The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2013 Hypatia Contest

Thursday, April 18, 2013
(in North America and South America)

Friday, April 19, 2013
(outside of North America and South America)

Solutions

1. (a) Pulling out two of each type of bill gives Rad, $2\times(\$5+\$10+\$20+\$50) = 2\times(\$85) = \$170$. Since his total sum of money is \$175, the only other bill that Rad pulled out must have a value of $\$175 - \$170 = \$5$.
That is, Rad pulled out three \$5 bills, two \$10 bills, two \$20 bills, and two \$50 bills, for a total of $3 + 2 + 2 + 2 = 9$ bills.

(b) Sandy pulls out at least one of each type of bill and so she must have at least $\$5 + \$10 + \$20 + \$50 = \$85$.
Thus, we know four of the five bills that Sandy pulls out and that these four bills total \$85. The fifth bill that Sandy pulls could be any one of the four different types of bills.
If this fifth bill is a \$5 bill, then Sandy's total sum of money is $\$85 + \$5 = \$90$.
If this fifth bill is a \$10 bill, then Sandy's total sum of money is $\$85 + \$10 = \$95$.
If this fifth bill is a \$20 bill, then Sandy's total sum of money is $\$85 + \$20 = \$105$.
Finally, if this fifth bill is a \$50 bill, then Sandy's total sum of money is $\$85+\$50 = \$135$.
Therefore, the sums of money that Sandy could have are \$90, \$95, \$105, and \$135.

(c) Lino could have at most three \$50 bills since four \$50 bills exceeds his total sum of money ($4 \times \$50 = \$200 > \$160$).
If Lino had no \$50 bills, then the bills (6 bills at most) each have value at most \$20 and would total \$160.
However, this is not possible since $6 \times \$20 = \$120$ which is less than the required \$160.
If Lino had one \$50 bill, then the remaining bills (5 bills at most) would total $\$160 - \$50 = \$110$.
However, this is not possible since the largest denomination of the remaining bills is \$20 and $5 \times \$20 = \$100$ which is less than the required \$110.
Therefore, we proceed by considering the cases where Lino has two or three \$50 bills.
These two cases are summarized in the table below.

Number of \$50s	Money in \$50s	Money remaining	Number of \$20s	Number of \$10s	Number of \$5s	Number of bills used
3	\$150	\$10		1		4
3	\$150	\$10			2	5
2	\$100	\$60	3			5
2	\$100	\$60	2	2		6

In each case from the table above, Lino has a total sum of \$160 and has pulled out 6 or fewer bills.
Since we are given that there are only four possibilities, then we have found them all.

2. (a) The equation of a parabola, written in the form $y = (x - p)^2 + q$, has its vertex at (p, q).
Thus, the parabola with equation $y = (x - 3)^2 + 1$, has its vertex at $(3, 1)$.

(b) *Solution 1*
Under a translation, the shape of a parabola remains unchanged.
That is, the new parabola is congruent to the original parabola.
After a translation of 3 units to the left and 3 units up, the original vertex $(3, 1)$ moves to the point $(3 - 3, 1 + 3)$ or $(0, 4)$.
Since this new parabola is congruent to the original, but it has its vertex at $(0, 4)$, then its equation is $y = (x - 0)^2 + 4$ or $y = x^2 + 4$.

Solution 2

Under a translation of 3 units left and 3 units up, the equation $y = (x-3)^2 + 1$ becomes

$$\begin{aligned} y - 3 &= ((x+3)-3)^2 + 1 \\ y - 3 &= x^2 + 1 \\ y &= x^2 + 4 \end{aligned}$$

(c) At the point of intersection of these two parabolas, their y values must be equal. Thus,

$$\begin{aligned} (x-3)^2 + 1 &= x^2 + 4 \\ x^2 - 6x + 9 + 1 &= x^2 + 4 \\ -6x &= -6 \\ x &= 1 \end{aligned}$$

Substituting $x = 1$ into the equation $y = x^2 + 4$, we determine the y value of the point of intersection to be $y = 5$.
Therefore, the two parabolas intersect at the point $(1, 5)$.

(d) At the point of intersection of these two parabolas, their y values must be equal. Thus,

$$\begin{aligned} (x-3)^2 + 1 &= ax^2 + 4 \\ x^2 - 6x + 9 + 1 &= ax^2 + 4 \\ 0 &= ax^2 - x^2 + 6x - 6 \\ 0 &= (a-1)x^2 + 6x - 6 \end{aligned}$$

Since the two parabolas intersect at exactly one point, then the resulting equation $(a-1)x^2 + 6x - 6 = 0$ (which is quadratic since $a < 0$), has exactly one solution.
Thus, the *discriminant* of this equation must equal zero.
(Note: The discriminant of a quadratic equation of the form $ax^2 + bx + c = 0$, is $b^2 - 4ac$.)
Solving $6^2 - 4(-6)(a-1) = 0$, we get $36 + 24(a-1) = 0$ or $24a = -12$, and so $a = -\frac{1}{2}$.
That is, the parabolas with equations $y = ax^2 + 4$ and $y = (x-3)^2 + 1$ touch at exactly one point when $a = -\frac{1}{2}$.

3. (a) If the sequence begins with 3 or more P's, then it cannot be non-predictive since there are only 2 Q's.
Thus, there are two possible cases to consider; the sequence begins with exactly 1 P (that is, the sequence begins PQ since the second letter is not P), or the sequence begins with exactly 2 P's (that is, the sequence begins PPQ since the third letter is not P).

Case 1: The sequence begins PQ

Any sequence that begins PQ is already non-predictive, and so the remaining 7 letters can be in any order.
Therefore the number of non-predictive sequences beginning PQ, with $m = 7$ and $n = 2$, is equal to the number of ways of arranging 6 P's and 1 Q in the remaining 7 positions.
There are 7 distinct sequences having 6 P's and 1 Q since the Q can be placed in any one of the 7 remaining positions (while the P's are placed in all other positions without choice).
Thus the number of non-predictive sequences in this case is 7.

Case 2: The sequence begins PPQ

If the fourth letter in this sequence was P, then it would not be possible for the sequence to be non-predictive since there are only 2 Q's.
Therefore, the sequence must begin PPQQ.
Since the remaining 5 letters are all P's, then there is no choice in forming the rest of the sequence, so there is only one such sequence.

If $m = 7$ and $n = 2$, then the total number of non-predictive sequences beginning with P is $7 + 1 = 8$.

(b) Using a similar argument to that in part (a), if $m > 2$ and $n = 2$, then the non-predictive sequences beginning with P must either begin PQ or PPQQ.
In the case in which the sequence begins PQ, there are $(m-1)$ P's and 1 Q left to arrange in the remaining m positions.
Since the Q can be placed in any one of the m remaining positions, and each placement produces a distinct sequence, there are m non-predictive sequences that begin PQ.
In the case where the sequence begins PPQQ, there are only P's left to arrange.
There is only 1 way to do this.
Thus, there is only 1 non-predictive sequence that begins PPQQ.
Therefore the number of non-predictive sequences with $m > 2$ and $n = 2$, beginning with a P, is $(m + 1)$.

Any sequence that begins with a Q is a non-predictive sequence.
Therefore, the total number of non-predictive sequences that begin with Q is equal to the number of ways of arranging the remaining m P's and 1 Q in the remaining $(m + 1)$ positions.
Since the Q can be placed in any one of the $(m+1)$ positions, and each placement produces a distinct sequence, there are $(m + 1)$ non-predictive sequences that begin with a Q.

Therefore when $n = 2$, for every $m > 2$ the number of non-predictive sequences that begin with P is equal to the number of non-predictive sequences that begin with Q.

(c) If the sequence begins with 4 or more P's, then it cannot be non-predictive since there are only 3 Q's. (The number of Q's can never "catch up" to the number of P's.)
Consider the cases in which the sequence begins with $0, 1, 2,$ or 3 P's.

Case 1: The sequence begins with a Q (it begins with 0 P's)
Any sequence that begins with a Q is a non-predictive sequence.
Therefore, the total number of non-predictive sequences that begin with Q is equal to the number of ways of arranging the remaining 10 P's and 2 Q's in 12 positions.
We count these arrangements by considering the number of distinct ways to place the 2 Q's in the 12 positions and filling in the remaining positions with the 10 P's.
There are 12 possible positions to place the first Q, followed by 11 possible positions to place the second, or 12×11 arrangements.
However, since the 2 Q's cannot be distinguished from one another (i.e. they are identical), this counts each of the possible arrangements twice.
(To see this, consider for example that placing the first Q in the 4th position and the second Q in the 7th position is the same sequence as placing the first Q in the 7th position and the second Q in the 4th position.)
Therefore in this case, the number of ways to place the two Q's is $\frac{12\times 11}{2} = 66$ and so the number of non-predictive sequences is 66.

Case 2: The sequence begins PQ
Any sequence that begins with PQ is a non-predictive sequence.
Therefore, the total number of non-predictive sequences that begin with PQ is equal to the number of ways of arranging the remaining 9 P's and 2 Q's in 11 positions.
We count these arrangements by considering the number of distinct ways to place the 2 Q's in the 11 possible positions and filling in the remaining positions with the 9 P's.
Using the same argument as was used in Case 1, this can done in $\frac{11\times 10}{2} = 55$ possible ways.

Case 3: The sequence begins PPQ
There are two possibilities for the 4th letter in the sequence beginning PPQ.
The sequence beginning PPQQ is non-predictive, however, so is the sequence that begins PPQPQQ.
(Can you verify for yourself that these are the only two ways to begin a non-predictive sequence with $m = 10$, $n = 2$ starting with PPQ?)
The total number of non-predictive sequences that begin PPQQ is equal to the number of ways of arranging the remaining 8 P's and 1 Q in 9 positions.
Since the Q can be placed in any one of the 9 positions, and each placement produces a distinct sequence, there are 9 non-predictive sequences that begin PPQQ.
There is only 1 non-predictive sequence that begins PPQPQQ since the remaining 7 letters are all P's.
Therefore in this case, the number of non-predictive sequences is $9 + 1 = 10$.

Case 4: The sequence begins PPPQ
Since there are only 2 more Q's available, this sequence must begin PPPQQQ so that it is non-predictive.
There is only 1 non-predictive sequence that begins PPPQQQ since the remaining 7 letters are all P's.
Thus in this final case, there is only 1 non-predictive sequence.

The number of non-predictive sequences with $m = 10$ and $n = 2$ is $66 + 55 + 10 + 1 = 132$.

4. (a) Since the edge length of each cube is 1 cm, then the length, width and height of the rectangular prism are 5 cm, 4 cm and 1 cm, respectively.
The top face of the prism has dimensions 5 cm by 4 cm, and so has area 20 cm^2.
Similarly, the bottom face on the opposite side of the prism has the same area, 20 cm^2.
The front and back faces of the prism each have dimensions 4 cm by 1 cm, and so have area 4 cm^2.
The right and left faces of the prism each have dimensions 5 cm by 1 cm, and so have area 5 cm^2.
Therefore, the surface area of the rectangular prism is $2 \times (20+4+5) = 2 \times 29 = 58$ cm^2.

(b) Suppose that the rectangular prism in question is l cm (cubes) in length and w cm (cubes) wide, with $l \geq w$.
Then the top surface of the rectangular prism has area $(l \times w)$ cm^2, the front surface has area $(w \times 1)$ cm^2, and the right side has area $(l \times 1)$ cm^2.
Therefore, the surface area of the entire rectangular prism is $2 \times (lw + w + l)$ cm^2.
Since the surface area is 180 cm^2, then $2 \times (lw + w + l) = 180$ or $lw + w + l = 90$.
Adding 1 to both sides of this equation, we get $lw+w+l+1 = 91$, so $w(l+1)+1(l+1) = 91$.
Factoring the left side of this equation one step further, we have $(w + 1)(l + 1) = 91$.
Since both l and w are positive integers, then $(w+1)(l+1)$ is the product of two positive integers.

The right side, 91, can be written as the product of two positive integers in exactly two different ways, 1×91 and 7×13.
Since both l and w are positive integers, then $2 \leq (w+1) \leq (l+1)$ and so neither factor can be equal to 1.
Therefore $(w+1) = 7$ and $(l+1) = 13$ (since $l \geq w$), and so $w = 6$ and $l = 12$.
Since the rectangular prism is 6 cubes wide and 12 cubes in length, then it has $6 \times 12 = 72$ cubes in total.

(c) As shown in part (b), the surface area of the original prism without the removal of the internal prism, is $2 \times (lw + w + l)$ cm^2.
To find the surface area of the frame, we must account for the area that is lost and that which is gained by removing the internal prism.

The area of the original prism that is lost is equal to the front and back rectangular faces of the internal prism.
Since the width of the original prism is w cm and the internal prism is located k cm from each side of the original prism, then the width of the internal prism is $(w - 2k)$ cm.
Similarly, the length of the internal prism is $(l - 2k)$ cm.
(Note that $w - 2k > 0$ and $l - 2k > 0$, so $w > 2k$ and $l > 2k$.)
Thus, the area from the original prism that is lost by removing the internal prism is $2 \times (w - 2k) \times (l - 2k)$ cm^2.

The area that is gained by removing the internal prism is equal to the top, bottom, left, and right rectangular surfaces of the internal prism.
Since the width of the internal prism is $(w - 2k)$ cm and its thickness is 1 cm, then the total area of the top and bottom faces is $2 \times (w - 2k) \times 1$ cm^2.
Similarly, since the length of the internal prism is $(l - 2k)$ cm and its thickness is 1 cm, then the total area of the right and left faces is $2 \times (l - 2k) \times 1$ cm^2.

Summarizing, the surface area of the original prism without the removal of the internal prism, is $2 \times (lw + w + l)$ cm^2.
The area of the original prism that is lost by removing the internal prism is, $2 \times (w - 2k) \times (l - 2k)$ cm^2.
The area that is added to the area of the original prism by removing the internal prism is, $(2 \times (w - 2k) \times 1 + 2 \times (l - 2k) \times 1)$ cm^2 or $2 \times (w - 2k + l - 2k) = 2 \times (w + l - 4k)$ cm^2.
That is, the surface area of the frame, in cm^2, is

$$2 \times (lw + w + l) - 2 \times (w - 2k) \times (l - 2k) + 2 \times (w + l - 4k).$$

Since the surface area of the frame is 532 cm^2, then we equate this with the expression for the surface area and simplify the resulting equation.

$$\begin{aligned}
2 \times (lw + w + l) - 2 \times (w - 2k) \times (l - 2k) + 2 \times (w + l - 4k) &= 532 \\
lw + w + l - (w - 2k) \times (l - 2k) + (w + l - 4k) &= \tfrac{532}{2} \\
lw + w + l - (lw - 2kw - 2kl + 4k^2) + (w + l - 4k) &= 266 \\
2w + 2l + 2kw + 2kl - 4k^2 - 4k &= 266 \\
w + l + kw + kl - 2k^2 - 2k &= \tfrac{266}{2} \\
kw + kl - 2k^2 + w + l - 2k &= 133 \\
k(w + l - 2k) + 1(w + l - 2k) &= 133 \\
(w + l - 2k)(k + 1) &= 133
\end{aligned}$$

Recall that $w > 2k$ and $l > 2k$, so then $(w + l - 2k)$ is a positive integer since w, l and k are positive integers.
Therefore, $(w + l - 2k)(k + 1)$ is the product of two positive integers.
Expressed as the product of two positive integers, 133 can only be written as 1×133 or 7×19.
Note that $(k + 1) > 1$, and $(w + l - 2k) > 2k + 2k - 2k = 2k > 1$, since k is a positive integer.
That is, $(k + 1) \neq 1$ and $(w + l - 2k) \neq 1$, and so either $k + 1 = 7$ or $k + 1 = 19$.
If $k + 1 = 7$, then $k = 6$ and $w + l - 2k = 19$ or $w + l - 12 = 19$, and so $w + l = 31$.
Since $w > 2k = 12$, we require all possible values for w and l such that $w \geq 13$, $l \geq w$, and $w + l = 31$.
Written as ordered pairs (w, l) the only possibilities are $(13, 18), (14, 17)$ and $(15, 16)$.

If $k + 1 = 19$, then $k = 18$ and $w + l - 2k = 7$ or $w + l - 36 = 19$, and so $w + l = 55$.
Since $w > 2k = 36$, then $l < 55 - 36 = 29$.
This is not possible since $l \geq w$.
Therefore, the only possible values for w and l such that the frame has surface area 532 cm^2 are 13 and 18, or 14 and 17, or 15 and 16.