)) = \frac{4}{3}\pi$ cm. (We can think of a wheel turning through 30° and the related horizontal distance through which it travels.)
Thus, $PQ = QZ - XY = 4\sqrt{3} - \frac{4}{3}\pi$ cm.
Therefore, the required distance double this, or $8\sqrt{3} - \frac{8}{3}\pi$ cm or about 5.4788 cm, which is closest to 55 mm.

Answer: (A)