Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest

Wednesday, April 19, 2006

Time: $2\frac{1}{2}$ hours

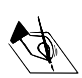
Calculators are permitted, provided they are non-programmable and without graphic displays.

Do not open this booklet until instructed to do so. The paper consists of 10 questions, each worth 10 marks. Parts of each question can be of two types. **SHORT ANSWER** parts are worth 2 marks each (questions 1-2) or 3 marks each (questions 3-7). **FULL SOLUTION** parts are worth the remainder of the 10 marks for the question.

Instructions for SHORT ANSWER parts:

1. **SHORT ANSWER** parts are indicated like this:  .
2. **Enter the answer in the appropriate box in the answer booklet.** For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

Instructions for FULL SOLUTION parts:

1. **FULL SOLUTION** parts are indicated like this:  .
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name, school name and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.

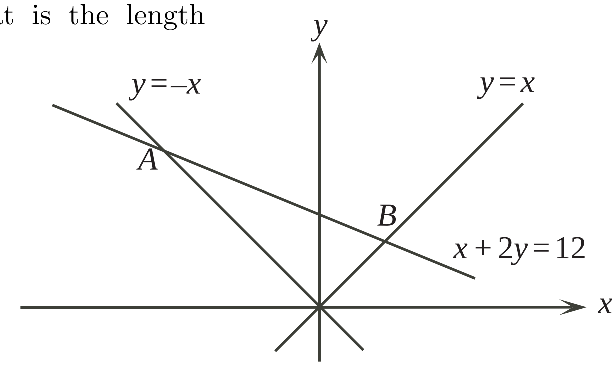
NOTE: At the completion of the Contest, insert the information sheet inside the answer booklet.

NOTES:	1. Please read the instructions on the front cover of this booklet.
	2. Write all answers in the answer booklet provided.
	3. For questions marked " 💡 ", full marks will be given for a correct answer placed in the appropriate box in the answer booklet. **If an incorrect answer is given, marks may be given for work shown.** Students are strongly encouraged to show their work.
	4. All calculations and answers should be expressed as exact numbers such as 4π, $2 + \sqrt{7}$, etc., except where otherwise indicated.

1. (a) What is the sum of the x-intercept and the y-intercept of the line $3x - 3y = 24$?

   (b) If the lines $px = 12$ and $2x + qy = 10$ intersect at $(1, 1)$, what is the value of $p + q$?

   (c) In the diagram, the line $x + 2y = 12$ intersects the lines $y = -x$ and $y = x$ at points A and B, respectively. What is the length of AB?



2. (a) The average of the digits of the integer 46 is 5. Including 46, how many two-digit positive integers have the average of their digits equal to 5?

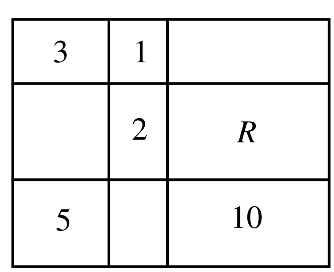
   (b) When a decimal point is placed between the digits of the two-digit integer n, the resulting number is equal to the average of the digits of n. What is the value of n?

   (c) The average of three positive integers is 28. When two additional integers, s and t, are included, the average of all five integers is 34. What is the average of s and t?

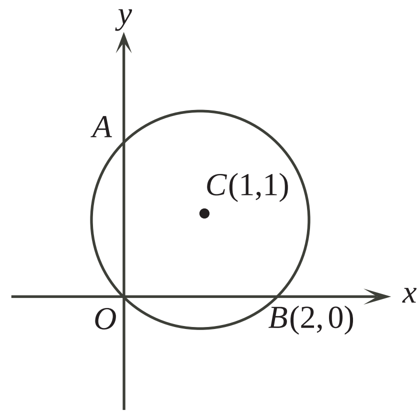
3. (a) Determine the coordinates of the vertex of the parabola $y = (x - 20)(x - 22)$.

   (b) Point A is the vertex of the parabola $y = x^2 + 2$, point B is the vertex of the parabola $y = x^2 - 6x + 7$, and O is the origin. Determine the area of $\triangle OAB$.

4. (a) In the diagram, the rectangle is divided into nine smaller rectangles. The areas of five of these rectangles are given. Determine the area of the rectangle labelled R.



(b) In the diagram, the circle with centre $C(1,1)$ passes through the point $O(0,0)$, intersects the y-axis at A, and intersects the x-axis at $B(2,0)$. Determine, with justification, the coordinates of A and the area of the part of the circle that lies in the first quadrant.



5. (a) If a is chosen randomly from the set $\{1,2,3,4,5\}$ and b is chosen randomly from the set $\{6,7,8\}$, what is the probability that a^b is an even number?

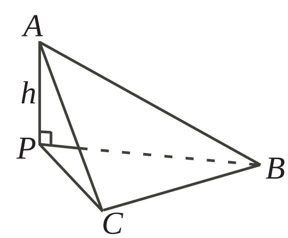
(b) A bag contains some blue and some green hats. On each turn, Julia removes one hat without looking, with each hat in the bag being equally likely to be chosen. If it is green, she adds a blue hat into the bag from her supply of extra hats, and if it is blue, she adds a green hat to the bag. The bag initially contains 4 blue hats and 2 green hats. What is the probability that the bag again contains 4 blue hats and 2 green hats after two turns?

6. (a) Suppose that, for some angles x and y,

$$\begin{aligned} \sin^2 x + \cos^2 y &= \tfrac{3}{2}a \\ \cos^2 x + \sin^2 y &= \tfrac{1}{2}a^2 \end{aligned}$$

Determine the possible value(s) of a.

(b) Survivors on a desert island find a piece of plywood (ABC) in the shape of an equilateral triangle with sides of length 2 m. To shelter their goat from the sun, they place edge BC on the ground, lift corner A, and put in a vertical post PA which is h m long above ground. When the sun is directly overhead, the shaded region ($\triangle PBC$) on the ground directly underneath the plywood is an isosceles triangle with largest angle ($\angle BPC$) equal to 120°. Determine the value of h, to the nearest centimetre.

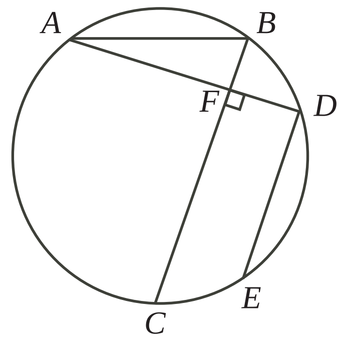


7. (a) The sequence 2, 5, 10, 50, 500, … is formed so that each term after the second is the product of the two previous terms. The 15th term ends with exactly k zeroes. What is the value of k?

(b) Suppose that a, b, c are three consecutive terms in an arithmetic sequence. Prove that $a^2 - bc$, $b^2 - ac$, and $c^2 - ab$ are also three consecutive terms in an arithmetic sequence.
(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7 is an arithmetic sequence with three terms.)

8. (a) If $\log_2 x - 2\log_2 y = 2$, determine y as a function of x, and sketch a graph of this function on the axes in the answer booklet.

(b) In the diagram, AB and BC are chords of the circle with $AB < BC$. If D is the point on the circle such that AD is perpendicular to BC and E is the point on the circle such that DE is parallel to BC, carefully prove, explaining all steps, that $\angle EAC + \angle ABC = 90°$.



9. Define $f(x) = \sin^6 x + \cos^6 x + k\left(\sin^4 x + \cos^4 x\right)$ for some real number k.

(a) Determine all real numbers k for which $f(x)$ is constant for all values of x.

(b) If $k = -0.7$, determine all solutions to the equation $f(x) = 0$.

(c) Determine all real numbers k for which there exists a real number c such that $f(c) = 0$.

10. Points A_1, A_2, …, A_N are equally spaced around the circumference of a circle and $N \geq 3$. Three of these points are selected at random and a triangle is formed using these points as its vertices.

(a) If $N = 7$, what is the probability that the triangle is acute? (A triangle is acute if each of its three interior angles is less than $90°$.)

(b) If $N = 2k$ for some positive integer $k \geq 2$, determine the probability that the triangle is acute.

(c) If $N = 2k$ for some positive integer $k \geq 2$, determine all possible values of k for which the probability that the triangle is acute can be written in the form $\dfrac{a}{2007}$ for some positive integer a.

Canadian Mathematics Competition

For students...

Thank you for writing the 2006 Euclid Contest!
In 2005, more than 15 600 students around the world registered to write the Euclid Contest.

If you are graduating from secondary school, good luck in your future endeavours!
If you will be returning to secondary school next year, encourage your teacher to register you for the 2006 Canadian Open Mathematics Challenge, which will be written in late November.
Visit our website

www.cemc.uwaterloo.ca

to find

- More information about the Canadian Open Mathematics Challenge
- Free copies of past Contests
- Workshops to help you prepare for future Contests
- Information about our publications for math enrichment and Contest preparation
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For teachers...

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