The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

2011 Euclid Contest

Tuesday, April 12, 2011

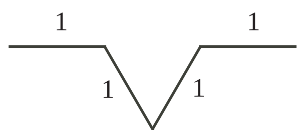
Solutions

1. (a) Since $(x+1)+(x+2)+(x+3)=8+9+10$, then $3x+6=27$ or $3x=21$ and so $x=7$.

   (b) Since $\sqrt{25+\sqrt{x}}=6$, then squaring both sides gives $25+\sqrt{x}=36$ or $\sqrt{x}=11$.
   Since $\sqrt{x}=11$, then squaring both sides again, we obtain $x=11^2=121$.
   Checking, $\sqrt{25+\sqrt{121}}=\sqrt{25+11}=\sqrt{36}=6$, as required.

   (c) Since $(a,2)$ is the point of intersection of the lines with equations $y=2x-4$ and $y=x+k$, then the coordinates of this point must satisfy both equations.
   Using the first equation, $2=2a-4$ or $2a=6$ or $a=3$.
   Since the coordinates of the point $(3,2)$ satisfy the equation $y=x+k$, then $2=3+k$ or $k=-1$.

2. (a) Since the side length of the original square is 3 and an equilateral triangle of side length 1 is removed from the middle of each side, then each of the two remaining pieces of each side of the square has length 1.
   Also, each of the two sides of each of the equilateral triangles that are shown has length 1.



   Therefore, each of the 16 line segments in the figure has length 1, and so the perimeter of the figure is 16.

   (b) Since $DC=DB$, then $\triangle CDB$ is isosceles and $\angle DBC=\angle DCB=15^\circ$.
   Thus, $\angle CDB=180^\circ-\angle DBC-\angle DCB=150^\circ$.
   Since the angles around a point add to 360°, then

$$\angle ADC=360^\circ-\angle ADB-\angle CDB=360^\circ-130^\circ-150^\circ=80^\circ\ .$$

   (c) By the Pythagorean Theorem in $\triangle EAD$, we have $EA^2+AD^2=ED^2$ or $12^2+AD^2=13^2$, and so $AD=\sqrt{169-144}=5$, since $AD>0$.
   By the Pythagorean Theorem in $\triangle ACD$, we have $AC^2+CD^2=AD^2$ or $AC^2+4^2=5^2$, and so $AC=\sqrt{25-16}=3$, since $AC>0$.
   (We could also have determined the lengths of AD and AC by recognizing 3-4-5 and 5-12-13 right-angled triangles.)
   By the Pythagorean Theorem in $\triangle ABC$, we have $AB^2+BC^2=AC^2$ or $AB^2+2^2=3^2$, and so $AB=\sqrt{9-4}=\sqrt{5}$, since $AB>0$.

3. (a) *Solution 1*
   Since we want to make $15-\dfrac{y}{x}$ as large as possible, then we want to subtract as little as possible from 15.
   In other words, we want to make $\dfrac{y}{x}$ as small as possible.
   To make a fraction with positive numerator and denominator as small as possible, we make the numerator as small as possible and the denominator as large as possible.
   Since $2\le x\le 5$ and $10\le y\le 20$, then we make $x=5$ and $y=10$.
   Therefore, the maximum value of $15-\dfrac{y}{x}$ is $15-\dfrac{10}{5}=13$.

Solution 2
Since y is positive and $2 \leq x \leq 5$, then $15 - \dfrac{y}{x} \leq 15 - \dfrac{y}{5}$ for any x with $2 \leq x \leq 5$ and positive y.
Since $10 \leq y \leq 20$, then $15 - \dfrac{y}{5} \leq 15 - \dfrac{10}{5}$ for any y with $10 \leq y \leq 20$.
Therefore, for any x and y in these ranges, $15 - \dfrac{y}{x} \leq 15 - \dfrac{10}{5} = 13$, and so the maximum possible value is 13 (which occurs when $x = 5$ and $y = 10$).

(b) *Solution 1*
First, we add the two given equations to obtain

$$(f(x) + g(x)) + (f(x) - g(x)) = (3x + 5) + (5x + 7)$$

or $2f(x) = 8x + 12$ which gives $f(x) = 4x + 6$.
Since $f(x) + g(x) = 3x + 5$, then $g(x) = 3x + 5 - f(x) = 3x + 5 - (4x + 6) = -x - 1$.
(We could also find $g(x)$ by subtracting the two given equations or by using the second of the given equations.)
Since $f(x) = 4x + 6$, then $f(2) = 14$.
Since $g(x) = -x - 1$, then $g(2) = -3$.
Therefore, $2f(2)g(2) = 2 \times 14 \times (-3) = -84$.

Solution 2
Since the two given equations are true for all values of x, then we can substitute $x = 2$ to obtain

$$\begin{aligned} f(2) + g(2) &= 11 \\ f(2) - g(2) &= 17 \end{aligned}$$

Next, we add these two equations to obtain $2f(2) = 28$ or $f(2) = 14$.
Since $f(2) + g(2) = 11$, then $g(2) = 11 - f(2) = 11 - 14 = -3$.
(We could also find $g(2)$ by subtracting the two equations above or by using the second of these equations.)
Therefore, $2f(2)g(2) = 2 \times 14 \times (-3) = -84$.

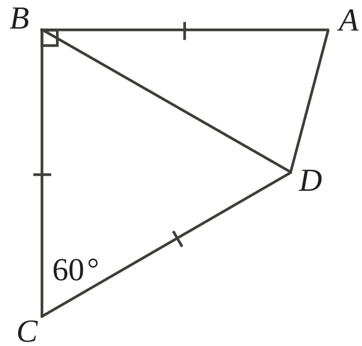
4. (a) We consider choosing the three numbers all at once.
We list the possible sets of three numbers that can be chosen:

$$\{1,2,3\} \ \{1,2,4\} \ \{1,2,5\} \ \{1,3,4\} \ \{1,3,5\} \ \{1,4,5\} \ \{2,3,4\} \ \{2,3,5\} \ \{2,4,5\} \ \{3,4,5\}$$

We have listed each in increasing order because once the numbers are chosen, we arrange them in increasing order.
There are 10 sets of three numbers that can be chosen.
Of these 10, the 4 sequences $1,2,3$ and $1,3,5$ and $2,3,4$ and $3,4,5$ are arithmetic sequences.
Therefore, the probability that the resulting sequence is an arithmetic sequence is $\frac{4}{10}$ or $\frac{2}{5}$.

(b) *Solution 1*

Join B to D.



Consider $\triangle CBD$.
Since $CB = CD$, then $\angle CBD = \angle CDB = \frac{1}{2}(180^\circ - \angle BCD) = \frac{1}{2}(180^\circ - 60^\circ) = 60^\circ$.
Therefore, $\triangle BCD$ is equilateral, and so $BD = BC = CD = 6$.
Consider $\triangle DBA$.
Note that $\angle DBA = 90^\circ - \angle CBD = 90^\circ - 60^\circ = 30^\circ$.
Since $BD = BA = 6$, then $\angle BDA = \angle BAD = \frac{1}{2}(180^\circ - \angle DBA) = \frac{1}{2}(180^\circ - 30^\circ) = 75^\circ$.
We calculate the length of AD.

<u>Method 1</u>

By the Sine Law in $\triangle DBA$, we have $\dfrac{AD}{\sin(\angle DBA)} = \dfrac{BA}{\sin(\angle BDA)}$.

Therefore, $AD = \dfrac{6\sin(30^\circ)}{\sin(75^\circ)} = \dfrac{6 \times \frac{1}{2}}{\sin(75^\circ)} = \dfrac{3}{\sin(75^\circ)}$.

<u>Method 2</u>

If we drop a perpendicular from B to P on AD, then P is the midpoint of AD since $\triangle BDA$ is isosceles. Thus, $AD = 2AP$.
Also, BP bisects $\angle DBA$, so $\angle ABP = 15^\circ$.
Now, $AP = BA\sin(\angle ABP) = 6\sin(15^\circ)$.
Therefore, $AD = 2AP = 12\sin(15^\circ)$.

<u>Method 3</u>

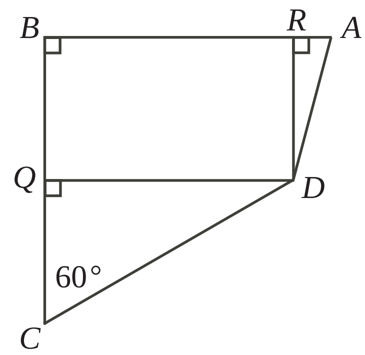
By the Cosine Law in $\triangle DBA$,

$$\begin{aligned} AD^2 &= AB^2 + BD^2 - 2(AB)(BD)\cos(\angle ABD) \\ &= 6^2 + 6^2 - 2(6)(6)\cos(30^\circ) \\ &= 72 - 72(\tfrac{\sqrt{3}}{2}) \\ &= 72 - 36\sqrt{3} \end{aligned}$$

Therefore, $AD = \sqrt{36(2-\sqrt{3})} = 6\sqrt{2-\sqrt{3}}$ since $AD > 0$.

Solution 2

Drop perpendiculars from D to Q on BC and from D to R on BA.



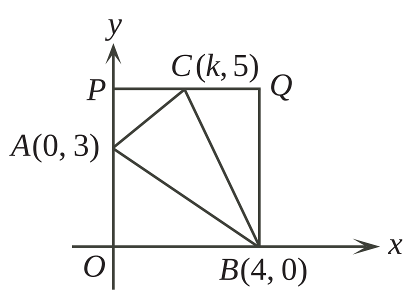
Then $CQ = CD\cos(\angle DCQ) = 6\cos(60^\circ) = 6 \times \frac{1}{2} = 3$.
Also, $DQ = CD\sin(\angle DCQ) = 6\sin(60^\circ) = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$.
Since $BC = 6$, then $BQ = BC - CQ = 6 - 3 = 3$.
Now quadrilateral $BQDR$ has three right angles, so it must have a fourth right angle and so must be a rectangle.
Thus, $RD = BQ = 3$ and $RB = DQ = 3\sqrt{3}$.
Since $AB = 6$, then $AR = AB - RB = 6 - 3\sqrt{3}$.
Since $\triangle ARD$ is right-angled at R, then using the Pythagorean Theorem and the fact that $AD > 0$, we obtain

$$AD = \sqrt{RD^2 + AR^2} = \sqrt{3^2 + (6 - 3\sqrt{3})^2} = \sqrt{9 + 36 - 36\sqrt{3} + 27} = \sqrt{72 - 36\sqrt{3}}$$

which we can rewrite as $AD = \sqrt{36(2 - \sqrt{3})} = 6\sqrt{2 - \sqrt{3}}$.

5. (a) Let n be the original number and N be the number when the digits are reversed. Since we are looking for the largest value of n, we assume that $n > 0$.
Since we want N to be 75% larger than n, then N should be 175% of n, or $N = \frac{7}{4}n$.
Suppose that the tens digit of n is a and the units digit of n is b. Then $n = 10a + b$.
Also, the tens digit of N is b and the units digit of N is a, so $N = 10b + a$.
We want $10b + a = \frac{7}{4}(10a + b)$ or $4(10b + a) = 7(10a + b)$ or $40b + 4a = 70a + 7b$ or $33b = 66a$, and so $b = 2a$.
This tells us that that any two-digit number $n = 10a + b$ with $b = 2a$ has the required property.
Since both a and b are digits then $b < 10$ and so $a < 5$, which means that the possible values of n are 12, 24, 36, and 48.
The largest of these numbers is 48.

(b) We "complete the rectangle" by drawing a horizontal line through C which meets the y-axis at P and the vertical line through B at Q.



Since C has y-coordinate 5, then P has y-coordinate 5; thus the coordinates of P are $(0,5)$.
Since B has x-coordinate 4, then Q has x-coordinate 4.
Since C has y-coordinate 5, then Q has y-coordinate 5.
Therefore, the coordinates of Q are $(4,5)$, and so rectangle $OPQB$ is 4 by 5 and so has area $4 \times 5 = 20$.
Now rectangle $OPQB$ is made up of four smaller triangles, and so the sum of the areas of these triangles must be 20.
Let us examine each of these triangles:

- $\triangle ABC$ has area 8 (given information)
- $\triangle AOB$ is right-angled at O, has height $AO = 3$ and base $OB = 4$, and so has area $\frac{1}{2} \times 4 \times 3 = 6$.
- $\triangle APC$ is right-angled at P, has height $AP = 5 - 3 = 2$ and base $PC = k - 0 = k$, and so has area $\frac{1}{2} \times k \times 2 = k$.
- $\triangle CQB$ is right-angled at Q, has height $QB = 5 - 0 = 5$ and base $CQ = 4 - k$, and so has area $\frac{1}{2} \times (4-k) \times 5 = 10 - \frac{5}{2}k$.

Since the sum of the areas of these triangles is 20, then $8 + 6 + k + 10 - \frac{5}{2}k = 20$ or $4 = \frac{3}{2}k$ and so $k = \frac{8}{3}$.

6. (a) *Solution 1*

Suppose that the distance from point A to point B is d km.
Suppose also that r_c is the speed at which Serge travels while not paddling (i.e. being carried by just the current), that r_p is the speed at which Serge travels with no current (i.e. just from his paddling), and r_{p+c} his speed when being moved by both his paddling and the current.
It takes Serge 18 minutes to travel from A to B while paddling with the current.
Thus, $r_{p+c} = \dfrac{d}{18}$ km/min.
It takes Serge 30 minutes to travel from A to B with just the current.
Thus, $r_c = \dfrac{d}{30}$ km/min.
But $r_p = r_{p+c} - r_c = \dfrac{d}{18} - \dfrac{d}{30} = \dfrac{5d}{90} - \dfrac{3d}{90} = \dfrac{2d}{90} = \dfrac{d}{45}$ km/min.
Since Serge can paddle the d km from A to B at a speed of $\dfrac{d}{45}$ km/min, then it takes him 45 minutes to paddle from A to B with no current.

Solution 2

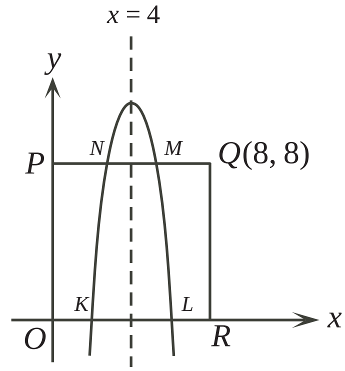
Suppose that the distance from point A to point B is d km, the speed of the current of the river is r km/h, and the speed that Serge can paddle is s km/h.
Since the current can carry Serge from A to B in 30 minutes (or $\frac{1}{2}$ h), then $\dfrac{d}{r} = \dfrac{1}{2}$.
When Serge paddles with the current, his speed equals his paddling speed plus the speed of the current, or $(s+r)$ km/h.
Since Serge can paddle with the current from A to B in 18 minutes (or $\dfrac{3}{10}$ h), then $\dfrac{d}{r+s} = \dfrac{3}{10}$.
The time to paddle from A to B with no current would be $\dfrac{d}{s}$ h.

Since $\dfrac{d}{r} = \dfrac{1}{2}$, then $\dfrac{r}{d} = 2$.
Since $\dfrac{d}{r+s} = \dfrac{3}{10}$, then $\dfrac{r+s}{d} = \dfrac{10}{3}$.
Therefore, $\dfrac{s}{d} = \dfrac{r+s}{d} - \dfrac{r}{d} = \dfrac{10}{3} - 2 = \dfrac{4}{3}$.
Thus, $\dfrac{d}{s} = \dfrac{3}{4}$, and so it would take Serge $\dfrac{3}{4}$ of an hour, or 45 minutes, to paddle from A to B with no current.

Solution 3
Suppose that the distance from point A to point B is d km, the speed of the current of the river is r km/h, and the speed that Serge can paddle is s km/h.
Since the current can carry Serge from A to B in 30 minutes (or $\dfrac{1}{2}$ h), then $\dfrac{d}{r} = \dfrac{1}{2}$ or $d = \frac{1}{2}r$.
When Serge paddles with the current, his speed equals his paddling speed plus the speed of the current, or $(s+r)$ km/h.
Since Serge can paddle with the current from A to B in 18 minutes (or $\dfrac{3}{10}$ h), then $\dfrac{d}{r+s} = \dfrac{3}{10}$ or $d = \frac{3}{10}(r+s)$.
Since $d = \frac{1}{2}r$ and $d = \frac{3}{10}(r+s)$, then $\frac{1}{2}r = \frac{3}{10}(r+s)$ or $5r = 3r + 3s$ and so $s = \frac{2}{3}r$.
To travel from A to B with no current, the time in hours that it takes is $\dfrac{d}{s} = \dfrac{\frac{1}{2}r}{\frac{2}{3}r} = \dfrac{3}{4}$, or 45 minutes.

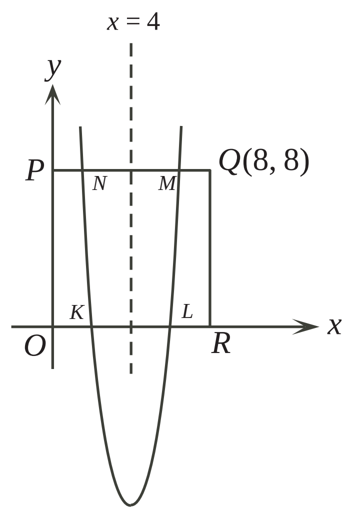
(b) First, we note that $a \neq 0$. (If $a = 0$, then the "parabola" $y = a(x-2)(x-6)$ is actually the horizontal line $y = 0$ which intersects the square all along OR.)
Second, we note that, regardless of the value of $a \neq 0$, the parabola has x-intercepts 2 and 6, and so intersects the x-axis at $(2,0)$ and $(6,0)$, which we call $K(2,0)$ and $L(6,0)$. This gives $KL = 4$.
Third, we note that since the x-intercepts of the parabola are 2 and 6, then the axis of symmetry of the parabola has equation $x = \frac{1}{2}(2+6) = 4$.
Since the axis of symmetry of the parabola is a vertical line of symmetry, then if the parabola intersects the two vertical sides of the square, it will intersect these at the same height, and if the parabola intersects the top side of the square, it will intersect it at two points that are symmetrical about the vertical line $x = 4$.
Fourth, we recall that a trapezoid with parallel sides of lengths a and b and height h has area $\frac{1}{2}h(a+b)$.
We now examine three cases.

Case 1: $a < 0$
Here, the parabola opens downwards.
Since the parabola intersects the square at four points, it must intersect PQ at points M and N. (The parabola cannot intersect the vertical sides of the square since it gets "narrower" towards the vertex.)



Since the parabola opens downwards, then $MN < KL = 4$.
Since the height of the trapezoid equals the height of the square (or 8), then the area of the trapezoid is $\frac{1}{2}h(KL + MN)$ which is less than $\frac{1}{2}(8)(4 + 4) = 32$.
But the area of the trapezoid must be 36, so this case is not possible.

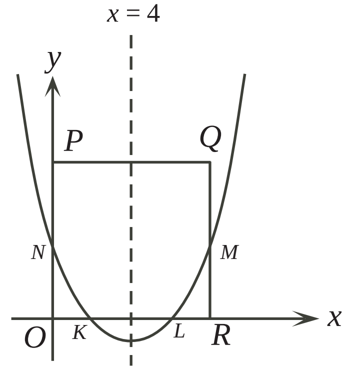
Case 2: $a > 0$; M and N on PQ
We have the following configuration:



Here, the height of the trapezoid is 8, $KL = 4$, and M and N are symmetric about $x = 4$.
Since the area of the trapezoid is 36, then $\frac{1}{2}h(KL + MN) = 36$ or $\frac{1}{2}(8)(4 + MN) = 36$ or $4 + MN = 9$ or $MN = 5$.
Thus, M and N are each $\frac{5}{2}$ units from $x = 4$, and so N has coordinates $(\frac{3}{2}, 8)$.
Since this point lies on the parabola with equation $y = a(x - 2)(x - 6)$, then $8 = a(\frac{3}{2} - 2)(\frac{3}{2} - 6)$ or $8 = a(-\frac{1}{2})(-\frac{9}{2})$ or $8 = \frac{9}{4}a$ or $a = \frac{32}{9}$.

Case 3: $a > 0$; M and N on QR and PO

We have the following configuration:



Here, $KL = 4$, $MN = 8$, and M and N have the same y-coordinate.
Since the area of the trapezoid is 36, then $\frac{1}{2}h(KL + MN) = 36$ or $\frac{1}{2}h(4 + 8) = 36$ or $6h = 36$ or $h = 6$.
Thus, N has coordinates $(0, 6)$.
Since this point lies on the parabola with equation $y = a(x - 2)(x - 6)$, then $6 = a(0 - 2)(0 - 6)$ or $6 = 12a$ or $a = \frac{1}{2}$.

Therefore, the possible values of a are $\frac{32}{9}$ and $\frac{1}{2}$.

7. (a) *Solution 1*

Consider a population of 100 people, each of whom is 75 years old and who behave according to the probabilities given in the question.
Each of the original 100 people has a 50% chance of living at least another 10 years, so there will be $50\% \times 100 = 50$ of these people alive at age 85.
Each of the original 100 people has a 20% chance of living at least another 15 years, so there will be $20\% \times 100 = 20$ of these people alive at age 90.
Since there is a 25% (or $\frac{1}{4}$) chance that an 80 year old person will live at least another 10 years (that is, to age 90), then there should be 4 times as many of these people alive at age 80 than at age 90.
Since there are 20 people alive at age 90, then there are $4 \times 20 = 80$ of the original 100 people alive at age 80.
In summary, of the initial 100 people of age 75, there are 80 alive at age 80, 50 alive at age 85, and 20 people alive at age 90.
Because 50 of the 80 people alive at age 80 are still alive at age 85, then the probability that an 80 year old person will live at least 5 more years (that is, to age 85) is $\frac{50}{80} = \frac{5}{8}$, or 62.5%.

Solution 2

Suppose that the probability that a 75 year old person lives to 80 is p, the probability that an 80 year old person lives to 85 is q, and the probability that an 85 year old person lives to 90 is r.
We want to the determine the value of q.
For a 75 year old person to live at least another 10 years, they must live another 5 years (to age 80) and then another 5 years (to age 85). The probability of this is equal to pq.
We are told in the question that this is equal to 50% or 0.5.
Therefore, $pq = 0.5$.

For a 75 year old person to live at least another 15 years, they must live another 5 years (to age 80), then another 5 years (to age 85), and then another 5 years (to age 90). The probability of this is equal to pqr. We are told in the question that this is equal to 20% or 0.2.
Therefore, $pqr = 0.2$
Similarly, since the probability that an 80 year old person will live another 10 years is 25%, then $qr = 0.25$.
Since $pqr = 0.2$ and $pq = 0.5$, then $r = \dfrac{pqr}{pq} = \dfrac{0.2}{0.5} = 0.4$.
Since $qr = 0.25$ and $r = 0.4$, then $q = \dfrac{qr}{r} = \dfrac{0.25}{0.4} = 0.625$.
Therefore, the probability that an 80 year old man will live at least another 5 years is 0.625, or 62.5%.

(b) Using logarithm rules, the given equation is equivalent to $2^{2\log_{10} x} = 3(2 \cdot 2^{\log_{10} x}) + 16$ or $(2^{\log_{10} x})^2 = 6 \cdot 2^{\log_{10} x} + 16$.
Set $u = 2^{\log_{10} x}$. Then the equation becomes $u^2 = 6u + 16$ or $u^2 - 6u - 16 = 0$.
Factoring, we obtain $(u - 8)(u + 2) = 0$ and so $u = 8$ or $u = -2$.
Since $2^a > 0$ for any real number a, then $u > 0$ and so we can reject the possibility that $u = -2$.
Thus, $u = 2^{\log_{10} x} = 8$ which means that $\log_{10} x = 3$.
Therefore, $x = 1000$.

8. (a) First, we determine the first entry in the 50th row.
Since the first column is an arithmetic sequence with common difference 3, then the 50th entry in the first column (the first entry in the 50th row) is $4 + 49(3) = 4 + 147 = 151$.
Second, we determine the common difference in the 50th row by determining the second entry in the 50th row.
Since the second column is an arithmetic sequence with common difference 5, then the 50th entry in the second column (that is, the second entry in the 50th row) is $7 + 49(5)$ or $7 + 245 = 252$.
Therefore, the common difference in the 50th row must be $252 - 151 = 101$.
Thus, the 40th entry in the 50th row (that is, the number in the 50th row and the 40th column) is $151 + 39(101) = 151 + 3939 = 4090$.

(b) We follow the same procedure as in (a).
First, we determine the first entry in the Rth row.
Since the first column is an arithmetic sequence with common difference 3, then the Rth entry in the first column (that is, the first entry in the Rth row) is $4 + (R - 1)(3)$ or $4 + 3R - 3 = 3R + 1$.
Second, we determine the common difference in the Rth row by determining the second entry in the Rth row.
Since the second column is an arithmetic sequence with common difference 5, then the Rth entry in the second column (that is, the second entry in the Rth row) is $7 + (R - 1)(5)$ or $7 + 5R - 5 = 5R + 2$.
Therefore, the common difference in the Rth row must be $(5R + 2) - (3R + 1) = 2R + 1$.
Thus, the Cth entry in the Rth row (that is, the number in the Rth row and the Cth column) is

$$3R + 1 + (C - 1)(2R + 1) = 3R + 1 + 2RC + C - 2R - 1 = 2RC + R + C$$

(c) Suppose that N is an entry in the table, say in the Rth row and Cth column.
From (b), then $N = 2RC + R + C$ and so $2N + 1 = 4RC + 2R + 2C + 1$.
Now $4RC + 2R + 2C + 1 = 2R(2C + 1) + 2C + 1 = (2R + 1)(2C + 1)$.
Since R and C are integers with $R \geq 1$ and $C \geq 1$, then $2R + 1$ and $2C + 1$ are each integers that are at least 3.
Therefore, $2N + 1 = (2R + 1)(2C + 1)$ must be composite, since it is the product of two integers that are each greater than 1.

9. (a) If $n = 2011$, then $8n - 7 = 16081$ and so $\sqrt{8n-7} \approx 126.81$.
Thus, $\dfrac{1+\sqrt{8n-7}}{2} \approx \dfrac{1+126.81}{2} \approx 63.9$.
Therefore, $g(2011) = 2(2011) + \left\lfloor \dfrac{1+\sqrt{8(2011)-7}}{2} \right\rfloor = 4022 + \lfloor 63.9 \rfloor = 4022 + 63 = 4085$.

(b) To determine a value of n for which $f(n) = 100$, we need to solve the equation

$$2n - \left\lfloor \frac{1+\sqrt{8n-7}}{2} \right\rfloor = 100 \qquad (*)$$

We first solve the equation

$$2x - \frac{1+\sqrt{8x-7}}{2} = 100 \qquad (**)$$

because the left sides of $(*)$ and $(**)$ do not differ by much and so the solutions are likely close together. We will try integers n in $(*)$ that are close to the solutions to $(**)$.
Manipulating $(**)$, we obtain

$$\begin{aligned} 4x - (1+\sqrt{8x-7}) &= 200 \\ 4x - 201 &= \sqrt{8x-7} \\ (4x-201)^2 &= 8x - 7 \\ 16x^2 - 1608x + 40401 &= 8x - 7 \\ 16x^2 - 1616x + 40408 &= 0 \\ 2x^2 - 202x + 5051 &= 0 \end{aligned}$$

By the quadratic formula,

$$x = \frac{202 \pm \sqrt{202^2 - 4(2)(5051)}}{2(2)} = \frac{202 \pm \sqrt{396}}{4} = \frac{101 \pm \sqrt{99}}{2}$$

and so $x \approx 55.47$ or $x \approx 45.53$.
We try $n = 55$, which is close to 55.47:

$$f(55) = 2(55) - \left\lfloor \frac{1+\sqrt{8(55)-7}}{2} \right\rfloor = 110 - \left\lfloor \frac{1+\sqrt{433}}{2} \right\rfloor$$

Since $\sqrt{433} \approx 20.8$, then $\dfrac{1+\sqrt{433}}{2} \approx 10.9$, which gives $\left\lfloor \dfrac{1+\sqrt{433}}{2} \right\rfloor = 10$.
Thus, $f(55) = 110 - 10 = 100$.
Therefore, a value of n for which $f(n) = 100$ is $n = 55$.

(c) We want to show that each positive integer m is in the range of f or the range of g, but not both.

To do this, we first try to better understand the "complicated" term of each of the functions – that is, the term involving the greatest integer function.

In particular, we start with a positive integer $k \geq 1$ and try to determine the positive integers n that give $\left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor = k$.

By definition of the greatest integer function, the equation $\left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor = k$ is equivalent to the inequality $k \leq \dfrac{1+\sqrt{8n-7}}{2} < k+1$, from which we obtain the following set of equivalent inequalities

$$\begin{array}{rcccc} 2k & \leq & 1+\sqrt{8n-7} & < & 2k+2 \\ 2k-1 & \leq & \sqrt{8n-7} & < & 2k+1 \\ 4k^2-4k+1 & \leq & 8n-7 & < & 4k^2+4k+1 \\ 4k^2-4k+8 & \leq & 8n & < & 4k^2+4k+8 \\ \frac{1}{2}(k^2-k)+1 & \leq & n & < & \frac{1}{2}(k^2+k)+1 \end{array}$$

If we define $T_k = \frac{1}{2}k(k+1) = \frac{1}{2}(k^2+k)$ to be the kth triangular number for $k \geq 0$, then $T_{k-1} = \frac{1}{2}(k-1)(k) = \frac{1}{2}(k^2-k)$.

Therefore, $\left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor = k$ for $T_{k-1}+1 \leq n < T_k+1$.

Since n is an integer, then $\left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor = k$ is true for $T_{k-1}+1 \leq n \leq T_k$.

When $k=1$, this interval is $T_0+1 \leq n \leq T_1$ (or $1 \leq n \leq 1$). When $k=2$, this interval is $T_1+1 \leq n \leq T_2$ (or $2 \leq n \leq 3$). When $k=3$, this interval is $T_2+1 \leq n \leq T_3$ (or $4 \leq n \leq 6$). As k ranges over all positive integers, these intervals include every positive integer n and do not overlap.

Therefore, we can determine the range of each of the functions f and g by examining the values $f(n)$ and $g(n)$ when n is in these intervals.

For each non-negative integer k, define $\mathcal{R}_k$ to be the set of integers greater than k^2 and less than or equal to $(k+1)^2$. Thus, $\mathcal{R}_k = \{k^2+1, k^2+2, \ldots, k^2+2k, k^2+2k+1\}$.

For example, $\mathcal{R}_0 = \{1\}$, $\mathcal{R}_1 = \{2,3,4\}$, $\mathcal{R}_2 = \{5,6,7,8,9\}$, and so on. Every positive integer occurs in exactly one of these sets.

Also, for each non-negative integer k define $\mathcal{S}_k = \{k^2+2, k^2+4, \ldots, k^2+2k\}$ and define $\mathcal{Q}_k = \{k^2+1, k^2+3, \ldots, k^2+2k+1\}$. For example, $\mathcal{S}_0 = \{\}$, $\mathcal{S}_1 = \{3\}$, $\mathcal{S}_2 = \{6,8\}$, $\mathcal{Q}_0 = \{1\}$, $\mathcal{Q}_1 = \{2,4\}$, $\mathcal{Q}_2 = \{5,7,9\}$, and so on. Note that $\mathcal{R}_k = \mathcal{Q}_k \cup \mathcal{S}_k$ so every positive integer occurs in exactly one $\mathcal{Q}_k$ or in exactly one $\mathcal{S}_k$, and that these sets do not overlap since no two $\mathcal{S}_k$'s overlap and no two $\mathcal{Q}_k$'s overlap and no $\mathcal{Q}_k$ overlaps with an $\mathcal{S}_k$.

We determine the range of the function g first.

For $T_{k-1}+1 \leq n \leq T_k$, we have $\left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor = k$ and so

$$\begin{array}{rcccc} 2T_{k-1}+2 & \leq & 2n & \leq & 2T_k \\ 2T_{k-1}+2+k & \leq & 2n + \left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor & \leq & 2T_k+k \\ k^2-k+2+k & \leq & g(n) & \leq & k^2+k+k \\ k^2+2 & \leq & g(n) & \leq & k^2+2k \end{array}$$

Note that when n is in this interval and increases by 1, then the $2n$ term causes the value of $g(n)$ to increase by 2.
Therefore, for the values of n in this interval, $g(n)$ takes precisely the values k^2+2, $k^2+4, k^2+6, \ldots, k^2+2k$.
In other words, the range of g over this interval of its domain is precisely the set $\mathcal{S}_k$.
As k ranges over all positive integers (that is, as these intervals cover the domain of g), this tells us that the range of g is precisely the integers in the sets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \ldots$.
(We could also include $\mathcal{S}_0$ in this list since it is the empty set.)

We note next that $f(1) = 2 - \left\lfloor \dfrac{1+\sqrt{8-7}}{2} \right\rfloor = 1$, the only element of $\mathcal{Q}_0$.

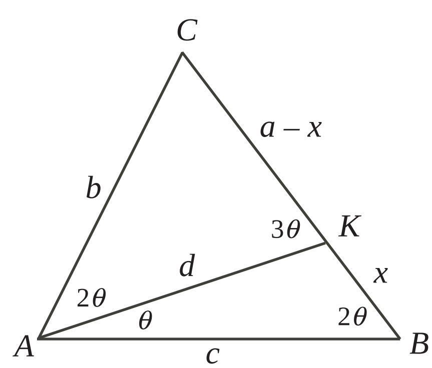
For $k \geq 1$ and $T_k + 1 \leq n \leq T_{k+1}$, we have $\left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor = k+1$ and so

$$\begin{array}{ccccc}
2T_k + 2 & \leq & 2n & \leq & 2T_{k+1} \\
2T_k + 2 - (k+1) & \leq & 2n - \left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor & \leq & 2T_{k+1} - (k+1) \\
k^2 + k + 2 - k - 1 & \leq & f(n) & \leq & (k+1)(k+2) - k - 1 \\
k^2 + 1 & \leq & f(n) & \leq & k^2 + 2k + 1
\end{array}$$

Note that when n is in this interval and increases by 1, then the $2n$ term causes the value of $f(n)$ to increase by 2.
Therefore, for the values of n in this interval, $f(n)$ takes precisely the values k^2+1, $k^2+3, k^2+5, \ldots, k^2+2k+1$.
In other words, the range of f over this interval of its domain is precisely the set $\mathcal{Q}_k$.
As k ranges over all positive integers (that is, as these intervals cover the domain of f), this tells us that the range of f is precisely the integers in the sets $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2, \ldots$.

Therefore, the range of f is the set of elements in the sets $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2, \ldots$ and the range of g is the set of elements in the sets $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \ldots$. These ranges include every positive integer and do not overlap.

10. (a) Suppose that $\angle KAB = \theta$.
Since $\angle KAC = 2\angle KAB$, then $\angle KAC = 2\theta$ and $\angle BAC = \angle KAC + \angle KAB = 3\theta$.
Since $3\angle ABC = 2\angle BAC$, then $\angle ABC = \frac{2}{3} \times 3\theta = 2\theta$.
Since $\angle AKC$ is exterior to $\triangle AKB$, then $\angle AKC = \angle KAB + \angle ABC = 3\theta$.
This gives the following configuration:



Now $\triangle CAK$ is similar to $\triangle CBA$ since the triangles have a common angle at C and $\angle CAK = \angle CBA$.

Therefore, $\dfrac{AK}{BA} = \dfrac{CA}{CB}$ or $\dfrac{d}{c} = \dfrac{b}{a}$ and so $d = \dfrac{bc}{a}$.

Also, $\dfrac{CK}{CA} = \dfrac{CA}{CB}$ or $\dfrac{a-x}{b} = \dfrac{b}{a}$ and so $a - x = \dfrac{b^2}{a}$ or $x = a - \dfrac{b^2}{a} = \dfrac{a^2-b^2}{a}$, as required.

(b) From (a), $bc = ad$ and $a^2 - b^2 = ax$ and so we obtain

$$\text{LS} = (a^2-b^2)(a^2-b^2+ac) = (ax)(ax+ac) = a^2x(x+c)$$

and

$$\text{RS} = b^2c^2 = (bc)^2 = (ad)^2 = a^2d^2$$

In order to show that LS = RS, we need to show that $x(x+c) = d^2$ (since $a > 0$).

Method 1: Use the Sine Law

First, we derive a formula for $\sin 3\theta$ which we will need in this solution:

$$\begin{aligned}
\sin 3\theta &= \sin(2\theta+\theta) \\
&= \sin 2\theta\cos\theta + \cos 2\theta \sin\theta \\
&= 2\sin\theta\cos^2\theta + (1-2\sin^2\theta)\sin\theta \\
&= 2\sin\theta(1-\sin^2\theta) + (1-2\sin^2\theta)\sin\theta \\
&= 3\sin\theta - 4\sin^3\theta
\end{aligned}$$

Since $\angle AKB = 180^\circ - \angle KAB - \angle KBA = 180^\circ - 3\theta$, then using the Sine Law in $\triangle AKB$ gives

$$\frac{x}{\sin\theta} = \frac{d}{\sin 2\theta} = \frac{c}{\sin(180^\circ - 3\theta)}$$

Since $\sin(180^\circ - X) = \sin X$, then $\sin(180^\circ - 3\theta) = \sin 3\theta$, and so $x = \dfrac{d\sin\theta}{\sin 2\theta}$ and $c = \dfrac{d\sin 3\theta}{\sin 2\theta}$. This gives

$$\begin{aligned}
x(x+c) &= \frac{d\sin\theta}{\sin 2\theta}\left(\frac{d\sin\theta}{\sin 2\theta} + \frac{d\sin 3\theta}{\sin 2\theta}\right) \\
&= \frac{d^2\sin\theta}{\sin^2 2\theta}(\sin\theta + \sin 3\theta) \\
&= \frac{d^2\sin\theta}{\sin^2 2\theta}(\sin\theta + 3\sin\theta - 4\sin^3\theta) \\
&= \frac{d^2\sin\theta}{\sin^2 2\theta}(4\sin\theta - 4\sin^3\theta) \\
&= \frac{4d^2\sin^2\theta}{\sin^2 2\theta}(1-\sin^2\theta) \\
&= \frac{4d^2\sin^2\theta\cos^2\theta}{\sin^2 2\theta} \\
&= \frac{4d^2\sin^2\theta\cos^2\theta}{(2\sin\theta\cos\theta)^2} \\
&= \frac{4d^2\sin^2\theta\cos^2\theta}{4\sin^2\theta\cos^2\theta} \\
&= d^2
\end{aligned}$$

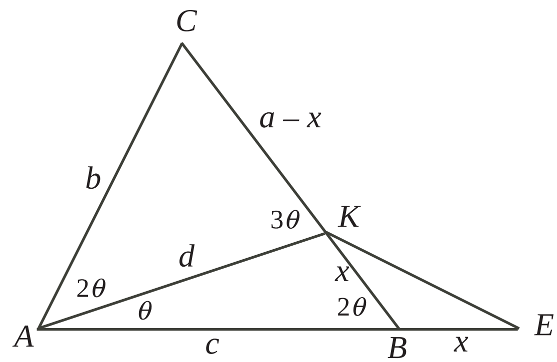
as required.

We could have instead used the formula $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ to show that $\sin 3\theta + \sin\theta = 2\sin 2\theta\cos\theta$, from which

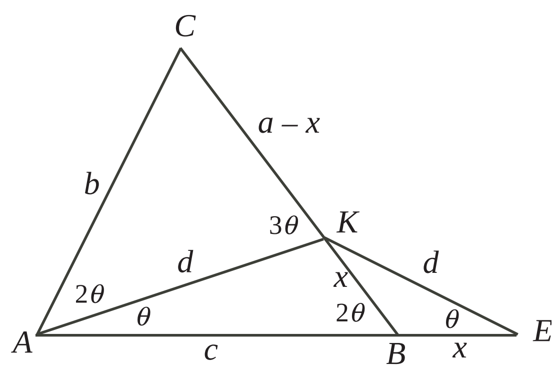
$$\sin\theta(\sin 3\theta + \sin\theta) = \sin\theta(2\sin 2\theta\cos\theta) = 2\sin\theta\cos\theta\sin 2\theta = \sin^2 2\theta$$

<u>Method 2: Extend AB</u>

Extend AB to E so that $BE = BK = x$ and join KE.



Now $\triangle KBE$ is isosceles with $\angle BKE = \angle KEB$.
Since $\angle KBA$ is the exterior angle of $\triangle KBE$, then $\angle KBA = 2\angle KEB = 2\theta$.
Thus, $\angle KEB = \angle BKE = \theta$.
But this also tells us that $\angle KAE = \angle KEA = \theta$.
Thus, $\triangle KAE$ is isosceles and so $KE = KA = d$.



So $\triangle KAE$ is similar to $\triangle BKE$, since each has two angles equal to θ.
Thus, $\frac{KA}{BK} = \frac{AE}{KE}$ or $\frac{d}{x} = \frac{c+x}{d}$ and so $d^2 = x(x+c)$, as required.

<u>Method 3: Use the Cosine Law and the Sine Law</u>

We apply the Cosine Law in $\triangle AKB$ to obtain

$$\begin{aligned} AK^2 &= BK^2 + BA^2 - 2(BA)(BK)\cos(\angle KBA) \\ d^2 &= x^2 + c^2 - 2cx\cos(2\theta) \\ d^2 &= x^2 + c^2 - 2cx(2\cos^2\theta - 1) \end{aligned}$$

Using the Sine Law in $\triangle AKB$, we get $\frac{x}{\sin\theta} = \frac{d}{\sin 2\theta}$ or $\frac{\sin 2\theta}{\sin\theta} = \frac{d}{x}$ or $\frac{2\sin\theta\cos\theta}{\sin\theta} = \frac{d}{x}$ and so $\cos\theta = \frac{d}{2x}$.

Combining these two equations,

$$\begin{aligned}
d^2 &= x^2 + c^2 - 2cx\left(\frac{2d^2}{4x^2} - 1\right) \\
d^2 &= x^2 + c^2 - \frac{cd^2}{x} + 2cx \\
d^2 + \frac{cd^2}{x} &= x^2 + 2cx + c^2 \\
d^2 + \frac{cd^2}{x} &= (x+c)^2 \\
xd^2 + cd^2 &= x(x+c)^2 \\
d^2(x+c) &= x(x+c)^2 \\
d^2 &= x(x+c)
\end{aligned}$$

as required (since $x + c \neq 0$).

(c) *Solution 1*

Our goal is to find a triple of positive integers that satisfy the equation in (b) and are the side lengths of a triangle.

First, we note that if (A, B, C) is a triple of real numbers that satisfies the equation in (b) and k is another real number, then the triple (kA, kB, kC) also satisfies the equation from (b), since

$$(k^2A^2 - k^2B^2)(k^2A^2 - k^2B^2 + kAkC) = k^4(A^2 - B^2)(A^2 - B^2 + AC) = k^4(B^2C^2) = (kB)^2(kC)^2$$

Therefore, we start by trying to find a triple (a, b, c) of rational numbers that satisfies the equation in (b) and forms a triangle, and then "scale up" this triple to form a triple (ka, kb, kc) of integers.

To do this, we rewrite the equation from (b) as a quadratic equation in c and solve for c using the quadratic formula.

Partially expanding the left side from (b), we obtain

$$(a^2 - b^2)(a^2 - b^2) + ac(a^2 - b^2) = b^2c^2$$

which we rearrange to obtain

$$b^2c^2 - c(a(a^2 - b^2)) - (a^2 - b^2)^2 = 0$$

By the quadratic formula,

$$c = \frac{a(a^2 - b^2) \pm \sqrt{a^2(a^2 - b^2)^2 + 4b^2(a^2 - b^2)^2}}{2b^2} = \frac{a(a^2 - b^2) \pm \sqrt{(a^2 - b^2)^2(a^2 + 4b^2)}}{2b^2}$$

Since $\angle BAC > \angle ABC$, then $a > b$ and so $a^2 - b^2 > 0$, which gives

$$c = \frac{a(a^2 - b^2) \pm (a^2 - b^2)\sqrt{a^2 + 4b^2}}{2b^2} = \frac{(a^2 - b^2)}{2b^2}(a \pm \sqrt{a^2 + 4b^2})$$

Since $a^2 + 4b^2 > 0$, then $\sqrt{a^2 + 4b^2} > a$, so the positive root is

$$c = \frac{(a^2 - b^2)}{2b^2}(a + \sqrt{a^2 + (2b)^2})$$

We try to find integers a and b that give a rational value for c. We will then check to see if this triple (a, b, c) forms the side lengths of a triangle, and then eventually scale these up to get integer values.
One way for the value of c to be rational (and in fact the only way) is for $\sqrt{a^2 + (2b)^2}$ to be an integer, or for a and $2b$ to be the legs of a Pythagorean triple.
Since $\sqrt{3^2 + 4^2}$ is an integer, then we try $a = 3$ and $b = 2$, which gives

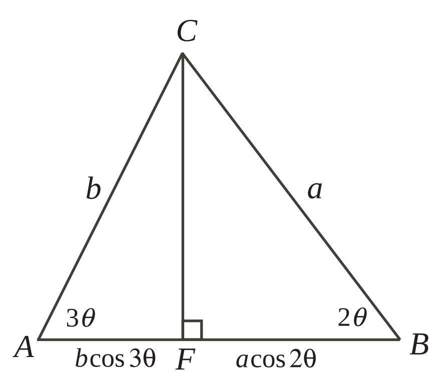
$$c = \frac{(3^2 - 2^2)}{2 \cdot 2^2}(3 + \sqrt{3^2 + 4^2}) = 5$$

and so $(a, b, c) = (3, 2, 5)$. Unfortunately, these lengths do not form a triangle, since $3 + 2 = 5$.
(The Triangle Inequality tells us that three positive real numbers a, b and c form a triangle if and only if $a + b > c$ and $a + c > b$ and $b + c > a$.)
We can continue to try small Pythagorean triples.
Now $15^2 + 8^2 = 17^2$, but $a = 15$ and $b = 4$ do not give a value of c that forms a triangle with a and b.
However, $16^2 + 30^2 = 34^2$, so we can try $a = 16$ and $b = 15$ which gives

$$c = \frac{(16^2 - 15^2)}{2 \cdot 15^2}(16 + \sqrt{16^2 + 30^2}) = \frac{31}{450}(16 + 34) = \frac{31}{9}$$

Now the lengths $(a, b, c) = (16, 15, \frac{31}{9})$ do form the sides of a triangle since $a + b > c$ and $a + c > b$ and $b + c > a$.
Since these values satisfy the equation from (b), then we can scale them up by a factor of $k = 9$ to obtain the triple $(144, 135, 31)$ which satisfies the equation from (b) and are the side lengths of a triangle.
(Using other Pythagorean triples, we could obtain other triples of integers that work.)

Solution 2
We note that the equation in (b) involves only a, b and c and so appears to depend only on the relationship between the angles $\angle CAB$ and $\angle CBA$ in $\triangle ABC$.
Using this premise, we use $\triangle ABC$, remove the line segment AK and draw the altitude CF.



Because we are only looking for one triple that works, we can make a number of assumptions that may or may not be true in general for such a triangle, but which will help us find an example.
We assume that 3θ and 2θ are both acute angles; that is, we assume that $\theta < 30^\circ$.
In $\triangle ABC$, we have $AF = b\cos 3\theta$, $BF = a\cos 2\theta$, and $CF = b\sin 3\theta = a\sin 2\theta$.
Note also that $c = b\cos 3\theta + a\cos 2\theta$.

One way to find the integers a, b, c that we require is to look for integers a and b and an angle θ with the properties that $b\cos 3\theta$ and $a\cos 2\theta$ are integers and $b\sin 3\theta = a\sin 2\theta$.
Using trigonometric formulae,

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos 2\theta &= 2\cos^2\theta - 1 \\
\sin 3\theta &= 3\sin\theta - 4\sin^3\theta \\
&\qquad \text{(from the calculation in (a), Solution 1, Method 1)} \\
\cos 3\theta &= \cos(2\theta+\theta) \\
&= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2(1-\cos^2\theta)\cos\theta \\
&= 4\cos^3\theta - 3\cos\theta
\end{aligned}$$

So we can try to find an angle $\theta < 30^\circ$ with $\cos\theta$ a rational number and then integers a and b that make $b\sin 3\theta = a\sin 2\theta$ and ensure that $b\cos 3\theta$ and $a\cos 2\theta$ are integers.
Since we are assuming that $\theta < 30^\circ$, then $\cos\theta > \frac{\sqrt{3}}{2} \approx 0.866$.
The rational number with smallest denominator that is larger than $\frac{\sqrt{3}}{2}$ is $\frac{7}{8}$, so we try the acute angle θ with $\cos\theta = \frac{7}{8}$.
In this case, $\sin\theta = \sqrt{1-\cos^2\theta} = \frac{\sqrt{15}}{8}$, and so

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta = 2\times\tfrac{7}{8}\times\tfrac{\sqrt{15}}{8} = \tfrac{7\sqrt{15}}{32} \\
\cos 2\theta &= 2\cos^2\theta - 1 = 2\times\tfrac{49}{64} - 1 = \tfrac{17}{32} \\
\sin 3\theta &= 3\sin\theta - 4\sin^3\theta = 3\times\tfrac{\sqrt{15}}{8} - 4\times\tfrac{15\sqrt{15}}{512} = \tfrac{33\sqrt{15}}{128} \\
\cos 3\theta &= 4\cos^3\theta - 3\cos\theta = 4\times\tfrac{343}{512} - 3\times\tfrac{7}{8} = \tfrac{7}{128}
\end{aligned}$$

To have $b\sin 3\theta = a\sin 2\theta$, we need $\frac{33\sqrt{15}}{128}b = \frac{7\sqrt{15}}{32}a$ or $33b = 28a$.
To ensure that $b\cos 3\theta$ and $a\cos 2\theta$ are integers, we need $\frac{7}{128}b$ and $\frac{17}{32}a$ to be integers, and so a must be divisible by 32 and b must be divisible by 128.
The integers $a = 33$ and $b = 28$ satisfy the equation $33b = 28a$.
Multiplying each by 32 gives $a = 1056$ and $b = 896$ which satisfy the equation $33b = 28a$ and now have the property that b is divisible by 128 (with quotient 7) and a is divisible by 32 (with quotient 33).
With these values of a and b, we obtain $c = b\cos 3\theta + a\cos 2\theta = 896\times\frac{7}{128} + 1056\times\frac{17}{32} = 610$.
We can then check that the triple $(a, b, c) = (1056, 896, 610)$ satisfies the equation from (b), as required.
As in our discussion in Solution 1, each element of this triple can be divided by 2 to obtain the "smaller" triple $(a, b, c) = (528, 448, 305)$ that satisfies the equation too.
Using other values for $\cos\theta$ and integers a and b, we could obtain other triples (a, b, c) of integers that work.