**Canadian Mathematics Competition**

***An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario***

# *2007 Pascal Contest*

**(Grade 9)**

**Tuesday, February 20, 2007**

*Solutions*

1. Calculating, $3 \times (7-5) - 5 = 3 \times 2 - 5 = 6 - 5 = 1$.

   Answer: (B)

2. Since $x$ is less than $-1$ and greater than $-2$, then the best estimate of the given choices is $-1.3$.

   Answer: (B)

3. The shaded square has side length 1 so has area $1^2 = 1$.
   The rectangle has dimensions 3 by 5 so has area $3 \times 5 = 15$.
   Thus, the fraction of the rectangle that is shaded is $\frac{1}{15}$.

   Answer: (A)

4. Calculating, $2^5 - 5^2 = 32 - 25 = 7$.

   Answer: (E)

5. In 3 hours, Leona earns \$24.75, so she makes $\$24.75 \div 3 = \$8.25$ per hour.
   Therefore, in a 5 hour shift, Leona earns $5 \times \$8.25 = \$41.25$.

   Answer: (E)

6. Calculating, $\dfrac{\sqrt{64}+\sqrt{36}}{\sqrt{64+36}} = \dfrac{8+6}{\sqrt{100}} = \dfrac{14}{10} = \dfrac{7}{5}$.

   Answer: (A)

7. *Solution 1*
   In total Megan and Dan inherit \$1 010 000.
   Since each donates 10%, then the total donated is 10% of \$1 010 000, or \$101 000.

   *Solution 2*
   Megan donates 10% of \$1 000 000, or \$100 000.
   Dan donates 10% of \$10 000, or \$1000.
   In total, they donate $\$100\,000 + \$1000 = \$101\,000$.

   Answer: (A)

8. We think of $BC$ as the base of $\triangle ABC$. Its length is 12.
   Since the $y$-coordinate of $A$ is 9, then the height of $\triangle ABC$ from base $BC$ is 9.
   Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(12)(9) = 54$.

   Answer: (B)

9. Calculating the given difference using a common denominator, we obtain $\frac{5}{8} - \frac{1}{16} = \frac{9}{16}$.
   Since $\frac{9}{16}$ is larger than each of $\frac{1}{2} = \frac{8}{16}$ and $\frac{7}{16}$, then neither (D) nor (E) is correct.
   Since $\frac{9}{16}$ is less than each of $\frac{3}{4} = \frac{12}{16}$, then (A) is not correct.
   As a decimal, $\frac{9}{16} = 0.5625$.
   Since $\frac{3}{5} = 0.6$ and $\frac{5}{9} = 0.\overline{5}$, then $\frac{9}{16} > \frac{5}{9}$, so (C) is the correct answer.

   Answer: (C)

10. Since $M = 2007 \div 3$, then $M = 669$.
    Since $N = M \div 3$, then $N = 669 \div 3 = 223$.
    Since $X = M - N$, then $X = 669 - 223 = 446$.

    Answer: (E)

11. The mean of 6, 9 and 18 is $\dfrac{6+9+18}{3} = \dfrac{33}{3} = 11$.
Thus the mean of 12 and $y$ is 11, so the sum of 12 and $y$ is $2(11) = 22$, so $y = 10$.
Answer: (C)

12. In $\triangle PQR$, since $PR = RQ$, then $\angle RPQ = \angle PQR = 48^\circ$.
Since $\angle MPN$ and $\angle RPQ$ are opposite angles, then $\angle MPN = \angle RPQ = 48^\circ$.
In $\triangle PMN$, $PM = PN$, so $\angle PMN = \angle PNM$.
Therefore, $\angle PMN = \frac{1}{2}(180^\circ - \angle MPN) = \frac{1}{2}(180^\circ - 48^\circ) = \frac{1}{2}(132^\circ) = 66^\circ$.
Answer: (D)

13. The prime numbers smaller than 10 are 2, 3, 5, and 7.
The two of these numbers which are different and add to 10 are 3 and 7.
The product of 3 and 7 is $3 \times 7 = 21$.
Answer: (B)

14. Since there were 21 males writing and the ratio of males to females writing is $3 : 7$, then there are $\frac{7}{3} \times 21 = 49$ females writing.
Therefore, the total number of students writing is $49 + 21 = 70$.
Answer: (D)

15. *Solution 1*
The first stack is made up of $1 + 2 + 3 + 4 + 5 = 15$ blocks.
The second stack is made up of $1 + 2 + 3 + 4 + 5 + 6 = 21$ blocks.
There are 36 blocks in total.
We start building the new stack from the top.
Since there are more than 21 blocks, we need at least 6 rows.
For 7 rows, $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$ blocks are needed.
For 8 rows, $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$ blocks are needed.
Therefore, Clara can build a stack with 0 blocks leftover.

*Solution 2*
Since the new stack will be larger than the second stack shown, let us think about adding new rows to this second stack using the blocks from the first stack.
The first stack contains $1 + 2 + 3 + 4 + 5 = 15$ blocks in total.
The first two rows that we would add to the bottom of the second stack would have 7 and 8 blocks in them, for a total of 15 blocks.
This uses all of the blocks from the first stack, with none left over, and creates a similar stack.
Therefore, there are 0 blocks left over.
Answer: (A)

16. *Solution 1*
The sum of the numbers in the second row is $10 + 16 + 22 = 48$, so the sum of the numbers in any row, column or diagonal is 48.
In the first row, $P + 4 + Q = 48$ so $P + Q = 44$.
In the third row, $R + 28 + S = 48$ so $R + S = 20$.
Therefore, $P + Q + R + S = 44 + 20 = 64$.

*Solution 2*
The sum of the numbers in the second row is $10 + 16 + 22 = 48$, so the sum of the numbers in any row, column or diagonal is 48.
From the first row, $P + 4 + Q = 48$ so $P + Q = 44$.
From the first column, $P + 10 + R = 48$ so $P + R = 38$.
Subtracting these two equations gives $(P + Q) - (P + R) = 44 - 38$ or $Q - R = 6$.
From one of the diagonals, $R + 16 + Q = 48$ or $Q + R = 32$.
Adding these last two equations, $2Q = 38$ or $Q = 19$, so $R = 32 - Q = 13$.
Also, $P = 44 - Q = 25$.
From the last row, $13 + 28 + S = 48$, or $S = 7$.
Thus, $P + Q + R + S = 25 + 19 + 13 + 7 = 64$.

*Solution 3*
The sum of all of the numbers in the grid is

$$P + Q + R + S + 10 + 16 + 22 + 28 + 4 = P + Q + R + S + 80$$

But the sum of the three numbers in the second column is $4 + 16 + 28 = 48$, so the sum of the three numbers in each column is 48.
Thus, the total of the nine numbers in the grid is $3(48) = 144$, so $P + Q + R + S + 80 = 144$ or $P + Q + R + S = 64$.

Answer: (C)

17. At present, the sum of Norine's age and the number of years that she has worked is $50+19 = 69$. This total must increase by $85 - 69 = 16$ before she can retire.
As every year passes, this total increases by 2 (as her age increases by 1 and the number of years that she has worked increases by 1).
Thus, it takes 8 years for her total to increase from 69 to 85, so she will be $50 + 8 = 58$ when she can retire.

Answer: (C)

18. By the Pythagorean Theorem in $\triangle PQR$, $PQ^2 = PR^2 - QR^2 = 13^2 - 5^2 = 144$, so $PQ = \sqrt{144} = 12$.
By the Pythagorean Theorem in $\triangle PQS$, $QS^2 = PS^2 - PQ^2 = 37^2 - 12^2 = 1225$, so $QS = \sqrt{1225} = 35$.
Therefore, the perimeter of $\triangle PQS$ is $12 + 35 + 37 = 84$.

Answer: (D)

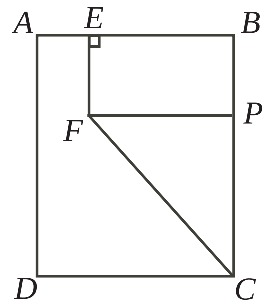
19. Since the reciprocal of $\frac{3}{10}$ is $\left(\frac{1}{x} + 1\right)$, then

$$\begin{aligned} \frac{1}{x} + 1 &= \frac{10}{3} \\ \frac{1}{x} &= \frac{7}{3} \\ x &= \frac{3}{7} \end{aligned}$$

so $x = \frac{3}{7}$.

Answer: (C)

20. Draw a line from $F$ to $BC$, parallel to $AB$, meeting $BC$ at $P$.




Since $EB$ is parallel to $FP$ and $\angle FEB = 90^\circ$, then $EBPF$ is a rectangle.
Since $EB = 40$, then $FP = 40$; since $EF = 30$, then $BP = 30$.
Since $AD = 80$, then $BC = 80$, so $PC = 80 - 30 = 50$.
Therefore, the area of $EBCF$ is sum of the areas of rectangle $EBPF$ (which is $30 \times 40 = 1200$) and $\triangle FPC$ (which is $\frac{1}{2}(40)(50) = 1000$), or $1200 + 1000 = 2200$.
Since the areas of $AEFCD$ and $EBCF$ are equal, then each is 2200, so the total area of rectangle $ABCD$ is 4400.
Since $AD = 80$, then $AB = 4400 \div 80 = 55$.
Therefore, $AE = AB - EB = 55 - 40 = 15$.

Answer: (D)

21. Let us first consider the possibilities for each integer separately:

- The two-digit prime numbers are 11, 13, 17, 19. The only one whose digits add up to a prime number is 11. Therefore, $P = 11$.
- Since $Q$ is a multiple of 5 between 2 and 19, then the possible values of $Q$ are 5, 10, 15.
- The odd numbers between 2 and 19 that are not prime are 9 and 15, so the possible values of $R$ are 9 and 15.
- The squares between 2 and 19 are 4, 9 and 16. Only 4 and 9 are squares of prime numbers, so the possible values of $S$ are 4 and 9.
- Since $P = 11$, the possible value of $Q$ are 5, 10 and 15, and $T$ is the average of $P$ and $Q$, then $T$ could be 8, 10.5 or 13. Since $T$ is also a prime number, then $T$ must be 13, so $Q = 15$.

We now know that $P = 11$, $Q = 15$ and $T = 13$.
Since the five numbers are all different, then $R$ cannot be 15, so $R = 9$.
Since $R = 9$, $S$ cannot be 9, so $S = 4$.
Therefore, the largest of the five integers is $Q = 15$.

Answer: (B)

22. By the Pythagorean Theorem, $PR = \sqrt{QR^2 + PQ^2} = \sqrt{15^2 + 8^2} = \sqrt{289} = 17$ km.
Asafa runs a total distance of $8 + 15 + 7 = 30$ km at 21 km/h in the same time that Florence runs a total distance of $17 + 7 = 24$ km.
Therefore, Asafa's speed is $\frac{30}{24} = \frac{5}{4}$ of Florence's speed, so Florence's speed is $\frac{4}{5} \times 21 = \frac{84}{5}$ km/h.

Asafa runs the last 7 km in $\frac{7}{21} = \frac{1}{3}$ hour, or 20 minutes.

Florence runs the last 7 km in $\frac{7}{\frac{84}{5}} = \frac{35}{84} = \frac{5}{12}$ hour, or 25 minutes.
Since Asafa and Florence arrive at $S$ together, then Florence arrived at $R$ 5 minutes before Asafa.

ANSWER: (E)

23. The total area of the larger circle is $\pi(2^2) = 4\pi$, so the total area of the shaded regions must be $\frac{5}{12}(4\pi) = \frac{5}{3}\pi$.

Suppose that $\angle ADC = x^\circ$.
The area of the unshaded portion of the inner circle is thus $\frac{x}{360}$ of the total area of the inner circle, or $\frac{x}{360}(\pi(1^2)) = \frac{x}{360}\pi$ (since $\angle ADC$ is $\frac{x}{360}$ of the largest possible central angle ($360^\circ$)).
The area of the shaded portion of the inner circle is thus $\pi - \frac{x}{360}\pi = \frac{360-x}{360}\pi$.

The total area of the outer ring is the difference of the areas of the outer and inner circles, or $\pi(2^2) - \pi(1^2) = 3\pi$.
The shaded area in the outer ring will be $\frac{x}{360}$ of this total area, since $\angle ADC$ is $\frac{x}{360}$ of the largest possible central angle ($360^\circ$).
So the shaded area in the outer ring is $\frac{x}{360}(3\pi) = \frac{3x}{360}\pi$.

So the total shaded area (which must equal $\frac{5}{3}\pi$) is, in term of $x$, $\frac{3x}{360}\pi + \frac{360-x}{360}\pi = \frac{360+2x}{360}\pi$.
Therefore, $\frac{360+2x}{360} = \frac{5}{3} = \frac{600}{360}$, so $360 + 2x = 600$ or $x = 120$.
Thus, $\angle ADC = 120^\circ$.

ANSWER: (B)

24. First, we complete the next several spaces in the spiral to try to get a better sense of the pattern:

| | | | | | |
|---|---|---|---|---|---|
| 17 | 16 | 15 | 14 | 13 | |
| 18 | 5 | 4 | 3 | 12 | |
| 19 | 6 | 1 | 2 | 11 | |
| 20 | 7 | 8 | 9 | 10 | |
| 21 | 22 | 23 | 24 | 25 | 26 |

We notice from this extended spiral that the odd perfect squares lie on a diagonal extending down and to the right from the 1, since 1, 9, 25 and so on will complete a square of numbers when they are written. (Try blocking out the numbers larger than each of these to see this.) This pattern does continue since when each of these odd perfect squares is reached, the number of spaces up to that point in the sequence actually does form a square.
The first odd perfect square larger than 2007 is $45^2 = 2025$.
2025 will lie 18 spaces to the left of 2007 in this row. (The row with 2025 will actually be long enough to be able to move 18 spaces to the left from 2025.)
The odd perfect square before 2025 is $43^2 = 1849$, so 1850 will be the number directly above 2025, as the row containing 1849 will continue one more space to the right before turning up.

Since 1850 is directly above 2025, then 1832 is directly above 2007.
The odd perfect square after 2025 is $47^2 = 2209$, so 2208 will be the number directly below 2025, since 2209 will be one space to the right and one down.
Since 2208 is directly below 2025, then 2190 is directly below 2007.
Therefore, the sum of the numbers directly above and below 2007 is $1832 + 2190 = 4022$.

Answer: (E)

25. For $x$ and $3x$ to each have even digits only, $x$ must be in one of the following forms. (Here, $a$, $b$, $c$ represent digits that can each be 0, 2 or 8, and $n$ is a digit that can only be 2 or 8.)

- $nabc$ ($2 \times 3 \times 3 \times 3 = 54$ possibilities)
- $na68$ ($2 \times 3 = 6$ possibilities)
- $n68a$ ($2 \times 3 = 6$ possibilities)
- $68ab$ ($3 \times 3 = 9$ possibilities)
- $n668$ (2 possibilities)
- $668a$ (3 possibilities)
- 6668 (1 possibility)
- 6868 (1 possibility)

In total, there are 82 possibilities for $x$.

In general terms, these are the only forms that work, since digits of 0, 2 and 8 in $x$ produce even digits with an even "carry" (0 or 2) thus keeping all digits in $3x$ even, while a 6 may be used, but must be followed by 8 or 68 or 668 in order to give a carry of 2.

More precisely, why do these forms work, and why are they the only forms that work?
First, we note that $3 \times 0 = 0$, $3 \times 2 = 6$, $3 \times 4 = 12$, $3 \times 6 = 18$ and $3 \times 8 = 24$.
Thus, each even digit of $x$ will produce an even digit in the corresponding position of $3x$, but may affect the next digit to the left in $3x$ through its "carry".
Note that a digit of 0 or 2 in $x$ produces no carry, while a digit of 8 in $x$ produces an even carry.
Therefore, none of these three digits can possibly create an odd digit in $3x$ (either directly or through carrying), as they each create an even digit in the corresponding position of $3x$ and do not affect whether the next digit to the left is even or odd. (We should note that the carry into any digit in $3x$ can never be more than 2, so we do not have to worry about creating a carry of 1 from a digit in $x$ of 0 or 2, or a carry of 3 from a digit of 8 in $x$ through multiple carries.)
So a digit of 2 or 8 can appear in any position of $x$ and a digit of 0 can appear in any position of $x$ except for the first position.

A digit of 4 can never appear in $x$, as it will always produce a carry of 1, and so will always create an odd digit in $3x$.

A digit of 6 can appear in $x$, as long as the carry from the previous digit is 2 to make the carry forward from the 6 equal to 2. (The carry into the 6 cannot be larger than 2.) When this happens, we have $3 \times 6 + \text{Carry} = 20$, and so a 2 is carried forward, which does not affect whether next digit is even or odd.
A carry of 2 can occur if the digit before the 6 is an 8, or if the digit before the 6 is a 6 which is preceded by 8 or by 68.

Combining the possible uses of 0, 2, 6, and 8 gives us the list of possible forms above, and hence 82 possible values for $x$.

ANSWER: (A)