Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2009 Fermat Contest

(Grade 11)

Wednesday, February 18, 2009

Solutions

1. Calculating, $3 + 3^3 = 3 + 27 = 30$.

   Answer: (D)

2. Since $3 \times 2 + 8 = \nabla + 5$, then $\nabla = 6 + 8 - 5 = 9$.

   Answer: (E)

3. Since $\angle TQR = 125^\circ$, then $\angle TQP = 180^\circ - \angle TQR = 180^\circ - 125^\circ = 55^\circ$.
   Since the sum of the angles in a triangle is 180°, then

   $$\angle PSQ = 180^\circ - \angle SPQ - \angle SQP = 180^\circ - 30^\circ - 55^\circ = 95^\circ$$

   Since $\angle TSU$ and $\angle PSQ$ are opposite, then $\angle TSU = \angle PSQ = 95^\circ$, so $x = 95$.

   Answer: (C)

4. Since $w = 4$, $x = 9$ and $z = 25$, then

   $$\sqrt{\frac{w}{x}} + \sqrt{\frac{x}{z}} = \sqrt{\frac{2^2}{3^2}} + \sqrt{\frac{3^2}{5^2}} = \frac{2}{3} + \frac{3}{5} = \frac{10}{15} + \frac{9}{15} = \frac{19}{15}$$

   Answer: (B)

5. Since $a^{-1} = \dfrac{1}{a}$, then $1 - 4(3-1)^{-1} = 1 - 4(2^{-1}) = 1 - \frac{4}{2} = 1 - 2 = -1$.

   Answer: (A)

6. There are 64 cubes to start.
   If we look at the bottom layer of cubes, we see that there are 6 uncovered cubes, each of which is missing 3 cubes above it. These are the only cubes that are missing.
   Thus, there are $6(3) = 18$ missing cubes, so there are $64 - 18 = 46$ cubes remaining.

   Answer: (A)

7. *Solution 1*
   Since $\sqrt{n^2 + n^2 + n^2 + n^2} = 64$, then $\sqrt{4n^2} = 64$ or $2n = 64$, since $n > 0$.
   Thus, $n = 32$.

   Solution 2
   Since $\sqrt{n^2 + n^2 + n^2 + n^2} = 64$, then $\sqrt{4n^2} = 64$.
   Thus, $4n^2 = 64^2 = 4096$, and so $n^2 = 1024$.
   Since $n > 0$, then $n = \sqrt{1024} = 32$.

   Answer: (D)

8. To maximize the number of songs used, Gavin should use as many of the shortest length songs as possible. (This is because he can always trade a longer song for a shorter song and shorten the total length used.)
   If Gavin uses all 50 songs of 3 minutes in length, this takes 150 minutes.
   There are $180 - 150 = 30$ minutes left, so he can play an additional $30 \div 5 = 6$ songs that are 5 minutes in length.
   In total, he plays $50 + 6 = 56$ songs.

   Answer: (C)

9. Since there are 4 ♠'s in each of the first three columns, then at least 1 ♠ must be moved out of each of these columns to make sure that each column contains exactly three ♠'s.
Therefore, we need to move at least 3 ♠'s in total.
If we move the ♠ from the top left corner to the bottom right corner

	♠	♠	♠	
♠	♠	♠		♠
♠	♠			
♠	♠	♠	♠	
		♠		♠

and the ♠ from the fourth row, third column to the fifth row, fourth column

	♠	♠	♠	
♠	♠	♠		♠
♠	♠			
♠	♠		♠	
		♠	♠	♠

and the ♠ from the second row, second column to the third row, fifth column

	♠	♠	♠	
♠		♠		♠
♠	♠			♠
♠	♠		♠	
		♠	♠	♠

then we have exactly three ♠'s in each row and each column.
Therefore, since we must move at least 3 ♠'s and we can achieve the configuration that we want by moving 3 ♠'s, then 3 is the smallest number.
(There are also other combinations of moves that will give the required result.)

Answer: (C)

10. Initially, the 25 m ladder has its bottom 7 m from the wall.
Suppose that its top is initially h m above the ground.
By the Pythagorean Theorem, $h^2 + 7^2 = 25^2$ or $h^2 + 49 = 625$.
Thus, $h^2 = 625 - 49 = 576$, so $h = \sqrt{576} = 24$, since $h > 0$.
When the top slides 4 m down the wall, the top is now $24 - 4 = 20$ m above the ground.
Suppose that the bottom ends up d m from the wall.
Again by the Pythagorean Theorem, $20^2 + d^2 = 25^2$, so $d^2 = 25^2 - 20^2 = 625 - 400 = 225$.
Since $d > 0$, then $d = 15$.
This means that the bottom is $15 - 7 = 8$ m farther from the wall than in its original position.

Answer: (E)

11. Since the result must work no matter what positive integers m and n we choose with $m < n$, we try $m = 1$ and $n = 2$.
In this case, $\dfrac{m}{n} = \dfrac{1}{2}$ and $\dfrac{m+3}{n+3} = \dfrac{4}{5}$.
Here, $\dfrac{m+3}{n+3} > \dfrac{m}{n}$, so the answer must be (D).

(We could also prove this algebraically by starting with $m < n$, which gives $3m < 3n$, which gives $mn + 3m < mn + 3n$, which gives $m(n+3) < n(m+3)$ which gives $\frac{m}{n} < \frac{m+3}{n+3}$.)

ANSWER: (D)

12. Between 5000 and 6000, every integer except for 6000 has thousands digit equal to 5.
Note that the number 6000 does not have the desired property.
Thus, we are looking for integers $5xyz$ with $x + y + z = 5$.
The possible combinations of three digits for x, y and z are: 5, 0, 0; 4, 1, 0; 3, 2, 0; 3, 1, 1; 2, 2, 1.
A combination of three different digits (like 4, 1, 0) can be arranged in 6 ways: 410, 401, 140, 104, 041, 014.
A combination of three digits with one repeated (like 5, 0, 0) can be arranged in 3 ways: 500, 050, 005.
Therefore, 5, 0, 0 and 3, 1, 1 and 2, 2, 1 each give 3 integers, and 4, 1, 0 and 3, 2, 0 each give 6 integers.
So the number of integers with the desired property is $3(3) + 2(6) = 21$.

ANSWER: (C)

13. Since x is an integer, then $x + 1$ is an integer.
Since $\frac{-6}{x+1}$ is to be integer, then $x + 1$ must be a divisor of -6.
Thus, there are 8 possible values for $x + 1$, namely -6, -3, -2, -1, 1, 2, 3, and 6.
This gives 8 possible values for x, namely -7, -4, -3, -2, 0, 1, 2, and 5.

ANSWER: (A)

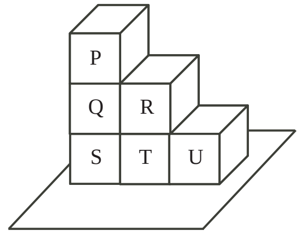
14. Since the three numbers in each straight line must have a product of 3240 and must include 45, then the other two numbers in each line must have a product of $\frac{3240}{45} = 72$.
The possible pairs of positive integers are 1 and 72, 2 and 36, 3 and 24, 4 and 18, 6 and 12, and 8 and 9.
The sums of the numbers in these pairs are 73, 38, 27, 22, 18, and 17.
To maximize the sum of the eight numbers, we want to choose the pairs with the largest possible sums, so we choose the first four pairs.
Thus, the largest possible sum of the eight numbers is $73 + 38 + 27 + 22 = 160$.

ANSWER: (E)

15. Suppose that there are 1000 students at Dunkley S.S.
On Monday, there were thus 100 students absent and 900 students present.
On Tuesday, 10% of the 900 students who were present on Monday, or $0.1(900) = 90$ students, were absent. The remaining $900 - 90 = 810$ students who were present on Monday were still present on Tuesday.
Similarly, 10% of the 100 students who were absent on Monday, or $0.1(100) = 10$ students, were present on Tuesday. The remaining $100 - 10 = 90$ students who were absent on Monday were still absent on Tuesday.
Thus, there were $810 + 10 = 820$ students present on Tuesday, or $\frac{820}{1000} \times 100\% = 82\%$ of the whole student population.

ANSWER: (B)

16. Label the six dice as shown:



The maximum overall exposed sum occurs when the sum of the exposed faces on each die is maximized.
Die P has 5 exposed faces. The sum of these faces is a maximum when the 1 is hidden, so the maximum exposed sum on die P is $2 + 3 + 4 + 5 + 6 = 20$.
Dice Q and S each have 3 exposed faces. Two of these are opposite to each other, so have a sum of 7. Thus, to maximize the exposed sum on these dice, we position them with the 6 as the unpaired exposed face. (This is on the left face of the stack.) Each of these dice has a maximum exposed sum of $6 + 7 = 13$.
Dice R and U each have 4 exposed faces. Two of these are opposite to each other, so have a sum of 7. Thus, to maximize the exposed sum on these dice, we position them with the 6 and the 5 as the unpaired exposed faces (on the top and right of the stack). Each of these dice have a maximum exposed sum of $5 + 6 + 7 = 18$.
Die T has 2 exposed faces, which are opposite each other, so have a sum of 7.
Therefore, the maximum possible sum of the exposed faces is $20 + 13 + 13 + 18 + 18 + 7 = 89$.

Answer: (C)

17. Suppose that the radius of the region is r.
The length of the semi-circle is half of the circumference, or $\frac{1}{2}(2\pi r) = \pi r$.
Thus, the perimeter of the shaded region is $\pi r + 2r$.
Since the perimeter is 20, then $\pi r + 2r = 20$ or $r(\pi + 2) = 20$ or $r = \dfrac{20}{\pi + 2}$.
Thus, the area of the semi-circle is half of the area of a circle with this radius, or $\frac{1}{2}\pi\left(\dfrac{20}{\pi+2}\right)^2$ which is approximately 23.768, which of the given choices is closest to 23.8.

Answer: (B)

18. Let the length of his route be d km.
Since he arrives 1 minute early when travelling at 75 km/h and 1 minute late when travelling at 70 km/h, then the difference between these times is 2 minutes, or $\frac{1}{30}$ of an hour.
The time that his trip takes while travelling at 75 km/h is $\dfrac{d}{75}$ hours, and at 70 km/h is $\dfrac{d}{70}$ hours.
Therefore,

$$\begin{aligned}
\frac{d}{70} - \frac{d}{75} &= \frac{1}{30} \\
75d - 70d &= \frac{75(70)}{30} \\
5d &= 25(7) \\
d &= 35
\end{aligned}$$

Therefore, the route is 35 km long.

Answer: (B)

19. Since $2^x = 15$ and $15^y = 32$, then $(2^x)^y = 32$ or $2^{xy} = 32$.
Since $2^5 = 32$, then $xy = 5$.

Answer: (A)

20. Since the circle has radius 1, then its area is $\pi(1^2) = \pi$.
Since the square and the circle have the same area, then the side length of the square is $\sqrt{\pi}$.
Let M be the midpoint of line segment PQ.
Since PQ is a chord of the circle, then OM is perpendicular to line segment PQ.
Since OM is perpendicular to PQ and O is the centre of the square, then OM is half of the length of one of the sides of the square, so $OM = \frac{1}{2}\sqrt{\pi}$.
By the Pythagorean Theorem in $\triangle OPM$, we have $PM^2 = OP^2 - OM^2 = 1^2 - \left(\frac{1}{2}\sqrt{\pi}\right)^2 = 1 - \frac{1}{4}\pi$.
Therefore, $PQ = 2PM = 2\sqrt{1 - \frac{1}{4}\pi} = \sqrt{4(1 - \frac{1}{4}\pi)} = \sqrt{4 - \pi}$.

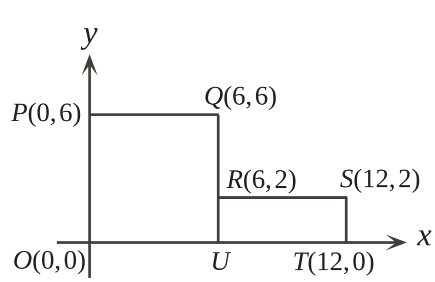
Answer: (A)

21. *Solution 1*
Since we want the maximum number of people, we start with the largest of the given choices and see if we can make it work.
Is it possible that 80 people could have eaten both ice cream and cake?
If so, there would be at least 80 people who ate cake and at least $\frac{3}{2}(80) = 120$ people who ate ice cream. (Note that there can be overlap between these two sets of people.)
Is this possible?
It is if we say that exactly 120 people ate ice cream and exactly 80 people ate cake, with all of these 80 eating ice cream as well.
So an overlap of 80 is possible, which means that the answer is (D).

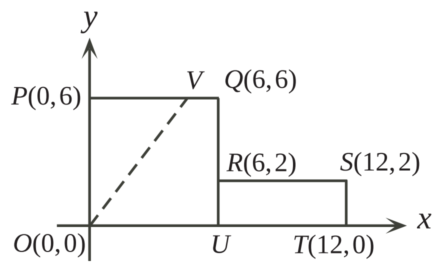
Solution 2
Suppose that the there were x people who ate cake only, y people who ate ice cream only, b people who ate both cake and ice cream, and n people who ate neither.
We know that $x + y + b + n = 120$ and so $x + y + b = 120 - n$.
We also know that the total number who ate cake was $x + b$ and the total number who ate ice cream was $y + b$, so $\dfrac{y+b}{x+b} = \dfrac{3}{2}$ or $2(y + b) = 3(x + b)$ or $2y = 3x + b$ or $y = \frac{3}{2}x + \frac{1}{2}b$.
Therefore, $\frac{3}{2}x + \frac{1}{2}b + x + b = 120 - n$ and so $\frac{5}{2}x + \frac{3}{2}b = 120 - n$.
Multiplying both sides by 2, we obtain $5x + 3b = 240 - 2n$.
Since x, b and n are non-negative, then the left side is at most 240 and so b can be at most $\frac{1}{3}(240) = 80$, if $y = n = 0$.
We saw in Solution 1 that $b = 80$ is possible.

Answer: (D)

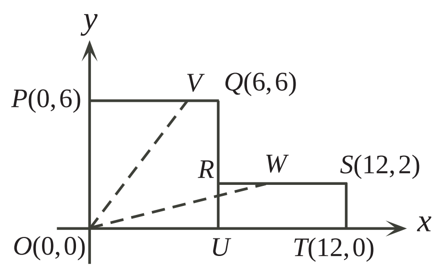
22. Extend QR downwards to meet the x-axis at $U(6, 0)$.



The area of figure $OPQRST$ equals the sum of the areas of square $OPQU$ (which has side length 6, so area 36) and rectangle $RSTU$ (which has height 2 and width 6, so area 12).
Thus, the area of figure $OPQRST$ is 48. If we are to divide the figure into three pieces of equal area, then each piece has area 16.
Let V be the first point on the perimeter (measured clockwise from P) so that the line through O and V cuts off an area of 16.
Note that the area of $\triangle OPQ$ is half of the area of square $OPQU$, or 18, so V is to the left of Q on PQ.
Thus, V has coordinates $(v, 6)$ for some number v.



Consider $\triangle OPV$ having base OP of length 6 and height PV of length v.
Since the area of $\triangle OPV$ is 16, then $\frac{1}{2}(6)(v) = 16$ or $3v = 16$ or $v = \frac{16}{3}$.
Therefore, the slope of OV is $\dfrac{6}{\frac{16}{3}} = \dfrac{9}{8}$.
Let W be the second desired point.
Since the area of $\triangle OTS$ is $\frac{1}{2}(OT)(TS) = \frac{1}{2}(12)(2) = 12$ (less than $\frac{1}{3}$ of the total area) and the area of trapezoid $ORST$ is $\frac{1}{2}(RS + OT)(ST) = \frac{1}{2}(6 + 12)(2) = 18$ (more than $\frac{1}{3}$ of the total area), then W lies on RS.



Suppose that W has coordinates $(w, 2)$ for some number w.
We want the area of trapezoid $WSTO$ to be 16.
Therefore,

$$\begin{aligned}
\tfrac{1}{2}(WS + OT)(ST) &= 16\\
\tfrac{1}{2}(12 - w + 12)(2) &= 16\\
24 - w &= 16\\
w &= 8
\end{aligned}$$

Thus, the coordinates of W are $(8, 2)$, and so the slope of OW is $\frac{2}{8} = \frac{1}{4}$.
Thus, the sum of the two required slopes is $\frac{9}{8} + \frac{1}{4} = \frac{11}{8}$.

ANSWER: (E)

23. Adding the second and third equations, we obtain

$$\begin{aligned} ac+bd+ad+bc &= 77 \\ ac+ad+bc+bd &= 77 \\ a(c+d)+b(c+d) &= 77 \\ (a+b)(c+d) &= 77 \end{aligned}$$

Since each of a, b, c and d is a positive integer, then $a+b$ and $c+d$ are each positive integers and are each at least 2.
Since the product of $a+b$ and $c+d$ is $77 = 7 \times 11$ (with 7 and 11 both prime), then one must equal 7 and the other must equal 11.
Therefore, $a+b+c+d = 7+11 = 18$.
(We can check with some work that $(a,b,c,d) = (5,2,4,7)$ is a solution to the system.)

ANSWER: (D)

24. The three machines operate in a way such that if the two numbers in the output have a common factor larger than 1, then the two numbers in the input would have to have a common factor larger than 1.
To see this, let us look at each machine separately. We use the fact that if two numbers are each multiples of d, then their sum and difference are also multiples of d.
Suppose that (m,n) is input into Machine A. The output is (n,m). If n and m have a common factor larger than 1, then m and n do as well.
Suppose that (m,n) is input into Machine B. The output is $(m+3n,n)$. If $m+3n$ and n have a common factor d, then $(m+3n)-n-n-n = m$ has a factor of d as each part of the subtraction is a multiple of d. Therefore, m and n have a common factor of d.
Suppose that (m,n) is input into Machine C. The output is $(m-2n,n)$. If $m-2n$ and n have a common factor d, then $(m-2n)+n+n = m$ has a factor of d as each part of the addition is a multiple of d. Therefore, m and n have a common factor of d.
In each case, any common factor that exists in the output is present in the input.

Let us look at the numbers in the five candidates.
After some work, we can find the prime factorizations of the six integers:

$$\begin{aligned} 2009 &= 7(287) = 7(7)(41) \\ 1016 &= 8(127) = 2(2)(2)(127) \\ 1004 &= 4(251) = 2(2)(251) \\ 1002 &= 2(501) = 2(3)(167) \\ 1008 &= 8(126) = 8(3)(42) = 16(3)(3)(7) = 2(2)(2)(2)(3)(3)(7) \\ 1032 &= 8(129) = 8(3)(43) = 2(2)(2)(3)(43) \end{aligned}$$

Therefore, the only one of 1002, 1004, 1008, 1016, 1032 that has a common factor larger than 1 with 2009 is 1008, which has a common factor of 7 with 2009.
How does this help? Since 2009 and 1008 have a common factor of 7, then whatever pair was input to produce $(2009,1008)$ must have also had a common factor of 7. Also, the pair that was input to create this pair also had a common factor of 7. This can be traced back through every step to say that the initial pair that produces the eventual output of $(2009,1008)$ must have a common factor of 7.
Thus, $(2009,1008)$ cannot have come from $(0,1)$.

Notes:

- This does not tell us that the other four pairs necessarily work. It does tell us, though, that $(2009, 1008)$ cannot work.
- We can trace the other four outputs back to $(0, 1)$ with some effort. (This process is easier to do than it is to describe!)
  To do this, we notice that if the output of Machine A was (a, b), then its input was (b, a), since Machine A switches the two entries.
  Also, if the output of Machine B was (a, b), then its input was $(a - 3b, b)$, since Machine B adds three times the second number to the first.
  Lastly, if the output of Machine C was (a, b), then its input was $(a + 2b, b)$, since Machine C subtracts two times the second number from the first.
  Consider $(2009, 1016)$ for example. We try to find a way from $(2009, 1016)$ back to $(0, 1)$. We only need to find one way that works, rather than looking for a specific way.

  We note before doing this that starting with an input of (m, n) and then applying Machine B then Machine C gives an output of $((m + 3n) - 2n, n) = (m + n, n)$. Thus, if applying Machine B then Machine C (we call this combination "Machine BC") gives an output of (a, b), then its input must have been $(a - b, b)$. We can use this combined machine to try to work backwards and arrive at $(0, 1)$. This will simplify the process and help us avoid negative numbers.
  We do this by making a chart and by attempting to make the larger number smaller wherever possible:

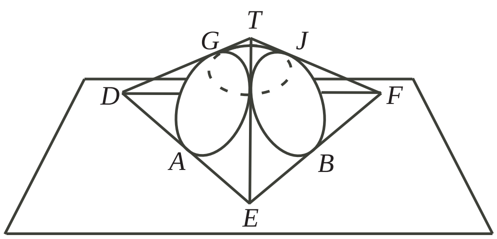
Output	Machine	Input
$(2009, 1016)$	BC	$(993, 1016)$
$(993, 1016)$	A	$(1016, 993)$
$(1016, 993)$	BC	$(23, 993)$
$(23, 993)$	A	$(993, 23)$
$(993, 23)$	BC, 43 times	$(4, 23)$
$(4, 23)$	A	$(23, 4)$
$(23, 4)$	BC, 5 times	$(3, 4)$
$(3, 4)$	A	$(4, 3)$
$(4, 3)$	BC	$(1, 3)$
$(1, 3)$	A	$(3, 1)$
$(3, 1)$	B	$(0, 1)$

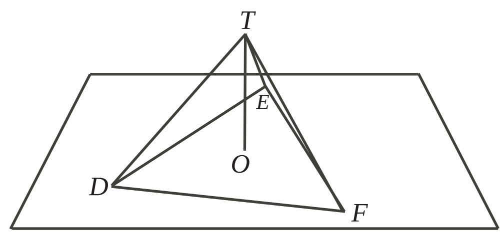
  Therefore, by going up through this table, we can see a way to get from an initial input of $(0, 1)$ to a final output of $(2009, 1016)$.
  In a similar way, we can show that we can obtain final outputs of each of $(2009, 1004)$, $(2009, 1002)$, and $(2009, 1032)$.

Answer: (D)

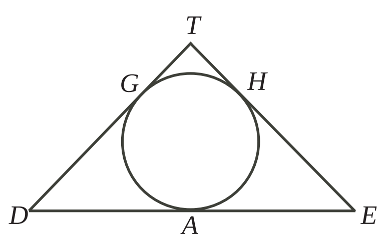
25. Let the three points at which the circles are tangent to the plane be A, B and C.
Each of the three circles is contained in a plane. This plane will intersect the original plane along a line that passes through A, B or C, and is tangent to the circle at one of these points.
Suppose that these three lines intersect at points D, E and F, with A on DE, B on EF, and C on FD.
By symmetry, $DE = EF = FD$. Also, A, B and C are the midpoints of these three segments.

(More formally, the configuration is not changed by rotating it through 120° or by reflecting it horizontally through the three vertical planes that pass through T (defined below) and the vertices of $\triangle DEF$, so these facts are true.)
Thus, $\triangle DEF$ is equilateral. Let O be its incentre. That is, O is the point of intersection of the three angle bisectors of $\triangle DEF$. Since $\triangle DEF$ is equilateral, then its angle bisectors, medians and altitudes are all the same.
Let the points where the three circles touch be G, H and J, with G the point of contact between the circles containing C and A, H the point of contact between the circles containing A and B, and J the point of contact between the circles containing B and C. Note that $GH = HJ = JG$ by symmetry. We want to determine the radius of the circle that passes through G, H and J.
Draw the lines through D, E and F passing through G, H and J, respectively. These lines will be tangent to the circles at points G, H and J, and will meet at a common point, which we call T. T is directly above point O, the centre of $\triangle DEF$. (Each of these facts is again true by the symmetry of rotation and reflection.) Now, $TDEF$ is a tetrahedron with its base ($\triangle DEF$) equilateral, and three congruent side faces.



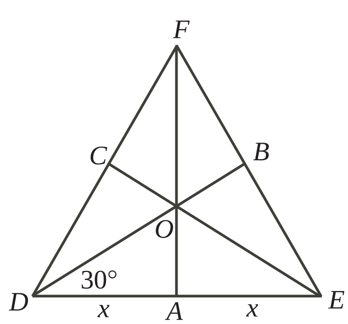


Consider the side face $\triangle DET$. The circle (of radius 10) containing A is tangent to the three sides of $\triangle DET$ at A, G and H.



Also, $\triangle DET$ is inclined at 45° to the horizontal.
Since A is the midpoint of DE, then OA is perpendicular to DE, as is TA.
Since $\triangle DET$ is inclined at 45° to the horizontal, then $\angle TAO = 45^\circ$.
Note that $\triangle TAO$ is right-angled at O (because T is directly above O), so the fact that $\angle TAO = 45^\circ$ tells us that $\triangle TAO$ is a 45°-45°-90° triangle, and so $TA = \sqrt{2}AO$.

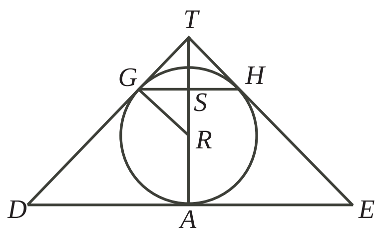
Next, we look at the dimensions of $\triangle DEF$.
Suppose that $DE = 2x$. Thus, $DA = AE = x$ and $\triangle DEF$ is equilateral with side length $2x$.



Now DO is the angle bisector of $\angle EDF$, so $\angle ODA = 30°$, and OA is perpendicular to DE, so $\triangle DOA$ is a 30°-60°-90° triangle. Thus, $OA = \frac{1}{\sqrt{3}}DA = \frac{1}{\sqrt{3}}x$, and $OD = \frac{2}{\sqrt{3}}DA = \frac{2}{\sqrt{3}}x$.
Since $TA = \sqrt{2}AO$, then $TA = \frac{\sqrt{2}}{\sqrt{3}}x$.
In $\triangle TAD$, the Pythagorean Theorem gives $TD = \sqrt{TA^2 + DA^2} = \sqrt{x^2 + \frac{2}{3}x^2} = \frac{\sqrt{5}}{\sqrt{3}}x$, since $x > 0$.

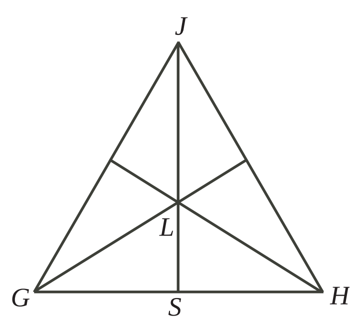
So, to summarize so far, $\triangle DET$ has $DE = 2x$, $DA = AE = x$, $TD = \frac{\sqrt{5}}{\sqrt{3}}x$ and $TA = \frac{\sqrt{2}}{\sqrt{3}}x$.

Let the centre of the circle contained in $\triangle DET$ be R (which will lie on TA by symmetry) and join R to G. Since RG is a radius of the circle, then $RG = 10$ and RG is perpendicular to DT.
Also, join GH and let the point of intersection of GH with TA be S. By symmetry, GH is perpendicular to TA.



Now $\triangle TSG$, $\triangle TGR$ and $\triangle TAD$ are all similar, since each is right-angled and shares a common angle at T.
We want to determine the length of SG.
By similar triangles, $\dfrac{SG}{TG} = \dfrac{AD}{TD} = \dfrac{x}{\frac{\sqrt{5}}{\sqrt{3}}x}$, so $SG = \frac{\sqrt{3}}{\sqrt{5}}TG$.
Also by similar triangles, $\dfrac{TG}{GR} = \dfrac{TA}{AD} = \dfrac{\frac{\sqrt{2}}{\sqrt{3}}x}{x}$, so $TG = \frac{\sqrt{2}}{\sqrt{3}}GR = 10\frac{\sqrt{2}}{\sqrt{3}}$.
Thus, $SG = \frac{\sqrt{3}}{\sqrt{5}}\left(10\frac{\sqrt{2}}{\sqrt{3}}\right) = 10\frac{\sqrt{2}}{\sqrt{5}} = 2\sqrt{10}$.

Finally, consider $\triangle GHJ$. This triangle is equilateral. S is the midpoint of GH and $SG = 2\sqrt{10}$. Let L be the incentre of this triangle.



Note that L is also the centre of the circle that passes through G, H and J because $LG = LJ = LH$, so LG is the radius of this circle.

Since above we saw that $OD = \frac{2}{\sqrt{3}}AD$, then $LG = \frac{2}{\sqrt{3}}SG$, since the configuration is the same in $\triangle DEF$ and $\triangle GHJ$. Thus, $LG = \frac{4\sqrt{10}}{\sqrt{3}} = \frac{4}{3}\sqrt{30}$, and this is the radius of the circle.
Since $\frac{4}{3}\sqrt{30} \approx 7.303$, the answer is closest to 7.3, of the given choices.

ANSWER: (C)