Canadian
Mathematics
Competition
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2006 Galois Contest

Thursday, April 20, 2006

Solutions

1. (a) The largest possible difference comes when one of Amelie and Bob chooses the slips with the three largest numbers and the other choose the slips with the three smallest numbers.
Thus, one of them chooses 1, 2 and 3 (for a total of 6) and the other chooses 4, 5 and 6 (for a total of 15).
The difference in the totals is 9.

(b) The total of the numbers on the six slips is $1 + 2 + 3 + 4 + 5 + 6 = 21$.
For Amelie's total to be one more than Bob's total, her total must be 11 and Bob's must be 10 (since the sum of their totals is 21).
The possible groups of three slips giving totals of 11 are

$$1, 4, 6 \quad 2, 3, 6 \quad 2, 4, 5$$

These are the possible groups that Amelie can choose.

(c) When Amelie and Bob each choose three of the slips, the sum of their totals is the sum of all of the numbers on the slips, or 21.
If they each had the same total, the sum of their totals would be even, so could not be 21.
Therefore, they cannot have the same total.

(d) Since Amelie and Bob must choose half of the slips, the total number of slips must be even.
Therefore, the smallest value that n could take is 8.
If $n = 8$, can they obtain the same total?
If $n = 8$, the sum of the numbers on the eight slips is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$, which is also even.
Here, Amelie and Bob could obtain the same total if Amelie chooses 1, 2, 7, 8 (and so Bob chooses 3, 4, 5, 6).
Therefore, the smallest value of n that works is $n = 8$.

2. (a) *Solution 1*
Since $DE = EF = 4$ and $\angle DEF = 90^\circ$, then by the Pythagorean Theorem,
$DF^2 = DE^2 + EF^2 = 4^2 + 4^2 = 32$, so $DF = \sqrt{32} = 4\sqrt{2}$.

Solution 2
Since $\triangle DEF$ is right-angled and isosceles, its angles are 45°, 45° and 90°.
Therefore, $DF = \sqrt{2}(DE) = 4\sqrt{2}$.

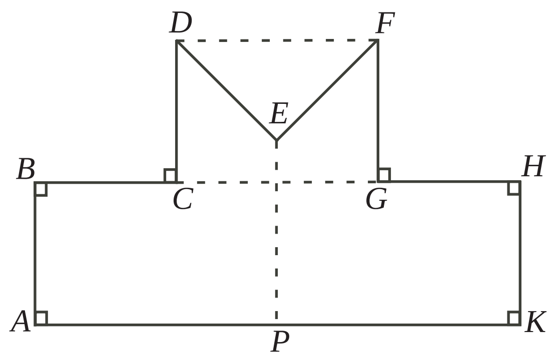
(b) *Solution 1*
Since $\triangle DEF$ is isosceles with $DE = EF$ and EM is perpendicular to DF, then
$DM = MF = \frac{1}{2}DF = 2\sqrt{2}$.
Since $\triangle DME$ is right-angled, then by the Pythagorean Theorem,
$EM^2 = DE^2 - DM^2 = 4^2 - (2\sqrt{2})^2 = 16 - 8 = 8$, so $EM = \sqrt{8} = 2\sqrt{2}$.

Solution 2
Since $\triangle DEF$ is isosceles with $DE = EF$ and EM is perpendicular to DF, then
$DM = MF = \frac{1}{2}DF = 2\sqrt{2}$.

Since $\triangle DEF$ is isosceles and right-angled, then $\angle EDF = 45°$, so $\triangle DME$ is also isosceles and right-angled.
Therefore, $EM = DM = 2\sqrt{2}$.

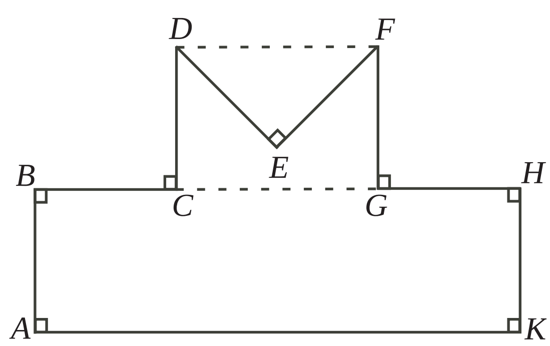
Solution 3
Since DE and EF are perpendicular, the area of $\triangle DEF$ is $\frac{1}{2}(DE)(EF) = \frac{1}{2}(4)(4) = 8$.
Since DF and ME are perpendicular, the area of $\triangle DEF$ is also $\frac{1}{2}(DF)(ME)$, so
$\frac{1}{2}(4\sqrt{2})(ME) = 8$ or $ME = \dfrac{8}{2\sqrt{2}} = 2\sqrt{2}$.

(c) Join DF and CG.



Each of $CGFD$ and $AKHB$ is a rectangle, as each has 4 right angles.
The height of rectangle $CGFD$ is 4 since $DC = FG = 4$, and the height of rectangle $AKHB$ is also 4 since $AB = HK = 4$.
The length of EP is thus the distance of E to CG plus the height of the bottom rectangle.
The distance of E to CG is the difference between the height of rectangle $CGFD$ and the length of EM, or $4 - 2\sqrt{2}$.
Thus, the length of EP is $4 + (4 - 2\sqrt{2}) = 8 - 2\sqrt{2}$.

(d) The area of the figure is equal to the area of rectangle $AKHB$ plus the area of rectangle $CGFD$ minus the area of triangle DEF.



Since the length of DF is $4\sqrt{2}$ and the length of CD is 4, the area of rectangle $CGFD$ is $4(4\sqrt{2}) = 16\sqrt{2}$.
Now $BH = BC + CG + GH = 4 + DF + 4 = 8 + 4\sqrt{2}$.
Since $AB = 4$, the area of rectangle $AKHB$ is $4(8 + 4\sqrt{2}) = 32 + 16\sqrt{2}$.
From Solution 3 to part (b), the area of $\triangle DEF$ is 8.
Thus, the area of figure $ABCDEFGHK$ is $(16\sqrt{2}) + (32 + 16\sqrt{2}) - 8 = 24 + 32\sqrt{2}$.

3. (a) Since A has coordinates $(0,16)$ and B has coordinates $(8,0)$, the slope of the line through A and B is $\dfrac{16-0}{0-8}=-2$.
Since the line passes through the y-axis at $A(0,16)$, then its y-intercept is 16, so the line has equation $y=-2x+16$.

(b) Suppose that P has coordinates (c,d).
Since P lies on the line $y=-2x+16$, then $d=-2c+16$, so P has coordinates $(c,-2c+16)$.
For $PDOC$ to be a square, $PD=PC$.
But PD is the distance of P from the y-axis, so $PD=c$ and PC is the distance of P from the x-axis, so $PC=-2c+16$.
Therefore, $c=-2c+16$ or $3c=16$ or $c=\frac{16}{3}$.
Thus, P has coordinates $\left(\frac{16}{3},\frac{16}{3}\right)$.

(c) *Solution 1*
As in (b), we may suppose that P has coordinates $(c,-2c+16)$.
The area of rectangle $PDOC$ is $PD\times PC$, or $c(-2c+16)$.
Since the area is 30, then

$$\begin{aligned}30&=c(-2c+16)\\30&=-2c^2+16c\\2c^2-16c+30&=0\\c^2-8c+15&=0\\(c-3)(c-5)&=0\end{aligned}$$

so $c=3$ or $c=5$.
Therefore, the two possible points P are $(3,10)$ and $(5,6)$.
(We can check that each gives a rectangle of area 30.)

Solution 2
For the area of rectangle $PDOC$ to be 30, the coordinates of P are $\left(c,\dfrac{30}{c}\right)$.
For P to lie on the line $y=-2x+16$,

$$\begin{aligned}\frac{30}{c}&=-2c+16\\30&=-2c^2+16c\\c^2-8c+15&=0\\(c-3)(c-5)&=0\end{aligned}$$

so $c=3$ or $c=5$.
Therefore, the two possible points P are $(3,10)$ and $(5,6)$.

4. (a) Suppose that we start with the 2 digit integer $\underline{a}\,\underline{b}=10a+b$ and reverse the order of its digits to obtain $\underline{b}\,\underline{a}=10b+a$.
The difference is $(10b+a)-(10a+b)=9b-9a=9(b-a)$.
For this difference to equal 27, we must have $9(b-a)=27$ or $b-a=3$ or $b=a+3$. That is, the second digit of the original number is 3 larger than the first digit.

Thus, the possible beginning numbers are 14, 25, 36, 47, 58, and 69.

(b) *Solution 1*
Start with $\underline{a}\,\underline{b}\,\underline{c} = 100a + 10b + c$ and reverse the order of the digits to obtain $\underline{c}\,\underline{b}\,\underline{a} = 100c + 10b + a$.
We may assume, without loss of generality, that the first number is larger than the second number (otherwise, we simply reverse the roles of the numbers).
Then their difference is

$$\underline{r}\,\underline{s}\,\underline{t} = \underline{a}\,\underline{b}\,\underline{c} - \underline{c}\,\underline{b}\,\underline{a} = (100a + 10b + c) - (100c + 10b + a) = 99a - 99c = 99(a - c)$$

Since a and c are distinct digits, then the possible values of $a - c$ are 1 through 9, so the possible values for the integer $\underline{r}\,\underline{s}\,\underline{t}$ are 99 times the numbers 1 through 9. We show the possible values, their reverses and the sums in the table:

$\underline{r}\,\underline{s}\,\underline{t}$	099	198	297	396	495	594	693	792	891
$\underline{t}\,\underline{s}\,\underline{r}$	990	891	792	693	594	495	396	297	198
Sum	1089	1089	1089	1089	1089	1089	1089	1089	1089

Therefore, the required sum is always 1089.

Solution 2
Start with $\underline{a}\,\underline{b}\,\underline{c} = 100a + 10b + c$ and reverse the order of the digits to obtain $\underline{c}\,\underline{b}\,\underline{a} = 100c + 10b + a$.
We may assume, without loss of generality, that the first number is larger than the second number (otherwise, we simply reverse the roles of the numbers).
Then their difference is

$$\underline{r}\,\underline{s}\,\underline{t} = \underline{a}\,\underline{b}\,\underline{c} - \underline{c}\,\underline{b}\,\underline{a} = (100a + 10b + c) - (100c + 10b + a) = 99a - 99c = 99(a - c)$$

Since a and c are distinct digits, then the possible values of $a - c$ are the integers 1 through 9.
In any of these cases, $\underline{r}\,\underline{s}\,\underline{t} = 99(a - c) = 100(a - c) - (a - c)$, that is, is $a - c$ less than $100(a - c)$ and so has hundreds digit $a - c - 1$, tens digit 9 and units digit $10 - (a - c)$.
In other words, $\underline{r}\,\underline{s}\,\underline{t} = \underline{a - c - 1}\ \ \underline{9}\ \ \underline{10 - (a - c)}$.
When the order of the digits is reversed, we obtain $\underline{t}\,\underline{s}\,\underline{r} = \underline{10 - (a - c)}\ \ \underline{9}\ \ \underline{a - c - 1}$.
Adding these numbers,

$$\begin{array}{cccc} & \underline{a-c-1} & \underline{9} & \underline{10-(a-c)} \\ + & \underline{10-(a-c)} & \underline{9} & \underline{a-c-1} \\ \hline \underline{1} & \underline{0} & \underline{8} & \underline{9} \end{array}$$

(Note that a 1 has been "carried" from the tens column to the hundreds column, and also from the hundreds column to the thousands column.)

(c) Since $N = \underline{a}\,\underline{b}\,\underline{c}\,\underline{d} = 1000a + 100b + 10c + d$, then $M = \underline{d}\,\underline{c}\,\underline{b}\,\underline{a} = 1000d + 100c + 10b + a$.

Then

$$\begin{aligned} P &= M - N \\ &= (1000d + 100c + 10b + a) - (1000a + 100b + 10c + d) \\ &= 1000(d-a) + 100(c-b) + 10(b-c) + (a-d) \\ &= 999d + 90c - 90b - 999a \\ &= 999(d-a) + 90(c-b) \end{aligned}$$

Since $a \leq b \leq c \leq d$, then $d - a \geq 0$ and $c - b \geq 0$.
(This tells us that while the third line of the equations above looks like it represents P in terms of its digits, two of the digits are possibly negative.)
We notice also that $d - a \geq c - b$.

Case 1: $a = b = c = d$
In this case, $P = 0$ so $Q = 0$, so $P + Q = 0$.

Case 2: $d - a = 1$
In this case, $c - b$ can only be 0 or 1, so the two possible values of P are $P = 999 = 0999$ and $P = 999 + 90 = 1089$.
Reversing these, we obtain $Q = 9990$ and $Q = 9801$, giving $P + Q = 999 + 9990 = 10\,989$ and $P + Q = 1089 + 9801 = 10\,890$.

Case 3: $d - a > 1$, $c - b = 0$
In this case, $P = 999(d-a) = 1000(d-a) - (d-a)$, so has digits $\underline{d-a-1}$ $\underline{9}$ $\underline{9}$ $\underline{10-(d-a)}$.
Thus, Q has digits $\underline{10-(d-a)}$ $\underline{9}$ $\underline{9}$ $\underline{d-a-1}$, and so adding P and Q, we obtain

$$\begin{array}{cccccc} & & \underline{d-a-1} & \underline{9} & \underline{9} & \underline{10-(d-a)} \\ + & & \underline{10-(d-a)} & \underline{9} & \underline{9} & \underline{d-a-1} \\ \hline & \underline{1} & \underline{0} & \underline{9} & \underline{8} & \underline{9} \end{array}$$

Alternatively, we could write

$$\begin{aligned} P &= 999(d-a) \\ &= 1000(d-a) - (d-a) \\ &= 1000(d-a-1) + 1000 - (d-a) \\ &= 1000(d-a-1) + 100(9) + 100 - (d-a) \\ &= 1000(d-a-1) + 100(9) + 10(9) + (10 - (d-a)) \end{aligned}$$

where we are writing out the "borrowing" process explicitly.
Here, the digits are $d - a - 1$, 9, 9 and $10 - (d - a)$ and each digit is at least 0.
Thus, $Q = 1000(10 - (d - a)) + 100(9) + 10(9) + (d - a - 1)$ and so

$$P + Q = 1000(9) + 100(18) + 10(18) + 9 = 10\,989$$

Case 4: $d-a>1$, $c-b>0$

In this case,

$$\begin{aligned} P &= 999(d-a)+90(c-b) \\ &= 1000(d-a)-(d-a)+100(c-b)-10(c-b) \\ &= \underline{d-a-1}\ \ \underline{9}\ \ \underline{9}\ \ \underline{10-(d-a)}+\underline{c-b-1}\ \ \underline{10-(c-b)}\ \ \underline{0} \\ &= \underline{d-a-1}\ \ \underline{9+c-b-1}\ \ \underline{9+10-(c-b)}\ \ \underline{10-(d-a)+0} \\ &= \underline{d-a-1}\ \ \underline{9+c-b-1+1}\ \ \underline{9-(c-b)}\ \ \underline{10-(d-a)} \\ &= \underline{d-a}\ \ \underline{c-b-1}\ \ \underline{9-(c-b)}\ \ \underline{10-(d-a)} \end{aligned}$$

where some "carrying" has been done in the last two lines.
Therefore, $Q=\underline{10-(d-a)}\ \ \underline{9-(c-b)}\ \ \underline{c-b-1}\ \ \underline{d-a}$.
Thus, adding P and Q we obtain

$$\begin{array}{cccccc} & & \underline{d-a} & \underline{c-b-1} & \underline{9-(c-b)} & \underline{10-(d-a)} \\ + & & \underline{10-(d-a)} & \underline{9-(c-b)} & \underline{c-b-1} & \underline{d-a} \\ \hline & \underline{1} & \underline{0} & \underline{8} & \underline{9} & \underline{0} \end{array}$$

Alternatively, we could write

$$\begin{aligned} P &= 999(d-a)+90(c-b) \\ &= 1000(d-a)+100(c-b)-10(c-b)-(d-a) \\ &= 1000(d-a)+100(c-b-1)+100-10(c-b)-(d-a) \\ &= 1000(d-a)+100(c-b-1)+10(10-(c-b))-(d-a) \\ &= 1000(d-a)+100(c-b-1)+10(9-(c-b))+(10-(d-a)) \end{aligned}$$

where we are writing out the "borrowing" process explicitly.
Here, the digits are $d-a$, $c-b-1$, $9-(c-b)$ and $10-(d-a)$ and each digit is at least 0.
Thus, $Q=1000(10-(d-a))+100(9-(c-b))+10(c-b-1)+(d-a)$ and so

$$P+Q=1000(10)+100(8)+10(8)+10=10\,890$$

Therefore, the possible values for $P+Q$ are 0, 10 890 and 10 989.