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2019 Canadian Team Mathematics Contest
Answer Key for Team Problems

Question	**Answer**
1	10
2	25
3	Greta
4	$\frac{7}{24}$
5	2
6	9
7	46
8	210
9	48 m
10	$\frac{3}{2}$
11	28
12	1
13	3
14	43
15	96
16	$h(x) = x^4 + x^3 - 1$
17	$70\sqrt{3}$
18	$\sqrt[3]{9} : 1$
19	-128
20	2024
21	3916
22	$(6053, 6056)$
23	$\frac{17}{2}$
24	169
25	$\frac{1}{5}$

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Answer Key for Individual Problems

Question	Answer
1	6
2	83
3	3:00 a.m.
4	$\frac{1}{3}$
5	6
6	-1
7	1
8	19.6 L
9	1409
10	77.3°

Answer Key for Relays

Question	Answer
0	$7, 140, 20^\circ$
1	$120, 16, 5492$
2	$132, 11, 21$
3	$11, 5, \frac{3}{2}$

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2019
Canadian Team Mathematics Contest

April 2019

Solutions

Individual Problems

1. Philippe connects AB, AC, AD, BC, BD, and CD.
   He draws 6 line segments.

   ANSWER: 6

2. In order for $\sqrt{2019-n}$ to be an integer, $2019-n$ must be a perfect square.
   Since n is a positive integer, then $2019-n$ is a perfect square less than 2019.
   Since n is to be as small as possible, $2019-n$ must be the largest perfect square less than 2019.
   Since $44^2=1936$ and $45^2=2025$, then 1936 is the largest perfect square less than 2019.
   Thus, $2019-n=1936$ and so $n=83$.

   ANSWER: 83

3. Since $100=4\cdot 24+4$, then 100 hours is 4 days and 4 hours.
   The time 4 days before 7:00 a.m. is also 7:00 a.m.
   The time 4 hours earlier than this is 3:00 a.m.
   Thus, the time 100 hours before 7:00 a.m. is 3:00 a.m.

   ANSWER: 3:00 a.m.

4. The total number of dots on the six faces of a standard six-faced die is $1+2+3+4+5+6=21$.
   When one face is lying on a table, the total number of dots visible equals 21 minus the number of dots on the face that is lying on the table.
   For this total to be at least 19, then the number of dots on the face lying on the table must be 1 or 2.
   Therefore, the total is at least 19 when either of 2 of the 6 faces is lying on the table.
   The probability of this is $\frac{2}{6}$ which equals $\frac{1}{3}$.

   ANSWER: $\frac{1}{3}$

5. First, we note that $\dfrac{a-b}{c-d}=\dfrac{b-a}{d-c}$ and each of $b-a$ and $d-c$ is positive.
   For $\dfrac{b-a}{d-c}$ to be as large as possible, we want $b-a$ to be as large as possible and $d-c$ to be as small as possible.
   Since d and c are integers with $d>c$, then the smallest possible value of $d-c$ is 1.
   For $b-a$ to be as large as possible, we want b to be as large as possible and a to be as small as possible.
   Since a is an integer with $0<a$, then $1\le a$.
   Since b, c and d are integers with $b<c<d<10$, then $b\le 7$.
   Therefore, the largest possible value of $b-a$ is $7-1$ which equals 6.
   Note that, in this case, it must be the case that $d=9$ and $c=8$ which does give $d-c=1$.
   Thus, the largest possible value of $\dfrac{a-b}{c-d}$ is $\dfrac{6}{1}$ which equals 6.

   ANSWER: 6

6. When the line with equation $y=-2x+7$ is reflected across a vertical line, the sign of the slope is reversed, and so becomes 2.
   Since the new line has equation $y=ax+b$, then $a=2$.
   The point on the original line that has x-coordinate 3 has y-coordinate $y=-2(3)+7=1$.
   This means that the point $(3,1)$ is on the original line.
   When this line is reflected across $x=3$, this point $(3,1)$ must also be on the reflected line.
   Substituting into $y=2x+b$ gives $1=2(3)+b$ and so $b=-5$.
   This means that $2a+b=2(2)+(-5)=-1$.

   ANSWER: -1

7. We note that $(2^3)^x = 2^{3x}$ and that $4096 = 4 \cdot 41024 = 4 \cdot 4 \cdot 256 = 4 \cdot 4 \cdot 16 \cdot 16 = 2^2 \cdot 2^2 \cdot 2^4 \cdot 2^4 = 2^{12}$.
Thus, $2^{3x} = 2^{12}$ which gives $3x = 12$ and so $x = 4$.
Since $y = x^3$, then $y = 4^3 = 64$.
We need to consider the integer 3^{64}.
The first few powers of 3 are

$$3^1 = 3, \quad 3^2 = 9, \quad 3^3 = 27, \quad 3^4 = 81, \quad 3^5 = 243, \quad 3^6 = 729, \quad \ldots$$

The units digits of powers of 3 repeat in a cycle of length 4, namely $3, 9, 7, 1, 3, 9, 7, 1, \ldots$.
(Since each power is obtained by multiplying the previous power by 3, then the units digit of each power is obtained by multiplying the units digit of the previous power by 3 (possibly keeping only the units digit). This means that once a units digit recurs, then the units digits will form a cycle.)
Since 64 is a multiple of 4, then the units digit of 3^{64} will be the last in the cycle, or 1.

Answer: 1

8. Suppose that Yasmine uses m full bottles of milk and s full bottles of syrup.
The volume of milk that she uses is thus $2m$ L and the volume of syrup that she uses is $1.4s$ L.
For this ratio to equal $5 : 2$, we need $\dfrac{2m}{1.4s} = \dfrac{5}{2}$ or $4m = 7s$.
Since 4 and 7 have no common divisor larger than 1, then the smallest positive integers that satisfy this are $m = 7$ and $s = 4$.
Thus, the volume, in litres, of chocolate beverage that Yasmine makes is $2 \cdot 7 + 1.4 \cdot 4 = 19.6$.

Answer: 19.6 L

9. Suppose that $x_1 = a$.
Then

$$\begin{aligned}
x_2 &= 2x_1 = 2a \\
x_3 &= x_2 - 1 = 2a - 1 \\
x_4 &= 2x_3 = 4a - 2 \\
x_5 &= x_4 - 1 = 4a - 3 \\
x_6 &= 2x_5 = 8a - 6 \\
x_7 &= x_6 - 1 = 8a - 7 \\
x_8 &= 2x_7 = 16a - 14 \\
x_9 &= x_8 - 1 = 16a - 15 \\
x_{10} &= 2x_9 = 32a - 30 \\
x_{11} &= x_{10} - 1 = 32a - 31
\end{aligned}$$

The pattern above suggests that, for each integer $k \geq 1$, we have

$$x_{2k} = 2^k a - (2^k - 2) \qquad \text{and} \qquad x_{2k+1} = 2^k a - (2^k - 1)$$

We note that this is true for $k = 1, 2, 3, 4, 5$.
Also, if this is true for some value of $k \geq 1$, then

$$x_{2k+2} = 2x_{2k+1} = 2(2^k a - (2^k - 1)) = 2^{k+1}a - 2(2^k - 1) = 2^{k+1}a - (2^{k+1} - 2)$$

and

$$x_{2k+3} = x_{2k+2} - 1 = 2^{k+1}a - (2^{k+1} - 1)$$

which means that it is true for $k+1$.
This inductive reasoning shows that that this form is correct for all $k \geq 1$.
Consider the terms x_m when $m \geq 11$. We want to determine the smallest value larger than 1395 that such a term can equal.
Suppose that m is even, say $m = 2k$ for some integer $k \geq 6$.
Note that

$$x_m = x_{2k} = 2^k a - (2^k - 2) = 2^k(a-1) + 2 = 2^6 \cdot 2^{k-6}(a-1) + 2 = 64\left(2^{k-6}(a-1)\right) + 2$$

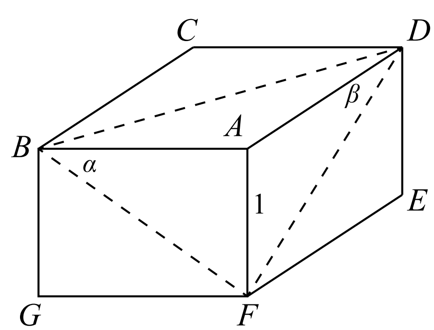
In other words, for every $k \geq 6$, the term x_{2k} is 2 more than a multiple of 64. (Note that 2^{k-6} is an integer since $k \geq 6$.)
Suppose that m is odd, say $m = 2k+1$ for some integer $k \geq 5$.
Note that

$$x_m = x_{2k+1} = 2^k a - (2^k - 1) = 2^k(a-1) + 1 = 2^5 \cdot 2^{k-5}(a-1) + 1 = 32\left(2^{k-5}(a-1)\right) + 1$$

In other words, for every $k \geq 5$, the term x_{2k+1} is 1 more than a multiple of 32 (Note that 2^{k-5} is an integer since $k \geq 5$.).
Now $43 \cdot 32 = 1376$ and $44 \cdot 32 = 1408$. Also, $21 \cdot 64 = 1344$ and $22 \cdot 64 = 1408$.
The smallest integer larger than 1395 that is 1 more than a multiple of 32 is 1409.
The smallest integer larger than 1395 that is 2 more than a multiple of 64 is 1410.
This means that 1409 is the smallest candidate for N, but we must confirm that there is a sequence with 1409 in it.
When $a = 45$, the 11th term is $32a - 31 = 1409$.
This means that 1409 is in such a sequence as a term past the 10th term when $a = 45$.
In particular, this sequence is $45, 90, 89, 178, 177, 354, 353, 706, 705, 1410, 1409$.

Answer: 1409

10. Without loss of generality, suppose that $AF = 1$.
We let $\alpha = \angle ABF = 40^\circ$ and $\beta = \angle ADF = 20^\circ$ and $\theta = \angle BFD$.



Using the cosine law in $\triangle BFD$, we obtain:

$$BD^2 = FB^2 + FD^2 - 2(FB)(FD)\cos\theta$$
$$\cos\theta = \frac{FB^2 + FD^2 - BD^2}{2(FB)(FD)}$$

Now, $\sin\alpha = \dfrac{AF}{FB}$. Since $AF = 1$, then $FB = \dfrac{1}{\sin\alpha}$.
Similarly, $FD = \dfrac{1}{\sin\beta}$.

Also, $\tan\alpha = \dfrac{AF}{AB}$. Since $AF = 1$, then $AB = \dfrac{1}{\tan\alpha} = \dfrac{\cos\alpha}{\sin\alpha}$.
Similarly, $AD = \dfrac{1}{\tan\beta} = \dfrac{\cos\beta}{\sin\beta}$.
By the Pythagorean Theorem, $BD^2 = AB^2 + AD^2 = \dfrac{\cos^2\alpha}{\sin^2\alpha} + \dfrac{\cos^2\beta}{\sin^2\beta}$.
Therefore,

$$\begin{aligned}
\cos\theta &= \frac{FB^2 + FD^2 - BD^2}{2(FB)(FD)} \\
&= \frac{\dfrac{1}{\sin^2\alpha} + \dfrac{1}{\sin^2\beta} - \dfrac{\cos^2\alpha}{\sin^2\alpha} - \dfrac{\cos^2\beta}{\sin^2\beta}}{2\cdot\dfrac{1}{\sin\alpha}\cdot\dfrac{1}{\sin\beta}} \\
&= \frac{\sin^2\beta + \sin^2\alpha - \cos^2\alpha\sin^2\beta - \cos^2\beta\sin^2\alpha}{2\sin\alpha\sin\beta} \\
&= \frac{\sin^2\beta(1-\cos^2\alpha) + \sin^2\alpha(1-\cos^2\beta)}{2\sin\alpha\sin\beta} \\
&= \frac{\sin^2\beta\sin^2\alpha + \sin^2\alpha\sin^2\beta}{2\sin\alpha\sin\beta} \\
&= \sin\alpha\sin\beta
\end{aligned}$$

Since $\alpha = 40^\circ$ and $\beta = 20^\circ$, then $\cos\theta = \sin(40^\circ)\sin(20^\circ) \approx 0.21985$.
Therefore, $\theta = \cos^{-1}(\sin(40^\circ)\sin(20^\circ)) \approx 77.3^\circ$.

Answer: 77.3°

Team Problems

1. Simplifying, we obtain $7x - 8 = 12 + 5x$ and so $2x = 20$ or $x = 10$.

   Answer: 10

2. Using the common factor of 2.5, we see that

$$3.5 \times 2.5 + 6.5 \times 2.5 = (3.5 + 6.5) \times 2.5 = 10 \times 2.5 = 25$$

   Answer: 25

3. Since Ada is younger than Darwyn, Ada cannot be the oldest.
   Since Max is younger than Greta, Max cannot be the oldest.
   Since James is older than Darwyn, then Darywn cannot be the oldest.
   Since Max and James are the same age and Max is not the oldest, then James cannot be the oldest.
   By elimination, Greta must be the oldest.
   (The order of ages, from oldest to youngest, could be Greta, Max/James, Darwyn, Ada.)

   Answer: Greta

4. By definition, the mean is $\dfrac{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}}{3} = \dfrac{\frac{4}{8} + \frac{2}{8} + \frac{1}{8}}{3} = \dfrac{7/8}{3} = \dfrac{7}{24}$.

   Answer: $\dfrac{7}{24}$

5. Since $M = 1^5 + (2^4 \times 3^3) - (4^2 \div 5^1)$ and $N = 1^5 - (2^4 \times 3^3) + (4^2 \div 5^1)$, then when M and N are added the terms $(2^4 \times 3^3)$ and $(4^2 \div 5^1)$ "cancel" out.
   Thus, $M + N = 1^5 + 1^5 = 2$.

   Answer: 2

6. If ab and ba are both prime numbers, then neither is even which means that neither digit a or b is even and neither equals 5, otherwise ab or ba would be even or divisible by 5 and so not prime.
   Therefore, each of a and b equals 1, 3, 7, or 9.
   The two-digit primes using these digits are $11, 13, 17, 19, 31, 37, 71, 73, 79, 97$.
   If ab equals one of these primes, then ba must be prime as well, which eliminates 19 as a possible value for ab, since 91 is not prime.
   Therefore, $abba$ could be $1111, 1331, 1771, 3113, 3773, 7117, 7337, 7997, 9779$.
   There are 9 such palindromes.

   Answer: 9

7. The integers less than 50 that can be written as a product of two consecutive positive integers are $1 \cdot 2 = 2$, $2 \cdot 3 = 6$, $3 \cdot 4 = 12$, $4 \cdot 5 = 20$, $5 \cdot 6 = 30$, and $6 \cdot 7 = 42$.
   Therefore, there are $50 - 6 = 44$ positive integers less than or equal to 50 that cannot be written as the product of two consecutive positive integers.
   This means that 50 would be the 44th integer in Adia's list.
   Counting backwards from 50, this means that the 40th integer in Adia's list is $50 - 4 = 46$. (None of the numbers eliminated are between 46 and 50.)

   Answer: 46

8. If $a = 1$, we get $1 < 1 + b < 22$ or $0 < b < 21$, which means that b can equal $1, 2, 3, \ldots, 19, 20$.
If $a = 2$, we get $1 < 2 + b < 22$ or $-1 < b < 20$.
Since b is positive, then b can equal $1, 2, 3, \ldots, 18, 19$.
In general, $1 - a < b < 22 - a$. Since b is positive, then b satisfies $1 \leq b \leq 21 - a$, which means that there are $21 - a$ possible values for b for a given a.
As a runs from 1 to 20, there are thus $20, 19, 18, \ldots, 3, 2, 1$ possible values for b in these 20 cases.
This means that the total number of pairs (a, b) is $20+19+18+\cdots+3+2+1 = \frac{1}{2}(20)(21) = 210$.
Answer: 210

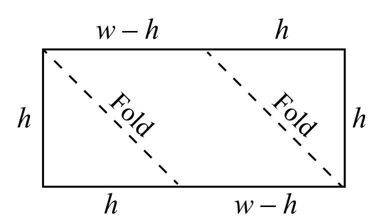
9. Suppose that the length of the trail is d m.
On the muddy day, Shelly-Ann runs $\frac{1}{3}d$ m at 2 m/s and $\frac{2}{3}d$ m at 8 m/s.
Since this takes 12 seconds in total, then $\dfrac{\frac{1}{3}d}{2} + \dfrac{\frac{2}{3}d}{8} = 12$.
Multiplying both sides by 8, we obtain $\frac{4}{3}d + \frac{2}{3}d = 96$ which gives $2d = 96$ or $d = 48$.
Therefore, the trail is 48 metres long.
Answer: 48 metres

10. Using exponent laws,

$$\begin{aligned} 5^a + 5^{a+1} &= \sqrt{4500} \\ 5^a(1 + 5^1) &= 30\sqrt{5} \\ 5^a \cdot 6 &= 30\sqrt{5} \\ 5^a &= 5\sqrt{5} \\ 5^a &= 5^1 \cdot 5^{1/2} \\ 5^a &= 5^{3/2} \end{aligned}$$

and so $a = \frac{3}{2}$.
Answer: $\frac{3}{2}$

11. Suppose that the original rectangle has height h and width w.
Since the area of the original rectangle is 40, then $hw = 40$.
Once the corners are folded, the height of the resulting parallelogram is still h and the new base is $w - h$, because the two legs of each folded triangle are equal since these triangles are isosceles.



Since the area of the parallelogram is 24, then $h(w - h) = 24$ or $hw - h^2 = 24$.
Since $hw = 40$, then $h^2 = 40 - 24 = 16$.
Since $h > 0$, then $h = 4$.
Since $hw = 40$ and $h = 4$, then $w = 10$.
The perimeter of the original rectangle is $2h + 2w$ which equals $2 \cdot 4 + 2 \cdot 10 = 28$.
Answer: 28

12. Let $x = 123456$.
Thus, $x - 1 = 123455$ and $x + 1 = 123457$.
Therefore,

$$123456^2 - 123455 \times 123457 = x^2 - (x-1)(x+1) = x^2 - (x^2 - 1) = 1$$

Answer: 1

13. Using the change of base formulas for logarithms,

$$(\log_2 4)(\log_4 6)(\log_6 8) = \frac{\log 4}{\log 2} \cdot \frac{\log 6}{\log 4} \cdot \frac{\log 8}{\log 6} = \frac{\log 8}{\log 2} = \log_2 8 = 3$$

Answer: 3

14. Since $\frac{x}{5} = \frac{6}{y}$, then $xy = 30$.
Since $\frac{6}{y} = \frac{z}{2}$, then $yz = 12$.
Since we would like the maximum value of $x + y + z$, we may assume that each of x, y and z is positive.
Since $xy = 30$ and $yz = 12$ and y is a positive integer, then y is a divisor of each of 30 and 12.
This means that y must equal one of 1, 2, 3, or 6.
If $y = 1$, then $x = 30$ and $z = 12$ which gives $x + y + z = 43$.
If $y = 2$, then $x = 15$ and $z = 6$ which gives $x + y + z = 23$.
If $y = 3$, then $x = 10$ and $z = 4$ which gives $x + y + z = 17$.
If $y = 6$, then $x = 5$ and $z = 2$ which gives $x + y + z = 13$.
Therefore, the maximum possible value of $x + y + z$ is 43.

Answer: 43

15. We note first that $1009^2 = (1000 + 9)^2 = 1000^2 + 2 \cdot 1000 \cdot 9 + 9^2 = 1\,018\,081$.
Therefore,

$$\begin{aligned}
\mathbf{G} - 1009^2 &= 10^{100} - 1\,018\,081 \\
&= (10^{100} - 1) - (1\,081\,081 - 1) \\
&= \underbrace{999\cdots999}_{\text{100 digits equal to 9}} - 1\,081\,080 \\
&= \underbrace{999\cdots999}_{\text{93 digits equal to 9}} 9\,999\,999 - 1\,081\,080 \\
&= \underbrace{999\cdots999}_{\text{93 digits equal to 9}} 8\,918\,919
\end{aligned}$$

and $\mathbf{G} - 1009^2$ has 96 digits equal to 9.

Answer: 96

16. Since $f(x)$ has degree 4, $g(x)$ has degree 8, and $h(x)$ is a polynomial with $g(x) = f(x)h(x)$, then $h(x)$ has degree 4.
We write $h(x) = ax^4 + bx^3 + cx^2 + dx + e$ and so

$$x^8 - x^6 - 2x^4 + 1 = (x^4 - x^3 + 0x^2 + 0x - 1)(ax^4 + bx^3 + cx^2 + dx + e)$$

Comparing coefficients of x^8 on the left and right sides, we obtain $1 = 1 \cdot a$ and so $a = 1$, giving

$$x^8 - x^6 - 2x^4 + 1 = (x^4 - x^3 + 0x^2 + 0x - 1)(x^4 + bx^3 + cx^2 + dx + e)$$

Comparing coefficients of x^7, we obtain $0 = 1 \cdot b + (-1) \cdot 1$ and so $b = 1$, giving

$$x^8 - x^6 - 2x^4 + 1 = (x^4 - x^3 + 0x^2 + 0x - 1)(x^4 + x^3 + cx^2 + dx + e)$$

Comparing coefficients of x^6, we obtain $-1 = 1 \cdot c + (-1) \cdot 1 + 0 \cdot 1$ and so $c = 0$, giving

$$x^8 - x^6 - 2x^4 + 1 = (x^4 - x^3 + 0x^2 + 0x - 1)(x^4 + x^3 + 0x^2 + dx + e)$$

Comparing coefficients of x^5, we obtain $0 = 1 \cdot d + (-1) \cdot 0 + 0 \cdot 1 + 0 \cdot 1$ and so $d = 0$, giving

$$x^8 - x^6 - 2x^4 + 1 = (x^4 - x^3 + 0x^2 + 0x - 1)(x^4 + x^3 + 0x^2 + 0x + e)$$

Comparing constant terms, we obtain $1 = (-1) \cdot e$ and so $e = -1$, giving

$$x^8 - x^6 - 2x^4 + 1 = (x^4 - x^3 + 0x^2 + 0x - 1)(x^4 + x^3 + 0x^2 + 0x - 1)$$

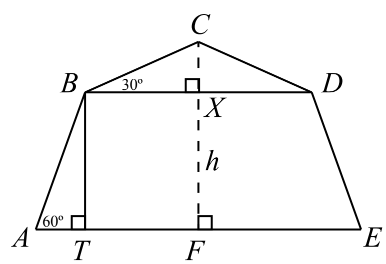
We can verify by expansion that the remaining terms match.
Therefore, $h(x) = x^4 + x^3 - 1$.
This result can also be obtained by polynomial long division.

Answer: $h(x) = x^4 + x^3 - 1$

17. Suppose that CF and BD cross at X.
Since the figure is symmetric about CF, then BD is perpendicular to CF and so is parallel to AE.
Let the distance from BD to AE be h; that is, $XF = h$.
Since $CF = 80\sqrt{3}$, then $CX = CF - XF = 80\sqrt{3} - h$.
Since the figure is symmetric about CF, then $CB = CD$ which means that $\triangle BCD$ is isosceles.
Since $\triangle BCD$ is isosceles, then $\angle CBD = \frac{1}{2}(180^\circ - \angle BCD) = 30^\circ$.
Since $\triangle BCX$ is right-angled at X, then it is a 30°-60°-90° triangle.
Using the ratios of sides in such a triangle, this means that

$$BX = \sqrt{3}CX = \sqrt{3}(80\sqrt{3} - h) = 240 - \sqrt{3}h$$

Since $\angle ABC = 150^\circ$ and $\angle CBD = 30^\circ$, then $\angle ABD = \angle ABC - \angle CBD = 120^\circ$.
Since BD and AE are parallel, then $\angle BAE = 180^\circ - \angle ABD = 60^\circ$.
We drop a perpendicular from B to T on AE.
This creates a rectangle $BXFT$ (it has three right angles and so must be a rectangle) and a 30°-60°-90° triangle, $\triangle BAT$.



Since $AE = 200$, then $AF = \frac{1}{2}AE = 100$.
Since $BX = 240 - \sqrt{3}h$, then $TF = BX = 240 - \sqrt{3}h$.
Thus, $AT = AF - TF = 100 - (240 - \sqrt{3}h) = \sqrt{3}h - 140$.
Also, $BT = XF = h$.
Finally, in $\triangle BAT$, we have $\dfrac{BT}{AT} = \sqrt{3}$, which gives $\dfrac{h}{\sqrt{3}h - 140} = \sqrt{3}$ and so $h = 3h - 140\sqrt{3}$.
Re-arranging, we obtain $2h = 140\sqrt{3}$ and finally that $h = 70\sqrt{3}$.

Answer: $70\sqrt{3}$

18. Suppose that the height, radius and diameter of Cylinder A are h, r and d, respectively.
Note that $d = h$ and $d = 2r$ and so $h = 2r$.
The volume of Cylinder A is $\pi r^2 h$ which equals $2\pi r^3$.
Since the height and diameter of Cylinder B are twice that of Cylinder A, then these are $2h$ and $2d$, respectively, which means that the radius of Cylinder B is $2r$ and the height of Cylinder B equals $4r$.
The volume of Cylinder B is $\pi(2r)^2(4r)$ which equals $16\pi r^3$.
Suppose that the height, radius and diameter of Cylinder C are H, R and D, respectively.
Note that $D = H$ and $D = 2R$ and so $H = 2R$.
The volume of Cylinder C is $\pi R^2 H$ which equals $2\pi R^3$.
Since the sum of the volumes of Cylinders A and B equals the that of Cylinder C, then we obtain $2\pi r^3 + 16\pi r^3 = 2\pi R^3$ and so $18\pi r^3 = 2\pi R^3$ or $R^3 = 9r^3$.
This means that $R = \sqrt[3]{9}r$ and so $2R = \sqrt[3]{9}(2r)$ which gives $D = \sqrt[3]{9}d$.
Therefore, the ratio of the diameter of Cylinder C to the diameter of Cylinder A is $\sqrt[3]{9} : 1$.

ANSWER: $\sqrt[3]{9} : 1$

19. Since $a > 0$, each parabola of the given form opens upwards and so its minimum occurs at its vertex.
The x-coordinate of the vertex will be the average of the x-coordinates of the roots, which in this case are $x = b$ and $x = c$. Thus, the x coordinate of the vertex is $x = \dfrac{b+c}{2}$.
Therefore, the minimum value of $f(x)$ is the value of $f(x)$ when $x = \dfrac{b+c}{2}$, which is

$$f\left(\frac{b+c}{2}\right) = a\left(\frac{b+c}{2} - b\right)\left(\frac{b+c}{2} - c\right) = a\left(\frac{-b+c}{2}\right)\left(\frac{b-c}{2}\right) = -\frac{a(b-c)^2}{4}$$

Since a and $(b-c)^2$ are non-negative, then $-\dfrac{a(b-c)^2}{4}$ is minimized when $a(b-c)^2$ is maximized.
Since a, b and c are distinct positive integers less than 10, then $a \leq 9$ and $-8 \leq b - c \leq 8$.
We suppose, without loss of generality, that $b > c$ and so $0 < b - c \leq 8$.
If $b - c = 8$, then we must have $b = 9$ and $c = 1$. Since a, b and c are distinct, then we cannot have both $a = 9$ and $b - c = 8$.
The maximum of $a(b-c)^2$ could occur when $a = 9$ and $b - c = 7$ (the maximum possible value if $a = 9$) or when $a = 8$ and $b - c = 8$. Any other combination of values of a and $b - c$ cannot give the maximum because we could increase one value or the other.
When $a = 9$ and $b - c = 7$ (which means $b = 8$ and $c = 1$), the value of $a(b-c)^2$ is 441.
When $a = 8$ and $b - c = 8$ (which means $b = 9$ and $c = 1$), the value of $a(b-c)^2$ is 512.
Therefore, the minimum of the minimum values of $f(x)$ is $-\dfrac{512}{4}$ which equals -128.

ANSWER: -128

20. We can write the integer N as

$$N = 1 \cdot 10^{a_1} + 2 \cdot 10^{a_2} + 3 \cdot 10^{a_3} + \cdots + (k-1) \cdot 10^{a_{k-1}} + k \cdot 10^0$$

for some positive integers $a_1, a_2, a_3, \ldots a_{k-1}$.
Suppose that

$$M = 1 \cdot (10^{a_1} - 1) + 2 \cdot (10^{a_2} - 1) + 3 \cdot (10^{a_3} - 1) + \cdots + (k-1) \cdot (10^{a_{k-1}} - 1)$$

Since the digits of $10^{a_j} - 1$ are all 9s for each j with $1 \leq j \leq k-1$, then each $10^{a_j} - 1$ is divisible by 9.
This means that M is divisible by 9.
Since M is divisible by 9, then N is divisible by 9 exactly when $N - M$ is divisible by 9.
But

$$N - M = 1 \cdot (10^{a_1} - (10^{a_1} - 1)) + 2 \cdot (10^{a_2} - (10^{a_2} - 1)) + \cdots + (k-1) \cdot (10^{a_{k-1}} - (10^{a_{k-1}} - 1)) + k \cdot 10^0$$

and so

$$N - M = 1 + 2 + \cdots + (k-1) + k = \tfrac{1}{2}k(k+1)$$

Therfore, N is divsible by 9 exactly when $\frac{1}{2}k(k+1)$ is divisible by 9.
Since $k > 2019$, we start checking values of k at $k = 2020$.
When $k = 2020, 2021, 2022, 2023$, the value of of $\frac{1}{2}k(k+1)$ is not divisible by 9.
When $k = 2024$, we see that $\frac{1}{2}k(k+1) = \frac{1}{2}(2024)(2025) = 1012 \cdot 9 \cdot 225$.
Therefore, $k = 2024$ is the smallest $k > 2019$ for which N is divisible by 9.

ANSWER: 2024

21. Since the numbers $x_1, x_2, \ldots, x_n$ form an arithmetic sequence, then for each integer k with $2 \leq k \leq n-1$, we have $x_k - x_{k-1} = x_{k+1} - x_k$.
Rearranging, we obtain $2x_k = x_{k-1} + x_{k+1}$ and so $\dfrac{x_k}{x_{k-1} + x_{k+1}} = \dfrac{1}{2}$ for each integer k with $2 \leq k \leq n-1$.
We note that there are $(n-1) - 2 + 1 = n - 2$ integers k in this range.
Therefore, starting with the given equation

$$\frac{x_2}{x_1 + x_3} + \frac{x_3}{x_2 + x_4} + \cdots + \frac{x_{n-2}}{x_{n-3} + x_{n-1}} + \frac{x_{n-1}}{x_{n-2} + x_n} = 1957$$

we obtain $(n-2) \cdot \dfrac{1}{2} = 1957$ which gives $n - 2 = 3914$ and so $n = 3916$.

ANSWER: 3916

22. Since $f_1(x) = \dfrac{1}{2-x}$, then $f_1(4) = \dfrac{1}{2-4} = -\dfrac{1}{2}$.
Since $f_2(x) = f_1(f_1(x))$, then $f_2(4) = f_1\left(-\dfrac{1}{2}\right) = \dfrac{1}{2 - (-1/2)} = \dfrac{1}{5/2} = \dfrac{2}{5}$.
Since $f_3(x) = f_1(f_2(x))$, then $f_3(4) = f_1\left(\dfrac{2}{5}\right) = \dfrac{1}{2 - (2/5)} = \dfrac{1}{8/5} = \dfrac{5}{8}$.
Since we can re-write $f_1(4) = \dfrac{-1}{2}$, then the values $f_1(4)$, $f_2(4)$, $f_3(4)$ each satisfy the equality $f_n(4) = \dfrac{3n-4}{3n-1}$.

Suppose that $f_k(4) = \dfrac{3k-4}{3k-1}$.
Then

$$\begin{aligned} f_{k+1}(4) &= f_1(f_k(4)) \\ &= f_1\left(\frac{3k-4}{3k-1}\right) \\ &= \frac{1}{2-(3k-4)/(3k-1)} \\ &= \frac{3k-1}{2(3k-1)-(3k-4)} \\ &= \frac{3k-1}{3k+2} \\ &= \frac{3(k+1)-4}{3(k+1)-1} \end{aligned}$$

This means that if one term in the sequence $f_1(4), f_2(4), f_3(4), \ldots$ is of this form, then the next term does, which means that all terms are of this form.
Therefore, $f_{2019}(4) = \dfrac{3(2019)-4}{3(2019)-1} = \dfrac{6053}{6056}$.
Since $6056 - 6053 = 3$, then any integer that is a divisor of both 6056 and 6053 is also a divisor of 3.
This means that the only possible positive common divisors of 6053 and 6053 are 1 and 3.
Since neither 6053 nor 6056 is divisible by 3, then they have no common divisor greater than 1.
Thus, if $f_{2019}(4) = \dfrac{a}{b}$ where a and b are positive integers with no common divisor larger than 1, then $(a, b) = (6053, 6056)$.

ANSWER: $(6053, 6056)$

23. Using the four numbers p, q, r, t, the possible sums of pairs are

$$p+q, p+r, p+t, q+r, q+t, r+t$$

Since $p < q < r < t$, then $p+q < p+r < p+t$.
Also, $p+t < q+t < r+t$, which gives $p+q < p+r < p+t < q+t < r+t$.
Now $p+r < q+r < r+t$, but the relative size of $p+t$ and $q+r$ is unknown at this time.
This means that the four largest sums are $p+r$, $p+t$, $q+t$, $r+t$, although we do not know the relative size of the first two of these.
Since the four largest sums are 19, 22, 25, and 28, then $r+t = 28$ and $q+t = 25$.
Also, $p+t$ and $q+r$ are 22 and 19 in some order.

<u>Case 1: $r+t = 28$ and $q+t = 25$ and $q+r = 22$ and $p+t = 19$</u>
Adding the first three of these equations, we obtain

$$(r+t) + (q+t) + (q+r) = 28 + 25 + 22$$

which gives $2q + 2r + 2t = 75$ and so $q + r + t = \frac{75}{2}$.
From this, we obtain $t = (q+r+t) - (q+r) = \frac{75}{2} - 22 = \frac{31}{2}$, from which $p = 19 - t = 19 - \frac{31}{2} = \frac{7}{2}$.
We can verify that when $q = \frac{19}{2}$ and $r = \frac{25}{2}$, the equations are satisfied.

Case 2: $r + t = 28$ and $q + t = 25$ and $q + r = 19$ and $p + t = 22$
Adding the first three of these equations, we obtain

$$(r + t) + (q + t) + (q + r) = 28 + 25 + 19$$

which gives $2q + 2r + 2t = 72$ and so $q + r + t = 36$.
From this, we obtain $t = (q + r + t) - (q + r) = 36 - 19 = 17$, from which $p = 22 - t = 5$.
We can verify that when $q = 8$ and $r = 11$, the equations are satisfied.

The sum of the possible values of p is thus $\frac{7}{2} + 5$ or $\frac{17}{2}$.

Answer: $\frac{17}{2}$

24. Using logarithm rules, the following equations are equivalent:

$$\begin{aligned} x^2 - \sqrt{13}x^{\log_{13} x} &= 0 \\ x^2 &= \sqrt{13}x^{\log_{13} x} \\ \log_{13}\left(x^2\right) &= \log_{13}\left(\sqrt{13}x^{\log_{13} x}\right) \\ 2\log_{13} x &= \log_{13}\left(\sqrt{13}\right) + \log_{13}\left(x^{\log_{13} x}\right) \\ 2\log_{13} x &= \tfrac{1}{2} + \log_{13} x \cdot \log_{13} x \\ 0 &= 2(\log_{13} x)^2 - 4\log_{13} x + 1 \end{aligned}$$

This equation is a quadratic equation in $\log_{13} x$.
If α and β are the two roots of the original equation, then

$$\alpha\beta = 13^{\log_{13}(\alpha\beta)} = 13^{\log_{13}\alpha + \log_{13}\beta}$$

But $\log_{13}\alpha + \log_{13}\beta$ is the sum of the two roots of the equation

$$2(\log_{13} x)^2 - 4\log_{13} x + 1 = 0$$

which is a quadratic equation in $\log_{13} x$.
Since the sum of the roots of the quadratic equation $ay^2 + by + c = 0$ is $-\dfrac{b}{a}$, then

$$\log_{13}\alpha + \log_{13}\beta = -\frac{-4}{2} = 2$$

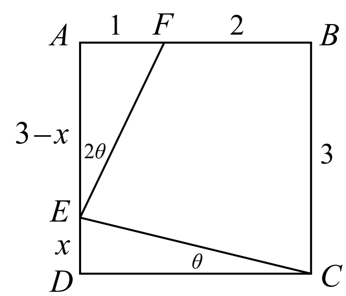
Finally, this gives $\alpha\beta = 13^2 = 169$.

Answer: 169

25. *Solution 1*

Suppose that $\angle ECD = \theta$.
Without loss of generality, suppose that the square has side length 3.
Therefore, $AB = AD = DC = 3$.
Since $BF = 2AF$ and $AF + BF = AB = 3$, then $AF = 1$ and $BF = 2$.
Suppose that $ED = x$. Since $AD = 3$, then $AE = 3 - x$.
Since $\angle ECD = \theta$, then $\angle BCE = 90^\circ - \theta$.
Since $\angle FEC = \angle BCE$, then $\angle FEC = 90^\circ - \theta$.
Since $\triangle EDC$ is right-angled at D, then $\angle DEC = 90^\circ - \angle ECD = 90^\circ - \theta$.
Since AED is a straight line, then

$$\angle AEF = 180^\circ - \angle FEC - \angle DEC = 180^\circ - 2(90^\circ - \theta) = 2\theta$$

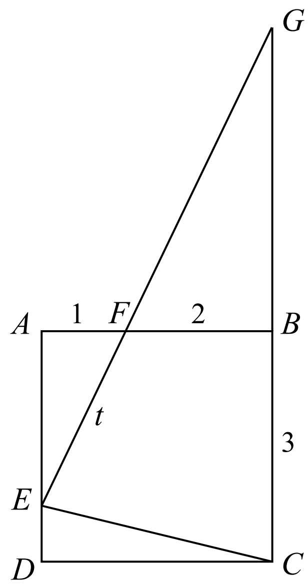


In $\triangle EDC$, we see that $\tan\theta = \dfrac{ED}{DC} = \dfrac{x}{3}$.
In $\triangle EAF$, we see that $\tan 2\theta = \dfrac{AF}{AE} = \dfrac{1}{3-x}$.
Since $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$, then

$$\begin{aligned}
\frac{1}{3-x} &= \frac{2x/3}{1-(x/3)^2} \\
\frac{1}{3-x} &= \frac{6x}{9-x^2} \\
9 - x^2 &= (3-x)(6x) \\
5x^2 - 18x + 9 &= 0 \\
(5x-3)(x-3) &= 0
\end{aligned}$$

Thus, $x = 3$ or $x = \dfrac{3}{5}$.
If $x = 3$, then EC is a diagonal of the square, which would make $\angle ECD = 45^\circ$, which is not allowed.
Therefore, $x = \dfrac{3}{5}$ and so $\tan(\angle ECD) = \tan\theta = \dfrac{3/5}{3} = \dfrac{1}{5}$.

Solution 2
Without loss of generality, suppose that the square has side length 3.
Therefore, $AB = AD = DC = 3$.
Since $BF = 2AF$ and $AF + BF = AB = 3$, then $AF = 1$ and $BF = 2$.
Extend EF and CB to meet at G. Let $EF = t$.



Now, $\triangle EAF$ is similar to $\triangle GBF$, since each is right-angled and $\angle AFE = \angle BFG$.
Since $BF : AF = 2 : 1$, then $GF : EF = 2 : 1$ and so $GF = 2t$.
Since $\angle FEC = \angle BCE$, then $\triangle GEC$ is isosceles with $GE = GC$.
Thus, $GB = GC - BC = GE - 3 = 3t - 3$.
Using the Pythagorean Theorem in $\triangle BGF$, we obtain

$$\begin{aligned} 2^2 + (3t-3)^2 &= (2t)^2 \\ 4 + 9t^2 - 18t + 9 &= 4t^2 \\ 5t^2 - 18t + 13 &= 0 \\ (5t-13)(t-1) &= 0 \end{aligned}$$

and so $t = \frac{13}{5}$ or $t = 1$.
If $t = 1$, then $EF = AF$ which means that point E is at A, which is not possible since we are told that $\angle ECD < 45^\circ$.
Thus, $t = \frac{13}{5}$.
Since $AF : EF = 1 : \frac{13}{5} = 5 : 13$, which means that $\triangle FAE$ is similar to a 5-12-13 triangle, and so $AE = \frac{12}{5}$.
Since $AD = 3$, then $ED = AD - AE = \frac{3}{5}$.
Finally, this means that $\tan(\angle ECD) = \dfrac{ED}{DC} = \dfrac{3/5}{3} = \dfrac{1}{5}$.

ANSWER: $\dfrac{1}{5}$

Relay Problems

(Note: Where possible, the solutions to parts (b) and (c) of each Relay are written as if the value of t is not initially known, and then t is substituted at the end.)

0. (a) Evaluating, $\dfrac{12+3\times 3}{3} = \dfrac{12+9}{3} = \dfrac{21}{3} = 7$.

   (b) The area of a triangle with base $2t$ and height $3t-1$ is $\frac{1}{2}(2t)(3t-1)$ or $t(3t-1)$.
   Since the answer to (a) is 7, then $t=7$, and so $t(3t-1) = 7(20) = 140$.

   (c) Since $AB = BC$, then $\angle BCA = \angle BAC$.
   Since $\angle ABC = t^\circ$, then $\angle BAC = \frac{1}{2}(180^\circ - \angle ABC) = \frac{1}{2}(180^\circ - t^\circ) = 90^\circ - \frac{1}{2}t^\circ$.
   Since the answer to (b) is 140, then $t = 140$, and so

$$\angle BAC = 90^\circ - \tfrac{1}{2}(140^\circ) = 20^\circ$$

Answer: $7, 140, 20^\circ$

1. (a) When $x=1$ and $y=630$, we get $2019x-3y-9 = 2019\cdot 1 - 3\cdot 630 - 9 = 2019-1890-9 = 120$.

   (b) At the beginning of 2018, there were 40 employees in Okotoks.
   At the end of 2018, there were 35% fewer employees in Okotoks, which is a total of $0.35\cdot 40 = 14$ fewer employees.
   At the beginning of 2018, there were t employees in Moose Jaw.
   At the end of 2018, there were 25% more employees in Moose Jaw, which is a total of $0.25t$ more employees.
   The net number of additional employees is thus $0.25t - 14$.
   Since the answer to (a) is 120, then $t = 120$ and so $0.25t - 14 = 0.25(120) - 14 = 16$.
   Thus, the "CEMC" had 16 more employees at the end of 2018 than it had at the beginning of 2018.

   (c) There are $4\cdot 3\cdot 2\cdot 1 = 24$ integers that Kolapo can make using the digits 2, 4, 5, and 9.
   These include $3\times 2\times 1 = 6$ integers beginning with 2, and 6 integers beginning with 4, and 6 integers beginning with 5, and 6 integers beginning with 9.
   Since the answer to (b) is 16, then $t = 16$.
   The 16th integer in Kolapo's list is in the third group (those beginning with 5), and is the 4th largest integer in this group.
   In increasing order, the integers beginning with 5 in Kolapo's list are 5249, 5294, 5429, 5492, 5924, 5942.
   Therefore, the 16th number is 5492.

Answer: $120, 16, 5492$

2. (a) Since $\triangle ABC$ is similar to $\triangle DEF$, then $\dfrac{AB}{BC} = \dfrac{DE}{EF}$, which means that $\dfrac{x}{33} = \dfrac{96}{24} = 4$.
   Thus, $x = 33\cdot 4 = 132$.

   (b) Manipulating the left side,

$$\begin{aligned}
2+4+6+\cdots+(2k-2)+2k &= t\\
2(1+2+3+\cdots+(k-1)+k) &= t\\
2\left(\tfrac{1}{2}k(k+1)\right) &= t\\
k(k+1) &= t
\end{aligned}$$

   Since the answer to (a) is 132, then $t = 132$.
   Since $k(k+1) = 132$ and k is positive, then $k = 11$.

(c) O has coordinates $(0,0)$ and Q has coordinates $(c,1)$.
Since the slope of OQ is 1, then $c=1$.
Since $a=2c$, then $a=2$ which means that the coordinates of P are $(2,b)$.
Since the slope of OP is t, then $t=\dfrac{b-0}{2-0}$, which means that $b=2t$.
P has coordinates $(2,2t)$ and Q has coordinates $(1,1)$.
The slope of PQ is thus $\dfrac{2t-1}{2-1}$ which equals $2t-1$.
Since the answer to (b) is 11, then $t=11$.
This means that the slope of PQ is $2\cdot 11-1$ which equals 21.

Answer: $132, 11, 21$

3. (a) Since $1^2=1$, $2^2=4$, $12^2=144$, and $13^2=169$, then the perfect squares between 2 and 150 are 2^2 through 12^2, of which there are 11.

(b) To find the points of intersection of the line with equation $y=-2x+t$ and the parabola with equation $y=(x-1)^2+1$, we equate values of y, to obtain

$$\begin{aligned}(x-1)^2+1&=-2x+t\\ x^2-2x+1+1&=-2x+t\\ x^2&=t-2\end{aligned}$$

The points of intersection thus have x-coordinates $x=\sqrt{t-2}$ and $x=-\sqrt{t-2}$.
The point P in the first quadrant has a positive x-coordinate, and so its x-coordinate is $\sqrt{t-2}$.
Thus, the y-coordinate of P is $y=-2\sqrt{t-2}+t$.
Since the answer to (a) is 11, then $t=11$.
This means that the y-coordinate of P is $y=-2\sqrt{11-2}+11=-2\cdot 3+11=5$.

(c) To find the x-intercept of the line with equation $(k-1)x+(k+1)y=t$, we set $y=0$ and obtain $(k-1)x=t$ or $x=\dfrac{t}{k-1}$. We assume that $k\neq 1$.
To find the y-intercept of the line with equation $(k-1)x+(k+1)y=t$, we set $x=0$ and obtain $(k+1)y=t$ or $y=\dfrac{t}{k+1}$. We assume that $k\neq -1$.
The triangle formed by the x-axis, the y-axis, and this line has vertices at the x-intercept, the y-intercept, and the origin.
This triangle is right-angled at the origin, so its area equals $\dfrac{1}{2}\cdot\dfrac{t}{k-1}\cdot\dfrac{t}{k+1}=\dfrac{t^2}{2(k^2-1)}$.
Since we are told that this area is 10, then $\dfrac{t^2}{2(k^2-1)}=10$ which gives $k^2-1=\dfrac{t^2}{20}$ or $k^2=1+\dfrac{t^2}{20}$.
Since the answer to (b) is 5, then $t=5$.
Therefore, $k^2=1+\dfrac{5^2}{20}=1+\dfrac{5}{4}=\dfrac{9}{4}$.
Since $t>0$, then for the x- and y-intercepts of the line to be positive (putting the triangle in the first quadrant), then $k>0$.
Since $k^2=\dfrac{9}{4}$, then $k=\dfrac{3}{2}$.

Answer: $11, 5, \dfrac{3}{2}$