The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2021 Canadian Intermediate Mathematics Contest

Wednesday, November 17, 2021
(in North America and South America)

Thursday, November 18, 2021
(outside of North America and South America)

Solutions

Part A

1. The area of a rectangle is equal to its length multiplied by its width, and so width equals area divided by length.
   The width of Rectangle A is equal to $\dfrac{36 \text{ cm}^2}{6 \text{ cm}} = 6$ cm.
   The width of Rectangle B is equal to $\dfrac{36 \text{ cm}^2}{12 \text{ cm}} = 3$ cm.
   The width of Rectangle C is equal to $\dfrac{36 \text{ cm}^2}{9 \text{ cm}} = 4$ cm.
   The smallest of these widths is 3 cm, and so $x = 3$.
   (Note that since the three rectangles have the same area, the smallest width would be the one with the largest length.)

   ANSWER: $x = 3$

2. The prime numbers that are greater than 10 and less than 20 are 11, 13, 17, 19.
   (Since $12 = 2 \times 6$ and $14 = 2 \times 7$ and $15 = 3 \times 5$ and $16 = 2 \times 8$ and $18 = 2 \times 9$, none of these integers is a prime number.)
   The sum of these prime numbers is
   $$11 + 13 + 17 + 19 = (11 + 19) + (13 + 17) = 30 + 30 = 60$$

   ANSWER: 60

3. From the 22nd floor to the nth floor, Taya and Jenna each go up $n - 22$ floors.
   Since Taya goes from each floor to the next in 15 seconds, this takes her $15 \times (n - 22)$ seconds.
   Since Jenna waits for 2 minutes, she waits for 120 seconds. Since Jenna goes from each floor to the next in 3 seconds, it takes a total of $120 + 3 \times (n - 22)$ seconds to reach the nth floor.
   Since their travel times are equal, then
   $$\begin{aligned} 15 \times (n - 22) &= 120 + 3 \times (n - 22) \\ 15 \times (n - 22) - 3 \times (n - 2) &= 120 \\ 12 \times (n - 22) &= 120 \\ n - 22 &= 10 \end{aligned}$$
   and so $n = 32$.

   ANSWER: $n = 32$

4. Since $\angle DQC$ is a straight angle, $\angle RQP = 180^\circ - \angle RQD - \angle PQC = 180^\circ - w^\circ - x^\circ$.
   Since $\angle BPC$ is a straight angle, $\angle RPQ = 180^\circ - \angle RPB - \angle QPC = 180^\circ - z^\circ - y^\circ$.
   The sum of the measures of the angles in $\triangle PQR$ is 180°, and so
   $$\begin{aligned} \angle RQP + \angle RPQ + \angle PRQ &= 180^\circ \\ (180^\circ - w^\circ - x^\circ) + (180^\circ - z^\circ - y^\circ) + 30^\circ &= 180^\circ \\ 390^\circ - w^\circ - x^\circ - y^\circ - z^\circ &= 180^\circ \\ 210^\circ &= w^\circ + x^\circ + y^\circ + z^\circ \end{aligned}$$
   and so $w + x + y + z = 210$.

   ANSWER: 210

5. Using the given rules, the first 7 numbers in the list are

$$\mathbf{3}, \mathbf{4}, \frac{\mathbf{5}}{\mathbf{3}}, \frac{(5/3)+1}{4} = \frac{8/3}{4} = \frac{\mathbf{2}}{\mathbf{3}}, \frac{(2/3)+1}{5/3} = \frac{5/3}{5/3} = \mathbf{1}, \frac{1+1}{2/3} = \frac{2}{2/3} = \mathbf{3}, \frac{3+1}{1} = \mathbf{4}$$

Since the 6th number equals the 1st number, the 7th number equals the 2nd number, and each number in the list depends on the previous two numbers, then the 8th number equals the 3rd number, which will mean that the 9th number will equal the 4th number, and so on.
In other words, the list will repeat every 5 numbers since each group of 5 numbers is produced in exactly the same way as the previous 5 numbers.
The sum of the first 5 numbers is $3 + 4 + \frac{5}{3} + \frac{2}{3} + 1 = 8 + \frac{7}{3} = 10\frac{1}{3}$.
Since $2021 \div 10\frac{1}{3} \approx 195.6$, then 195 groups of 5 numbers will still have a sum less than 2021.
In fact, since $195 \times 10\frac{1}{3} = 1950 + 65 = 2015$ and $195 \times 5 = 975$, then the sum of the first 975 numbers in the list is 2015.
We make a chart of the next terms and the sum to that point until we reach a sum that is greater than 2021 and is an odd integer:

Term #	976	977	978	979	980	981	982
Term	3	4	$\frac{5}{3}$	$\frac{2}{3}$	1	3	4
Sum to this point	2018	2022	$2023\frac{2}{3}$	$2024\frac{1}{3}$	$2025\frac{1}{3}$	$2028\frac{1}{3}$	$2032\frac{1}{3}$

Term #	983	984	985	986	987	988	989
Term	$\frac{5}{3}$	$\frac{2}{3}$	1	3	4	$\frac{5}{3}$	$\frac{2}{3}$
Sum to this point	2034	$2034\frac{2}{3}$	$2035\frac{2}{3}$	$2038\frac{2}{3}$	$2042\frac{2}{3}$	$2044\frac{1}{3}$	2045

Therefore, the smallest positive integer N for which the sum of the first N numbers is equal to an odd integer greater than 2021 is $N = 989$ when the sum is 2045.

ANSWER: 989

6. *Solution 1*

Suppose that Dragomir removes the socks one at a time. There are 12 socks that he can remove 1st, then 11 socks 2nd, then 10 socks 3rd, and finally 9 socks 4th.
This means that there are $12 \times 11 \times 10 \times 9$ ways in which he can remove 4 socks one at a time.
Suppose that there is exactly one pair among the 4 socks removed. There are 6 different possibilities for how this pair was formed: 1st and 2nd socks match, 1st and 3rd match, 1st and 4th match, 2nd and 3rd match, 2nd and 4th match, 3rd and 4th match.
We determine the number of ways in which each of these possibilities can arise.

Case 1: Matching pair 1st and 2nd
There are 12 socks that he can pick 1st (no restrictions).
There is 1 sock that he can pick 2nd (pair of 1st).
There are 10 socks that he can pick 3rd (any of the 10 remaining socks).
There are 8 socks that he can pick 4th (any of the 9 remaining socks other than pair of 3rd).
In this case, there are $12 \times 1 \times 10 \times 8$ ways.

Case 2: Matching pair 1st and 3rd
There are 12 socks that he can pick 1st (no restrictions).
There are 10 socks that he can pick 2nd (all but pair of 1st).
There is 1 sock that he can pick 3rd (pair of 1st).
There are 8 socks that he can pick 4th (any other than pair of 2nd).
In this case, there are $12 \times 10 \times 1 \times 8$ ways.

Case 3: Matching pair 1st and 4th
There are 12 socks that he can pick 1st (no restrictions).
There are 10 socks that he can pick 2nd (all but pair of 1st).
There are 8 socks that he can pick 3rd (all but pair of 1st and pair of 2nd).
There is 1 sock that he can pick 4th (pair of 1st).
In this case, there are $12 \times 10 \times 8 \times 1$ ways.

Case 4: Matching pair 2nd and 3rd
There are 12 socks that he can pick 1st (no restrictions).
There are 10 socks that he can pick 2nd (all but pair of 1st).
There is 1 sock that he can pick 3rd (pair of 2nd).
There are 8 socks that he can pick 4th (all but pair of 1st).
In this case, there are $12 \times 10 \times 1 \times 8$ ways.

Case 5: Matching pair 2nd and 4th
There are 12 socks that he can pick 1st (no restrictions).
There are 10 socks that he can pick 2nd (all but pair of 1st).
There are 8 socks that he can pick 3rd (all but pair of 1st and pair of 2nd).
There is 1 sock that he can pick 4th (pair of 2nd).
In this case, there are $12 \times 10 \times 8 \times 1$ ways.

Case 6: Matching pair 3rd and 4th
There are 12 socks that he can pick 1st (no restrictions).
There are 10 socks that he can pick 2nd (all but pair of 1st).
There are 8 socks that he can pick 3rd (all but pair of 1st and pair of 2nd).
There is 1 sock that he can pick 4th (pair of 3rd).
In this case, there are $12 \times 10 \times 8 \times 1$ ways.

In each of the six cases, there are $12 \times 10 \times 8 \times 1$ ways in which the socks can be chosen.

Therefore, the overall probability is $\dfrac{6 \times 12 \times 10 \times 8 \times 1}{12 \times 11 \times 10 \times 9} = \dfrac{6 \times 8}{11 \times 9} = \dfrac{2 \times 8}{11 \times 3} = \dfrac{16}{33}$.

Solution 2

Since there are 12 socks in the drawer, there are $\dbinom{12}{4} = \dfrac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = 11 \times 5 \times 9 = 495$ ways of choosing 4 socks.

Next, we count the number of ways in which the 4 chosen socks can include exactly 1 matching pair.

There are 6 possible matching pairs that can be chosen.

The remaining 2 socks must come from 2 different pairs of the remaining 5 pairs.

If we label the remaining pairs A, B, C, D, E, there are 10 possible two-pair combinations from which these 2 socks can be chosen: AB, AC, AD, AE, BC, BD, BE, CD, CE, DE. (We could also use a binomial coefficient to calculate this total.)

For each of the 2 pairs from which these 2 socks will be chosen, there are in fact 2 socks to choose.

Thus, the total number of ways in which the 4 socks can be chosen that satisfy the given conditions is $6 \times 10 \times 2 \times 2 = 240$.

This means that the probability is $\dfrac{240}{495} = \dfrac{16 \times 15}{33 \times 15} = \dfrac{16}{33}$.

Solution 3

Since there are 12 socks in the drawer, there are $\dbinom{12}{4} = \dfrac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = 11 \times 5 \times 9 = 495$ ways of choosing 4 socks.

These 4 socks can include 0 matching pairs, 1 matching pair, or 2 matching pairs.

We count the number of ways in which the 4 chosen socks can include 0 matching pairs or 2 matching pairs and subtract this from the total number of ways of choosing 4 socks to obtain the number of ways of choosing 1 matching pair.

To choose 0 matching pairs, we choose 4 of the 6 pairs and choose 1 of the 2 socks in each pair.

There are $\dbinom{6}{4} = \dfrac{6 \times 5 \times 4 \times 3}{4 \times 3 \times 2 \times 1} = 15$ ways of choosing 4 of 6 pairs, and $2^4 = 16$ ways of choosing 1 of the 2 socks in each of 4 pairs.

Thus, there are $15 \times 16 = 240$ ways of choosing exactly 0 matching pairs.

To choose 2 matching pairs, we choose 2 of the 6 pairs and choose both socks in each pair.

There are $\dbinom{6}{2} = \dfrac{6 \times 5}{2 \times 1} = 15$ ways of choosing 2 of 6 pairs, and 1 way to choose both socks in each pair.

Thus, there are $15 \times 1 = 15$ ways of choosing exactly 2 matching pairs.

This means that there are $495 - 240 - 15 = 240$ ways of choosing exactly 1 matching pair.

This means that the probability is $\dfrac{240}{495} = \dfrac{16 \times 15}{33 \times 15} = \dfrac{16}{33}$.

Answer: $\dfrac{16}{33}$

Part B

1. (a) Since Karol uses 28 cups of Gluze, and the recipe calls for 4 cups of Gluze, he must have repeated the recipe $28 \div 4 = 7$ times.
Since the recipe calls for 3 cups of Blurpos, and the recipe was repeated 7 times, he must have used $7 \times 3 = 21$ cups of Blurpos.

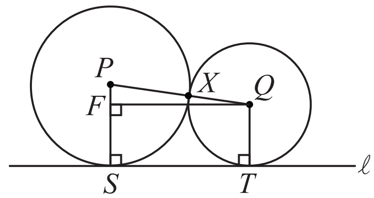
   (b) Suppose that Karol uses 48 cups of Gluze.
Since $48 \div 4 = 12$, he uses his recipe 12 times and so uses $12 \times 3 = 36$ cups of Blurpos.
Suppose instead that Karol uses 48 cups of Blurpos.
Since $48 \div 3 = 16$, he uses his recipe 16 times and so uses $16 \times 4 = 64$ cups of Gluze.
Therefore, the two possible values of N are 36 and 64.

   (c) Karol starts with 64 cups of Gluze and 42 cups of Blurpos. One of these ingredients will be the "limiting ingredient"; that is, when Karol repeats his recipe, he will probably run out of one of the ingredients before the other.
If Karol used all 64 cups of Gluze, then he must have used his recipe $64 \div 4 = 16$ times.
If Karol used all 42 cups of Blurpos, then he must have used his recipe $42 \div 3 = 14$ times.
Therefore, he can make the recipe at most 14 times since after the 14th time, he would run out of Blurpos.
Each time Karol uses his recipe, he makes 60 Zippies.
When he uses his recipe 14 times, he makes $14 \times 60 = 840$ Zippies.

   (d) We start by assuming that Karol makes 1 recipe worth of Zippies.
In other words, suppose that Karol makes 60 Zippies using 4 cups of Gluze and 3 cups of Blurpos.
He sells each Zippie for \$0.50, which earns him $60 \times \$0.50 = \$30.00$.
His profit on each Zippie is \$0.30, which means that his total profit is $60 \times \$0.30 = \$18.00$.
Since he earns \$30.00 and his profit is \$18.00, his costs must be $\$30.00 - \$18.00 = \$12.00$.
Since Karol pays \$1.80 for each cup of Gluze, he pays a total of $4 \times \$1.80 = \$7.20$ for Gluze.
This means that he pays $\$12.00 - \$7.20 = \$4.80$ for Blurpos.
Since he uses 3 cups of Blurpos, Karol pays $\$4.80 \div 3 = \$1.60$ per cup of Blurpos.

2. (a) Since $ABCD$ is a rectangle and $AB = 3$, then $DC = 3$.
Thus, $DE = DC + CE = 3 + 6 = 9$.
Now, the area of trapezoid $ABED$ is 48.
Since $ABCD$ is a rectangle, then AD is perpendicular to DE which means that AD is a height of trapezoid $ABED$.
The area of trapezoid $ABED$ equals $\frac{1}{2}(AB + DE) \times AD$.
(We could instead note that trapezoid $ABED$ is made up of a rectangle with base 3 and an unknown height, and a right-angled triangle with base 6 and the same unknown height.)
Therefore, $\frac{1}{2}(3 + 9) \times AD = 48$ and so $6 \times AD = 48$ which gives $AD = 8$.
Since $ABCD$ is a rectangle, then $BC = AD = 8$.
Finally, by the Pythagorean Theorem, $BE = \sqrt{BC^2 + CE^2} = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$.

(b) Draw a perpendicular from Q to F on PS.



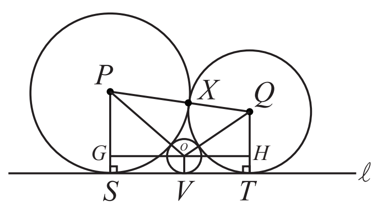
Quadrilateral $FQTS$ has three right angles and so it must be a rectangle.
Therefore, $FS = QT$. Since QT is a radius, then $FS = QT = 16$.
Since PS is a radius, then $PS = 25$.
Therefore, $PF = PS - FS = 25 - 16 = 9$.
We note that $PQ = PX + XQ = 25 + 16 = 41$ (they are both radii).
By the Pythagorean Theorem in $\triangle PFQ$, which is right-angled at F,

$$FQ = \sqrt{PQ^2 - PF^2} = \sqrt{41^2 - 9^2} = \sqrt{1681 - 81} = 40$$

Since $FQTS$ is a rectangle, $ST = FQ = 40$.
Finally, the area of trapezoid $PQTS$ equals

$$\tfrac{1}{2}(PS + QT) \times ST = \tfrac{1}{2}(25 + 16) \times 40 = 41 \times 20 = 820$$

(c) Suppose that the centre of the circle of radius r is O and that this circle touches the line ℓ at point V.
Join P to O and Q to O.
Join V to O. Since the circle is tangent to ℓ at V, OV is perpendicular to ℓ. Also, draw perpendiculars from O to G on PS and from O to H on QT.



As in (b), $GOVS$ is a rectangle with $GS = OV = r$ and $SV = GO$.
Since $PS = 25$, then $PG = PS - GS = 25 - r$.
Since PO joins the centres of two tangent circles, then the length of PO equals the sum of the radii of the two circles. In other words, $PO = 25 + r$.
By the Pythagorean Theorem in $\triangle POG$,

$$GO^2 = PO^2 - PG^2 = (25 + r)^2 - (25 - r)^2 = (625 + 50r + r^2) - (625 - 50r + r^2) = 100r$$

Since $GO > 0$, then $GO = \sqrt{100r} = \sqrt{100} \times \sqrt{r} = 10\sqrt{r}$.
Thus, $SV = GO = 10\sqrt{r}$.
Using a similar analysis,

$$HO^2 = QO^2 - QH^2 = (16 + r)^2 - (16 - r)^2 = (256 + 32r + r^2) - (256 - 32r + r^2) = 64r$$

Since $HO > 0$, then $HO = \sqrt{64r} = \sqrt{64} \times \sqrt{r} = 8\sqrt{r}$.
Thus, $TV = HO = 8\sqrt{r}$.
From (b), $ST = 40$ and so $SV + TV = 40$.
Therefore, $10\sqrt{r} + 8\sqrt{r} = 40$ which gives $18\sqrt{r} = 40$ and so $\sqrt{r} = \frac{40}{18} = \frac{20}{9}$.
Finally, $r = \left(\frac{20}{9}\right)^2 = \frac{400}{81}$.

3. (a) The key move in Beryl's strategy is to choose 2 on turn #2 (her first turn).
On turn #1, Alphonse must choose 1, for a running total of 1.
On turn #2, Beryl chooses 2, for a running total of 3.
On turn #3, Alphonse can choose 1, 2 or 3, for a running total of 4, 5 or 6, respectively.
On turn #4, Beryl can choose 1, 2, 3, or 4, and so chooses 4, 3 or 2 (corresponding to Alphonse's choices of 1, 2 or 3, respectively) to achieve a running total of 8 in each case.
Therefore, if Beryl chooses 2 on turn #2 and then chooses appropriately on turn #4, she is guaranteed to win.

(b) Alphonse has a winning strategy when $N = 17$.
To show this, we use the fact that $1 + 4 + 5 + 7 = 17$.

On turn #1, Alphonse must choose 1.

On turn #2, Beryl can choose 1 or 2; on turn #3, Alphonse can choose 1, 2 or 3.
No matter what Beryl chooses on turn #2, Alphonse can choose a number to make the sum of their numbers on those two turns equal to 4.
(If Beryl chooses 1, Alphonse chooses 3. If Beryl chooses 2, Alphonse chooses 2.)
Thus, Alphonse can ensure that the running total after 3 turns is $1 + 4 = 5$.

On turn #4, Beryl can choose 1, 2, 3, or 4; on turn #5, Alphonse can choose 1, 2, 3, 4, or 5.
No matter what Beryl chooses on turn #4, Alphonse can choose a number to make the sum of their numbers on those two turns equal to 5.
(If Beryl chooses 1, 2, 3, or 4, Alphonse chooses 4, 3, 2, or 1, respectively.)
Thus, Alphonse can ensure that the running total after 5 turns is $5 + 5 = 10$.

On turn #6, Beryl can choose 1, 2, 3, 4, 5, or 6; on turn #7, Alphonse can choose 1, 2, 3, 4, 5, 6, or 7.
No matter what Beryl chooses on turn #6, Alphonse can choose a number to make the sum of their numbers on those two turns equal to 7.
(If Beryl chooses x with $1 \leq x \leq 6$, Alphonse chooses $7 - x$. Note that $1 \leq 7 - x \leq 6$.)
Thus, Alphonse can ensure that the running total after 7 turns is $10 + 7 = 17$.
Note as well that, following this method, the largest that Beryl can make the running total after turn #6 is $10 + 6 = 16$ and so she cannot win "early" and thus break Alphonse's strategy.

Therefore, Alphonse has a winning strategy when $N = 17$.

(c) When $N = 1$, Alphonse wins automatically.
When $N = 2$ or $N = 3$, Beryl has a winning strategy by choosing 1 or 2, respectively, on her first turn.
When $N = 4$ or $N = 5$, Alphonse has a winning strategy by choosing 2 or 3 if Beryl chooses 1 (making the previous running total 2) or by choosing 1 or 2 if Beryl chooses 2 (making the previous running total 3).
When $N = 6$, Beryl has a winning strategy by choosing 1 on turn #2, which forces Alphonse to make the running total after turn #3 equal to 3, 4 or 5, after which Beryl can win by choosing 3, 2 or 1 on turn #4.
When $N = 7$, Beryl has a winning strategy by choosing 1 on turn #2, which forces Alphonse to make the running total after turn #3 equal to 3, 4 or 5, after which Beryl can win by choosing 4, 3 or 2 on turn #4.
We saw what happens when $N = 8$ in (a).
We can continue these types of argument to show that Alphonse has a winning strategy

when $N = 9$, $N = 10$, and $N = 11$, and Beryl has a winning strategy when $N = 12$, $N = 13$, $N = 14$, and $N = 15$.
So far, we have:

Winning strategy for Alphonse	Winnning strategy for Beryl
$N = 1$	$N = 2, 3$
$N = 4, 5$	$N = 6, 7, 8$
$N = 9, 10, 11$	$N = 12, 13, 14, 15$

These small values of N suggest that Alphonse has a winning strategy for k values of N starting with $N = k^2$ (that is, for $N = k^2$ up to and including $N = k^2 + k - 1$) and Beryl has a winning strategy for $k + 1$ values of N leading up to $N = (k + 1)^2 - 1$ (that is, for $N = k^2 + k$ up to and including $N = k^2 + 2k$).

We note that $45^2 = 2025$ and $46^2 = 2116$.
Based on our hypotheses, our guess is that Beryl has a winning strategy for

$$N = 1980, 1981, 1982, \ldots, 2023, 2024$$

while Alphonse has a winning strategy for

$$N = 2025, 2026, 2027, \ldots, 2068, 2069$$

and Beryl has a winning strategy for

$$N = 2070, 2071, 2072, \ldots, 2114, 2115$$

If this is true, the smallest positive integer m with $m > 2021$ for which Alphonse has a winning strategy when $N = m$ and Beryl has a winning strategy when $N = m + 1$ is $m = 2069$.
In order to prove that this is true, we need to show that none of $m = 2022$ through $m = 2068$ satisfy the given conditions and that $m = 2069$ does. To do this, we prove that Beryl has a winning strategy for $N = 2022$ and $N = 2023$, Alphonse has a winning strategy for $N = 2025$ through $N = 2069$, and Beryl has a winning strategy for $N = 2070$. (Can you explain why we do not need to consider $N = 2024$?)

Consider $N = 2025$.
Note that

$$\begin{aligned} 1 + 3 + 5 + \ldots + 85 + 87 + 89 &= 45 + (1 + 89) + (3 + 87) + \cdots (41 + 49) + (43 + 47) \\ &= 45 + 22 \cdot 90 \\ &= 2025 \end{aligned}$$

The following table summarizes Alphonse's strategy, based on Beryl's moves:

Turn(s)	Details	Condition
#1	A must choose 1	
#2 and #3	B can choose 1 or 2 A chooses 2 or 1, respectively	Sum over two turns is 3
#4 and #5	B can choose 1, 2, 3, or 4 A chooses 4, 3, 2, or 1, respectively	Sum over two turns is 5
$\vdots$	$\vdots$	$\vdots$
#$(2q)$ and #$(2q + 1)$ $(1 \leq q \leq 44)$	B can choose $1, 2, 3, \ldots, 2q - 1$, or $2q$ A chooses $2q, 2q - 1, \ldots, 2, 1$, respectively	Sum over two turns is $2q + 1$

Therefore, after 89 turns, using this strategy, Alphonse can ensure that the running total is

$$1 + 3 + 5 + \cdots + 87 + 89 = 2025$$

and so Alphonse has a winning strategy.

Consider the case where N equals one of $2026, 2027, \ldots, 2068, 2069$.
Let $k = N - 2025$. Note that $1 \leq k \leq 44$.
Alphonse's winning strategy for these values of N is to reproduce the strategy above for $N = 2025$ with the change that for k of his turns from turn #3, turn #5, …, turn #87, turn #89 (there are 44 turns in this list), he makes the combined sum 1 larger than the corresponding combined sum in strategy for $N = 2025$.
By doing this, after turn #89, Alphonse can ensure that the running total is $2025+k = N$, and so Alphonse has a winning strategy.
(Note that in the $N = 2025$ case, by choosing one of $2q, 2q-1, \ldots, 2, 1$ on turn $2q+1$, Alphonse could ensure that the sum of the numbers chosen on turns $\#(2q)$ and $\#(2q+1)$ is $2q+1$. Since Alphonse can choose up to $2q+1$ on turn $2q+1$, then he can make his choice one larger on turn $2q+1$ than he did in the $N = 2025$ case and so make the sum of the numbers on turns $2q$ and $2q+1$ equal to $(2q+1)+1$.)

Consider $N = 2070$.
Note that

$$\begin{aligned} 2+4+6+\ldots+86+88+90 &= 90+(2+88)+(4+86)+\cdots(42+48)+(44+46) \\ &= 90+22\cdot 90 \\ &= 2070 \end{aligned}$$

The following table summarizes Beryl's strategy, based on Alphonse's moves:

Turn(s)	Details	Condition
#1 and #2	A must choose 1 B chooses 1	Sum over two turns is 2
#3 and #4	A can choose 1, 2 or 3 B chooses 3, 2 or 1, respectively	Sum over two turns is 4
#5 and #6	A can choose 1, 2, 3, 4, or 5 B chooses 5, 4, 3, 2, or 1, respectively	Sum over two turns is 6
$\vdots$	$\vdots$	$\vdots$
$\#(2q-1)$ and $\#(2q)$ $(1 \leq q \leq 45)$	A can choose $1, 2, 3, \ldots, 2q-2$, or $2q-1$ A chooses $2q-1, 2q-2, \ldots, 2, 1$, respectively	Sum over two turns is $2q$

Therefore, after 90 turns, using this strategy, Beryl can ensure that the running total is

$$2+4+6+\ldots+86+88+90 = 2070$$

and so Beryl has a winning strategy.

Consider $N = 2022$ and $N = 2023$.
Note that

$$2+4+7+9+\cdots+87+89 = 6+(7+89)+(9+87)+\cdots+(47+49) = 6+21\times 96 = 2022$$

and

$$2+5+7+9+\cdots+87+89 = 7+(7+89)+(9+87)+\cdots+(47+49) = 7+21\times 96 = 2023$$

(The first sum starts with $2 + 4$ and continues with the sum of all of the odd numbers from 7 to 89, inclusive. The second sum starts with 2 and continues with the sum of all of the odd numbers from 5 to 89, inclusive.)
By following the idea of the cases above, in particular making the sum of the numbers on turn #1 and turn #2 equal to 2, the sum of the numbers on turn #3 and turn #4 equal to 4 or 5, and the sum of the numbers on the pairs of turns #5 and #6 through #87 and #88 equal to $7, 9, \ldots, 87, 89$, respectively, Beryl can guarantee that the running total after turn #88 is 2022 or 2023. (For example, on turns #5 and #6, if Alphonse chooses 1, 2, 3, 4, or 5 on turn #5, Beryl then chooses 6, 5, 4, 3, or 2, respectively, on turn #6.)

In summary, we have shown that Beryl has a winning strategy for $N = 2022$ and $N = 2023$, Alphonse has a winning strategy for $N = 2025$ through $N = 2069$, and Beryl has a winning strategy for $N = 2070$, which shows that the value of m in question is $m = 2069$.