The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Canadian Senior Mathematics Contest

Thursday, November 21, 2013
(in North America and South America)

Friday, November 22, 2013
(outside of North America and South America)

Time: 2 hours

Calculators are permitted, provided they are non-programmable and without graphic displays.

Do not open this booklet until instructed to do so.
There are two parts to this paper.

PART A

1. This part consists of six questions, each worth 5 marks.
2. **Enter the answer in the appropriate box in the answer booklet.**
   For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

PART B

1. This part consists of three questions, each worth 10 marks.
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, they will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name, school name and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

NOTES:

The questions in each part are arranged roughly in order of increasing difficulty.
The early problems in Part B are likely easier than the later problems in Part A.
At the completion of the contest, insert your student information form inside your answer booklet.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on the Web site, http://www.cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some students may be shared with other mathematical organizations for other recognition opportunities.

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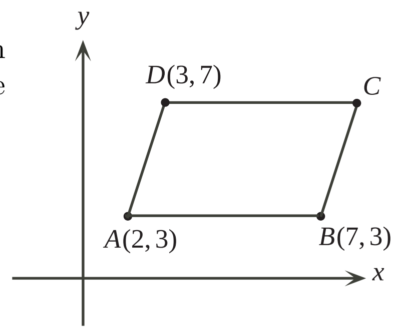
NOTE:	1.	Please read the instructions on the front cover of this booklet.
	2.	Write solutions in the answer booklet provided.
	3.	It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc., rather than as $12.566\ldots$ or $4.646\ldots$.
	4.	**Calculators are permitted**, provided they are non-programmable and without graphic displays.
	5.	Diagrams are not drawn to scale. They are intended as aids only.

PART A

For each question in Part A, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded only if relevant work is shown in the space provided in the answer booklet.

1. In the diagram, $ABCD$ is a parallelogram with A, B, C, D in the first quadrant, as shown. What are the coordinates of C?

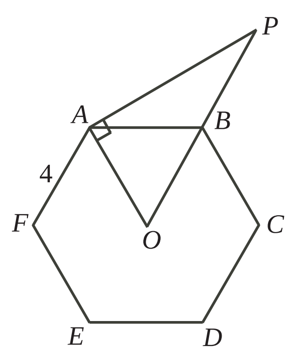


2. Mr. Matheson has four cards, numbered 1, 2, 3, 4. He gives one card each to Ben, Wendy, Riley, and Sara. Ben is not given the number 1. Wendy's number is 1 greater than Riley's number. Which number could Sara *not* have been given?

3. If $\dfrac{99!}{101! - 99!} = \dfrac{1}{n}$, determine the value of n.

   (If m is a positive integer, then $m!$ represents the product of the integers from 1 to m, inclusive. For example, $5! = 5(4)(3)(2)(1) = 120$ and $99! = 99(98)(97)\cdots(3)(2)(1)$.)

4. In the diagram, $ABCDEF$ is a regular hexagon with side length 4 and centre O. The line segment perpendicular to OA and passing through A meets OB extended at P. What is the area of $\triangle OAP$?



5. Each of the positive integers 2013 and 3210 has the following three properties:

   (i) it is an integer between 1000 and 10 000,

   (ii) its four digits are consecutive integers, and

   (iii) it is divisible by 3.

   In total, how many positive integers have these three properties?

6. If p and q are positive integers, $\max(p, q)$ is the maximum of p and q and $\min(p, q)$ is the minimum of p and q. For example, $\max(30, 40) = 40$ and $\min(30, 40) = 30$. Also, $\max(30, 30) = 30$ and $\min(30, 30) = 30$.

   Determine the number of ordered pairs (x, y) that satisfy the equation

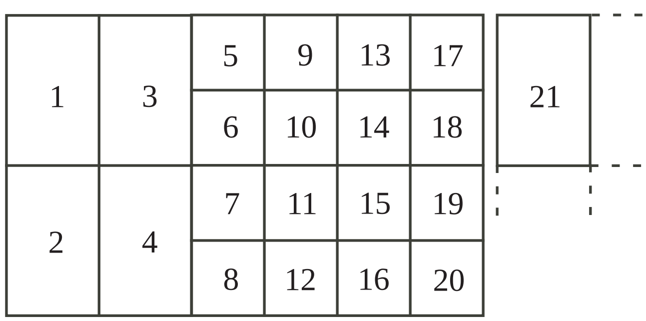
   $$\max(60, \min(x, y)) = \min(\max(60, x), y)$$

   where x and y are positive integers with $x \leq 100$ and $y \leq 100$.

PART B

For each question in Part B, your solution must be well organized and contain words of explanation or justification when appropriate. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

1. At Galbraith H.S., the lockers are arranged in banks of 20 lockers. Each bank of lockers consists of six columns of lockers; the first two columns in each bank consist of two larger lockers and the last four columns in each bank consist of four smaller lockers. The lockers are numbered consecutively starting at 1, moving down each column and then down the next column, and so on. The first twenty-one lockers and their locker numbers are shown in the diagram.



   (a) What is the sum of the locker numbers of the column of lockers that contain the number 24?

   (b) The sum of the locker numbers for one column is 123. What are the locker numbers in this column?

   (c) The sum of the locker numbers for another column is 538. What are the locker numbers in this column?

   (d) Explain why 2013 cannot be a sum of any column of locker numbers.

2. (a) Expand and simplify fully the expression $(a - 1)(6a^2 - a - 1)$.

   (b) Determine all values of θ with $6\cos^3\theta - 7\cos^2\theta + 1 = 0$ and $-180^\circ < \theta < 180^\circ$. Round each answer to 1 decimal place where appropriate.
   (Note that $\cos^3\theta = (\cos\theta)^3$.)

   (c) Determine all values of θ with $6\cos^3\theta - 7\cos^2\theta + 1 < 0$ and $-180^\circ < \theta < 180^\circ$.

3. If m and n are positive integers, an (m, n)-sequence is defined to be an infinite sequence $x_1, x_2, x_3, \ldots$ of A's and B's such that if $x_i = A$ for some positive integer i, then $x_{i+m} = B$ and if $x_i = B$ for some positive integer i, then $x_{i+n} = A$. For example, $ABABAB\ldots$ is a $(1, 1)$-sequence.

    (a) Determine all $(2, 2)$-sequences.

    (b) Show that there are no $(1, 2)$-sequences.

    (c) For every positive integer r, show that if there exists an (m, n)-sequence, then there exists an (rm, rn)-sequence.

    (d) Determine all pairs (m, n) for which there is an (m, n)-sequence.