The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Canadian Senior Mathematics Contest

Wednesday, November 17, 2021
(in North America and South America)

Thursday, November 18, 2021
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 2 hours

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Do not open this booklet until instructed to do so.
There are two parts to this paper. The questions in each part are arranged roughly in order of increasing difficulty. The early problems in Part B are likely easier than the later problems in Part A.

PART A

1. This part consists of six questions, each worth 5 marks.
2. **Enter the answer in the appropriate box in the answer booklet.**
   For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

PART B

1. This part consists of three questions, each worth 10 marks.
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, they will be supplied by your supervising teacher. Insert these pages into your answer booklet. Write your name, school name, and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

At the completion of the contest, insert your student information form inside your answer booklet.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on the website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some students may be shared with other mathematical organizations for other recognition opportunities.

Canadian Senior Mathematics Contest

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write solutions in the answer booklet provided.
3. Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.
4. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
5. Diagrams are not drawn to scale. They are intended as aids only.
6. No student may write both the Canadian Senior Mathematics Contest and the Canadian Intermediate Mathematics Contest in the same year.

PART A

For each question in Part A, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded only if relevant work is shown in the space provided in the answer booklet.

Useful Fact for Part A:
The volume of a square-based pyramid equals one-third times the area of its base times its height. Similarly, the volume of a triangular-based pyramid equals one-third times the area of its base times its height.

1. In the diagram, a row of 3 squares is made using 10 toothpicks. In total, how many toothpicks are needed to make a row of 11 squares?

2. The operation ∇ is defined by $a \nabla b = (a+1)(b-2)$ for real numbers a and b. For example, $4 \nabla 5 = (4+1)(5-2) = 15$. If $5 \nabla x = 30$, what is the value of x?

3. Consider the points $P(0,0)$, $Q(4,0)$ and $R(1,2)$. The line with equation $y = mx + b$ is parallel to PR and passes through the midpoint of QR. What is the value of b?

4. In an experiment, 1000 people receive Medicine A and 1000 different people receive Medicine B. The 2000 people are asked whether they have severe side effects, mild side effects, or no side effects. The following information is obtained from the experiment:

   (i) The probability that a random person has severe side effects is $\frac{3}{25}$.

   (ii) The probability that a random person with severe side effects was given Medicine A is $\frac{2}{3}$.

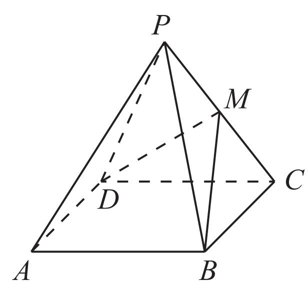
   (iii) The probability that a random person who was given Medicine A has severe or mild side effects is $\frac{19}{100}$.

   (iv) The probability that a random person who was given Medicine B has severe or mild side effects is $\frac{3}{20}$.

   What is the probability that a random person with mild side effects was given Medicine B?

5. What are all real numbers $x > 0$ for which $\log_2(x^2)+2\log_x 8 = \dfrac{392}{\log_2(x^3) + 20\log_x(32)}$?

6. In the diagram, $PABCD$ is a pyramid with square base $ABCD$ and with $PA = PB = PC = PD$. Suppose that M is the midpoint of PC and that $\angle BMD = 90°$. Triangular-based pyramid $MBCD$ is removed by cutting along the triangle defined by the points M, B and D. The volume of the remaining solid $PABMD$ is 288. What is the length of AB?



PART B

For each question in Part B, your solution must be well-organized and contain words of explanation or justification. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

1. (a) Factor $x^2 - 4$ as the product of two linear expressions in x.
   (b) Determine the integer k for which $98^2 - 4 = 100k$.
   (c) Determine the positive integer n for which $(20 - n)(20 + n) = 391$.
   (d) Prove that 3 999 991 is not a prime number. (A *prime number* is a positive integer greater than 1 whose only positive divisors are 1 and itself. For example, 7 is a prime number.)

2. If n is a positive integer, a *Leistra sequence* is a sequence $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ with n terms with the following properties:
   (P1) Each term $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ is an even positive integer.
   (P2) Each term $a_2, a_3, \ldots, a_{n-1}, a_n$ is obtained by dividing the previous term in the sequence by an integer between 10 and 50, inclusive. (For a specific sequence, the divisors used do not all have to be the same.)
   (P3) There is no integer m between 10 and 50, inclusive, for which $\dfrac{a_n}{m}$ is an even integer.

   For example,

Leistra sequences
1000, 50, 2
1000, 100, 4
3000, 300, 30, 2
106

Not Leistra sequences	Reason
1000, 50, 1	(P1) fails – includes an odd integer
1000, 200, 4	(P2) fails – divisor 5 falls outside range ($\frac{1000}{200} = 5$)
3000, 300, 30	(P3) fails – can be extended with $\frac{30}{15} = 2$
104	(P3) fails – can be extended with $\frac{104}{13} = 8$

   (a) Determine all Leistra sequences with $a_1 = 216$.
   (b) How many Leistra sequences have $a_1 = 2 \times 3^{50}$?
   (c) How many Leistra sequences have $a_1 = 2^2 \times 3^{50}$?
   (d) Determine the number of Leistra sequences with $a_1 = 2^3 \times 3^{50}$.

   (In parts (b) and (c), full marks will be given for a correct answer. Part marks may be awarded for an incomplete solution or work leading to an incorrect answer.)

3. A pair of functions $f(x)$ and $g(x)$ is called a *Payneful pair* if

   (i) $f(x)$ is a real number for all real numbers x,
   (ii) $g(x)$ is a real number for all real numbers x,
   (iii) $f(x+y) = f(x)g(y) + g(x)f(y)$ for all real numbers x and y,
   (iv) $g(x+y) = g(x)g(y) - f(x)f(y)$ for all real numbers x and y, and
   (v) $f(a) \neq 0$ for some real number a.

   For every Payneful pair of functions $f(x)$ and $g(x)$:

   (a) Determine the values of $f(0)$ and $g(0)$.
   (b) If $h(x) = (f(x))^2 + (g(x))^2$ for all real numbers x, determine the value of $h(5)h(-5)$.
   (c) If $-10 \leq f(x) \leq 10$ and $-10 \leq g(x) \leq 10$ for all real numbers x, determine the value of $h(2021)$.

2021
Canadian
Senior
Mathematics
Contest
(English)