The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

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2019 Canadian Team Mathematics Contest

Team Problems

IMPORTANT NOTES:

- Calculating devices are not permitted.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

PROBLEMS:

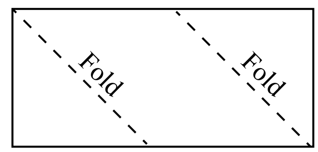
1. For what value of x is $4x - 8 + 3x = 12 + 5x$?
2. What is the value of $3.5 \times 2.5 + 6.5 \times 2.5$?
3. Ada is younger than Darwyn. Max is younger than Greta. James is older than Darwyn. Max and James are the same age. Which of the five people is the oldest?
4. Determine the average (mean) of $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$ as a fraction in lowest terms.
5. Suppose that

$$M = 1^5 + 2^4 \times 3^3 - 4^2 \div 5^1$$
$$N = 1^5 - 2^4 \times 3^3 + 4^2 \div 5^1$$

   What is the value of $M + N$?

6. How many four-digit palindromes $abba$ have the property that the two-digit integer ab and the two-digit integer ba are both prime numbers? (For example, 2332 does not have this property, since 23 is prime but 32 is not.)
7. Adia writes a list in increasing order of the integers between 1 and 100, inclusive, that cannot be written as the product of two consecutive positive integers. What is the 40th integer in her list?
8. For how many ordered pairs of positive integers (a, b) is $1 < a + b < 22$?
9. Shelly-Ann normally runs along the Laurel Trail at a constant speed of 8 m/s. One day, one-third of the trail is covered in mud, through which Shelly-Ann can only run one-quarter of her normal speed, and it takes her 12 s to run the entire length of the trail. How long is the trail, in metres?
10. Determine the value of a for which $5^a + 5^{a+1} = \sqrt{4500}$.

11. Ezekiel has a rectangular piece of paper with an area of 40. The width of the paper is more than twice the height. He folds the bottom left and top right corners at 45° and creates a parallelogram with an area of 24. What is the perimeter of the original rectangle?



12. What is the value of $123456^2 - 123455 \times 123457$?

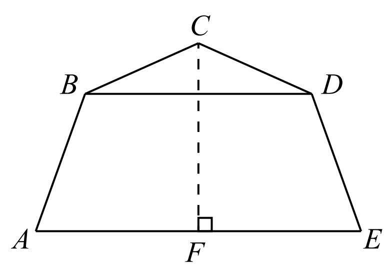
13. Determine the value of $(\log_2 4)(\log_4 6)(\log_6 8)$.

14. The integers x, y and z satisfy $\frac{x}{5} = \frac{6}{y} = \frac{z}{2}$. What is the largest possible value of $x + y + z$?

15. Suppose that $\mathbf{G} = 10^{100}$. ($\mathbf{G}$ is known as a *googol*.) How many times does the digit 9 occur in the integer equal to $\mathbf{G} - 1009^2$?

16. Suppose that $f(x) = x^4 - x^3 - 1$ and $g(x) = x^8 - x^6 - 2x^4 + 1$. If $g(x) = f(x)h(x)$, determine the polynomial function $h(x)$.

17. In the diagram, pentagon $ABCDE$ is symmetrical about altitude CF. Also, $AE = 200$, $CF = 80\sqrt{3}$, $\angle ABC = 150°$, and $\angle BCD = 120°$. Determine the vertical distance between AE and BD.



18. The height of Cylinder A is equal to its diameter. The height and diameter of Cylinder B are each twice those of Cylinder A. The height of Cylinder C is equal to its diameter. The volume of Cylinder C is the sum of the volumes of Cylinders A and B. What is the ratio of the diameter of Cylinder C to the diameter of Cylinder A?

19. Suppose that $f(x) = a(x-b)(x-c)$ is a quadratic function where a, b and c are distinct positive integers less than 10. For each choice of a, b and c, the function $f(x)$ has a minimum value. What is the minimum of these possible minimum values?

20. The integers from 1 to k are concatenated to form the integer $N = 123456789101112\ldots$. Determine the smallest integer value of $k > 2019$ such that N is divisible by 9.

21. The real numbers $x_1, x_2, x_3, \ldots, x_n$ are the consecutive terms of an arithmetic sequence. If

$$\frac{x_2}{x_1 + x_3} + \frac{x_3}{x_2 + x_4} + \frac{x_4}{x_3 + x_5} + \cdots + \frac{x_{n-2}}{x_{n-3} + x_{n-1}} + \frac{x_{n-1}}{x_{n-2} + x_n} = 1957$$

what is the value of n?

22. Suppose that $f_1(x) = \dfrac{1}{2-x}$. For each positive integer $n \geq 2$, define $f_n(x) = f_1\left(f_{n-1}(x)\right)$ for all real numbers x in the domain of $f_1\left(f_{n-1}(x)\right)$. The value of $f_{2019}(4)$ can be written as $\dfrac{a}{b}$ where a and b are positive integers with no common divisor larger than 1. What is (a, b)?

23. The numbers p, q, r, and t satisfy $p < q < r < t$. When these numbers are paired, each pair has a different sum and the four largest sums are 19, 22, 25, and 28. What is the the sum of the possible values for p?

24. Suppose that α and β are the two positive roots of the equation

$$x^2 - \sqrt{13}x^{\log_{13} x} = 0$$

Determine the value of $\alpha\beta$.

25. In the diagram, $ABCD$ is a square. Point F is on AB with $BF = 2AF$. Point E is on AD with $\angle FEC = \angle BCE$. If $0° < \angle ECD < 45°$, what is the value of $\tan(\angle ECD)$?