The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

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2021 Canadian Team Mathematics Contest

Individual Problems (45 minutes)

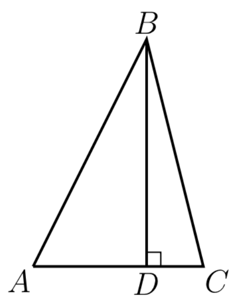
IMPORTANT NOTES:

- Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) previously stored information such as formulas, programs, notes, etc., (iv) a computer algebra system, (v) dynamic geometry software.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

PROBLEMS:

1. Determine the largest 6-digit positive integer that is divisible by 5.

2. In the diagram, $\triangle ABC$ has an area of 84 and $AC = 12$. Point D is on AC so that BD is perpendicular to AC. What is the length of BD?



3. Below are five facts about the ages of five students, Adyant, Bernice, Cici, Dara, and Ellis.

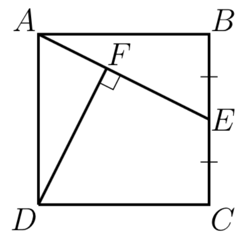
   - Adyant is older than Bernice.
   - Dara is the youngest.
   - Bernice is older than Ellis.
   - Bernice is younger than Cici.
   - Cici is not the oldest.

   Determine which of the five students is the third oldest.

4. For non-zero integers x and y we define $x\nabla y = x^y + 4x$. For example, $2\nabla 3 = 2^3 + 4(2) = 16$. Determine all real numbers x such that $x\nabla 2 = 12$.

5. The equations $x - 2y - 3 = 0$ and $18x - k^2y - 9k = 0$ represent two lines. For some real number k, these two lines are distinct and parallel. Determine the value of k.

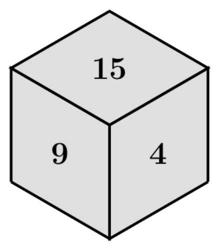
6. Square $ABCD$ has sides of length 2. The midpoint of BC is E. Point F is on AE so that DF is perpendicular to AE. Determine the length of DF.



7. Determine the number of ordered pairs of positive integers (a, b) for which $20a + 21b = 2021$.

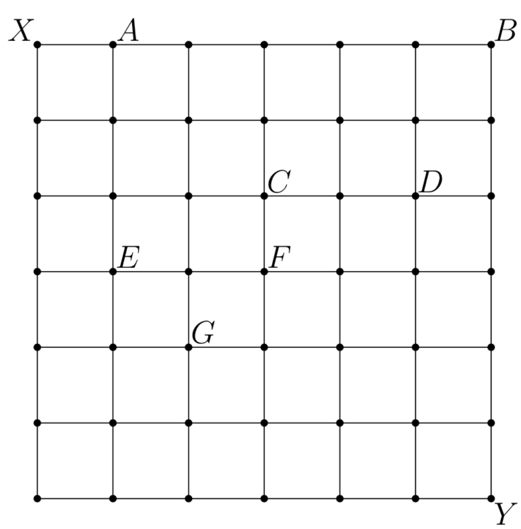
8. Amanda has two identical cubes. Each cube has one integer on each face so that the following statements are all true:

   - Three adjacent faces of each cube have the numbers 15, 9 and 4 as shown.
   - The numbers on all pairs of opposite faces have the same sum s.
   - When both cubes are rolled and the numbers on the top faces are added, the probability that the sum equals 24 is $\frac{1}{12}$.

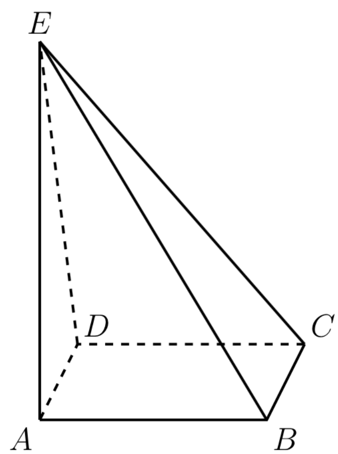


   Determine the sum of all possible values of s.

9. Celine traces paths on the grid below starting at point X and ending at point Y. Each path must follow the lines connecting the dots and only ever move horizontally to the right or vertically down. It may be useful to know that there are a total of 924 such paths. Consider the 7 points labelled A, B, C, D, E, F, and G. List these points in decreasing order of the number of paths passing through that point. For example, B is on exactly one path, so B should be the last point in your list.



10. $ABCDE$ is a pyramid with square base $ABCD$. Point E is directly above A with $AE = 1024$ and $AB = 640$. The pyramid is cut into two pieces by a horizontal plane parallel to $ABCD$. This horizontal plane is a distance h above the base $ABCD$. The portion of $ABCDE$ that is above the plane is a new pyramid. For how many integers h is the volume of the new pyramid an integer?



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Team Problems (45 minutes)

IMPORTANT NOTES:

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PROBLEMS:

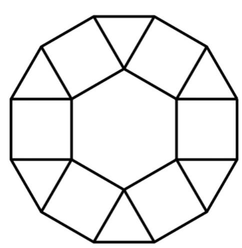
1. What integer is equal to $1^1 + 2^2 + 3^3$?

2. What is the smallest prime number that is greater than 50?

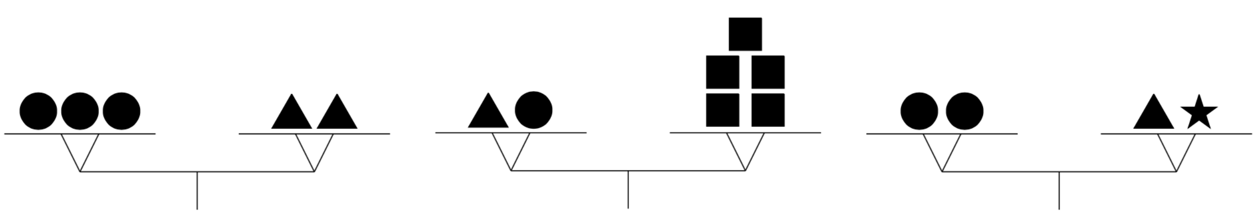
3. For each positive integer n, the expression $1 + 2 + 3 + \cdots + (n-1) + n$ represents the sum of all of the integers from 1 to n inclusive. What integer is equal to

$$(1 + 2 + 3 + \cdots + 2020 + 2021) - (1 + 2 + 3 + \cdots + 2018 + 2019)?$$

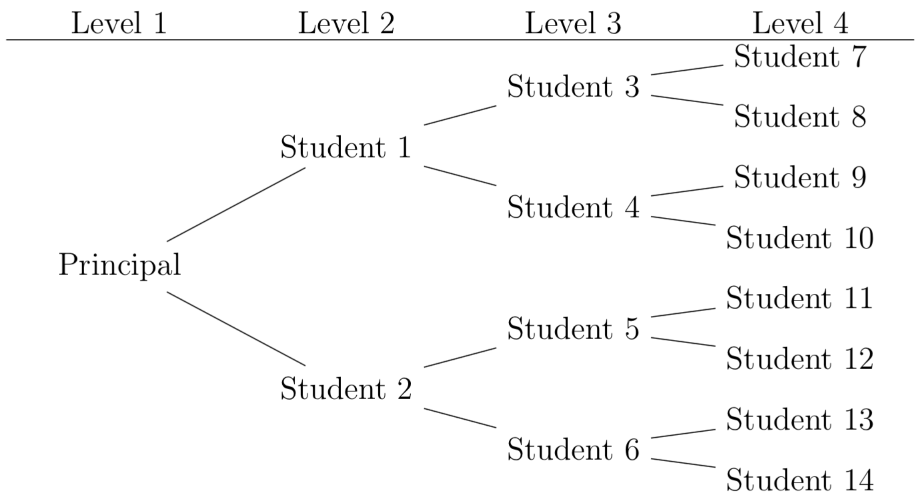
4. Tam created the mosaic shown using a regular hexagon, squares and equilateral triangles. If the side length of the hexagon is 20 cm, what is the outside perimeter of the mosaic?



5. The three scales shown below are balanced. The mass of ★ is 1 kg. Which of the other objects, (circle, square and triangle), has a mass of 1 kg?



6. The diagram shows the first four levels of a school's emergency telephone tree. In the case of an emergency, the principal calls two students. The first level consists of just the principal, and in the second level, two students are contacted. In the next level, each of these two students contacts two students who have not been contacted, so after the third level, a total of 6 students have been contacted. This continues so that each student contacts two students who have not yet been contacted. After the 8^{th} level, how many students in total have been contacted?



7. A group of n students doing an art project used red, blue, and yellow paint. Every student used at least one colour, and some students used more than one colour.

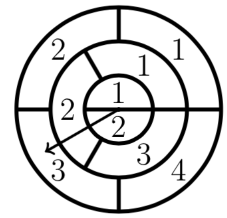
   - The yellow paint was used by a total of 46 students.
   - The red paint was used by a total of 69 students.
   - The blue paint was used by a total of 104 students.
   - Exactly 14 students used both yellow and blue and did not use red.
   - Exactly 13 students used both yellow and red and did not use blue.
   - Exactly 19 students used both blue and red and did not use yellow.
   - Exactly 16 students used all three colours.

   What is the value of n?

8. A real number x satisfies the equation $\dfrac{1}{x} + \dfrac{x}{80} = \dfrac{7}{30}$. What are the possible values of x?

9. Nabil has a tablet that starts with its battery fully charged to 100%. The battery life decreases at a constant rate as the tablet is being used. He uses the tablet for exactly 60 minutes, after which 68% of the battery life remains. For how many *more* minutes can Nabil use the tablet before the battery is at 0%?

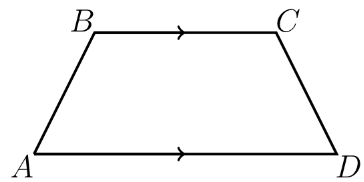
10. In the spinner shown, all circles have the same centre:



- The inner ring is divided into equal sections numbered 1 and 2.
- The middle ring is divided into equal sections numbered 1, 2 and 3.
- The outer ring is divided into equal sections numbered 1, 2, 3, and 4.

After the arrow is spun, it lies on one section of each of the three rings and the score is the total of the three numbers in these sections. For example, in the spinner shown, the score is $2+2+3=7$. The arrow is spun once. What is the probability that the score is odd?

11. In trapezoid $ABCD$, AD is parallel to BC. Also, $AB=CD$, $BC=9$, $AD=21$, and the perimeter of $ABCD$ is 50. What is the length of the diagonal AC?



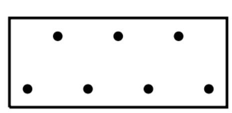
12. A parabola passes through the point of intersection of the lines with equations $y=-x+3$ and $x-2y-6=0$, as well as the the x-intercept of each line. If the parabola also passes through the point $(10,k)$, what is the value of k?

13. Ann, Bill and Carol each have some tokens. There are four exchanges of tokens that happen in the order below:

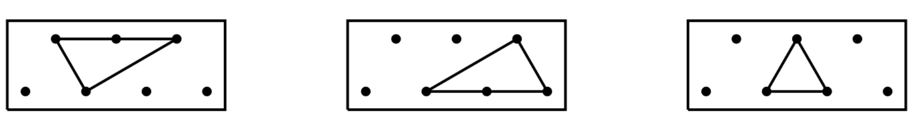
- First, Ann gives Bill the number of tokens that Bill has.
- Next, Ann gives Carol twice the number of tokens that Carol has.
- Next, Bill gives Ann the number of tokens that Ann has.
- Finally, Bill gives Carol twice the number of tokens that Carol has.

After these four exchanges of tokens, Ann, Bill, and Carol each have 36 tokens. How many tokens did Ann have before any were exchanged?

14. Jiawei is drawing triangles using three of the points in the box below as vertices.



Below are three triangles that Jiawei might draw.

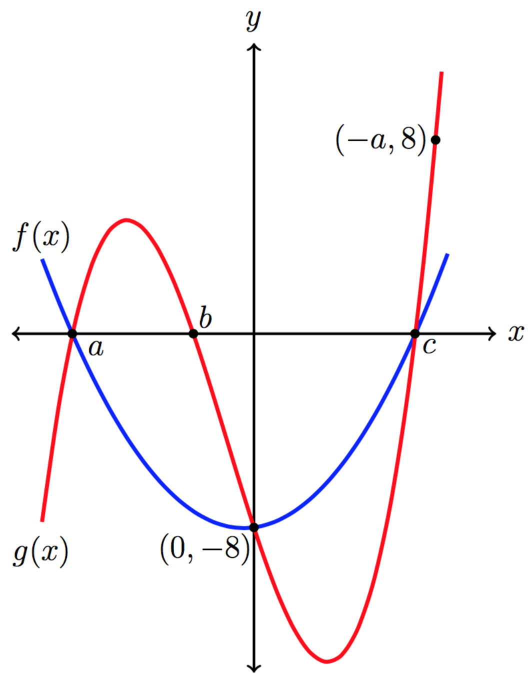


If the first two of these are considered to be different, how many triangles of positive area can Jiawei draw in this way?

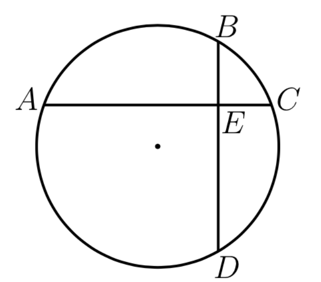
15. How many positive integers x with $200 \leq x \leq 600$ have exactly one digit that is a prime number?

16. Suppose f is a function that satisfies $f(2) = 20$ and $f(2n) + nf(2) = f(2n + 2)$ for all positive integers n. What is the value of $f(10)$?

17. Suppose a, b, and c are real numbers with $a < b < 0 < c$. Let $f(x)$ be the quadratic function $f(x) = (x - a)(x - c)$ and $g(x)$ be the cubic function $g(x) = (x - a)(x - b)(x - c)$. Both $f(x)$ and $g(x)$ have the same y-intercept of -8 and $g(x)$ passes through the point $(-a, 8)$. Determine the value of c.



18. AC and BD are two perpendicular chords in a circle. The chords intersect at E, as shown, such that $BE = 3$, $EC = 2$, and $AE = 6$. The exact perimeter of the quadrilateral $ABCD$ may be written in the form $m\sqrt{n} + p\sqrt{q}$, where m, n, p, and q are positive integers, $q > n$, and neither q nor n has a divisor that is a perfect square greater than 1. Determine the value of $\sqrt{mn} + \sqrt{p + q}$.



19. The table shown is to be filled in with integers so that

- the integers in each row form an arithmetic sequence that increases from left to right,
- the integers in each column form a geometric sequence that is increasing from top to bottom, and
- the three geometric sequences have the same common ratio, which is an integer.

Determine the sum of all possible values of b.

5		
		b
		900

20. Suppose that $f(x) = \dfrac{2x+1}{x-2}$ and that $y = g(x)$ is a linear function. If $f^{-1}(g(2)) = 7$ and $g^{-1}(f(1)) = \dfrac{4}{5}$, what is the x-intercept of $y = g(x)$?

21. If n is a positive integer, the symbol $n!$ (read "n factorial") represents the product of the integers from 1 to n. For example, $4! = (1)(2)(3)(4)$ or $4! = 24$.
Determine
$$\frac{1}{\log_2 100!} + \frac{1}{\log_3 100!} + \frac{1}{\log_4 100!} + \cdots + \frac{1}{\log_{99} 100!} + \frac{1}{\log_{100} 100!}.$$

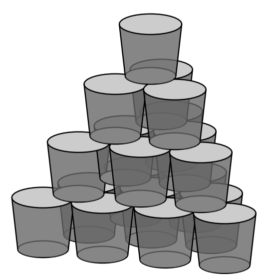
22. For how many integers n is $\dfrac{2n^3 - 12n^2 - 2n + 12}{n^2 + 5n - 6}$ equal to an integer?

23. An angle θ with $0^\circ \leq \theta \leq 180^\circ$ satisfies $\sqrt{2}\cos 2\theta = \cos\theta + \sin\theta$. Determine all possible values of θ.

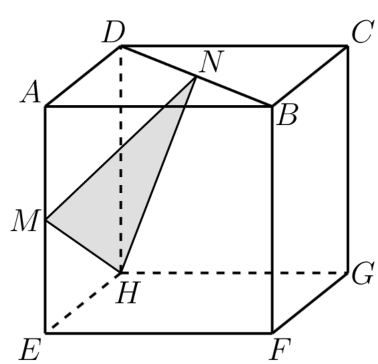
24. Twenty buckets, each with a volume of 6 L, are stacked in a pyramid as shown, with ten buckets in the bottom layer, six in the second layer, three in the third layer, and one in the top layer. Each layer is arranged in an equilateral triangle and the buckets are stacked so that each bucket, other than those in the bottom layer, rests on exactly three buckets in the layer below.

Water is poured into the top bucket one litre at a time. After a litre is poured, the water is allowed to settle before another litre is poured. If water is poured into a bucket that is full, the water spills over the rim of the bucket in such a way that exactly one third of the runoff goes into each of the three buckets on which it rests. For example, once the top bucket is full, additional water spills equally into the three buckets in the second-highest layer.

At the first instant when at least one bucket on the bottom layer is full, what is the total amount of water that has been poured into the top bucket?



25. The point N is the centre of the face $ABCD$ of the cube $ABCDEFGH$, as shown. Also, M is the midpoint of the edge AE. If the area of $\triangle MNH$ is $13\sqrt{14}$, what is the edge length of the cube?



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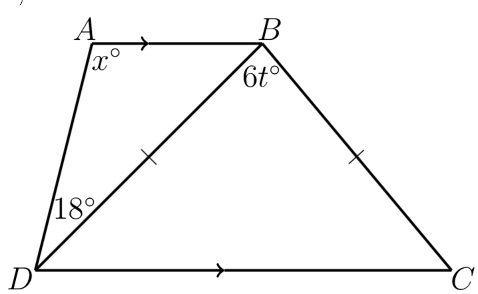
Relay Problem #1 (Seat a)

If $(y-5)^2 = (y-9)^2$, what is the value of y?

Relay Problem #1 (Seat b)

Let t be TNYWR.
In the diagram, $ABCD$ is a trapezoid with AB parallel to DC and $BC = BD$. If $\angle DAB = x^\circ$, $\angle ADB = 18^\circ$, and $\angle DBC = 6t^\circ$, what is the value of x?



Relay Problem #1 (Seat c)

Let t be TNYWR.
There are t cards. Each card has one picture printed on it. The picture on each card is either of a dinosaur or a robot. Each dinosaur and robot is coloured either blue or green.

- 16 cards have blue dinosaurs
- 14 cards have green robots
- 36 cards have blue robots

If a card is chosen at random, what is the probability, written as a fraction in lowest terms, that it has either a green dinosaur or a blue robot printed on it?

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Relay Problem #2 (Seat a)

A line passing through $(-5, k)$ and $(13, -7)$ has a slope of $-\frac{1}{2}$. What is the value of k?

Relay Problem #2 (Seat b)

Let t be TNYWR
Three buckets, labelled A, B, and C, are filled with water.
The amount of water in bucket A is 6 litres more than half of the amount in bucket C.
The amount of water in bucket B is the average (mean) of the amounts in buckets A and C.
The amount of water in bucket C is $18t + 8$ litres.
In total, how many litres of water are there in the three buckets?

Relay Problem #2 (Seat c)

Let t be TNYWR
The vertex and the two x-intercepts of the parabola with equation $y = ax^2 + 6ax$ are joined to form a triangle with an area of t square units. If $a < 0$, what is the value of a?

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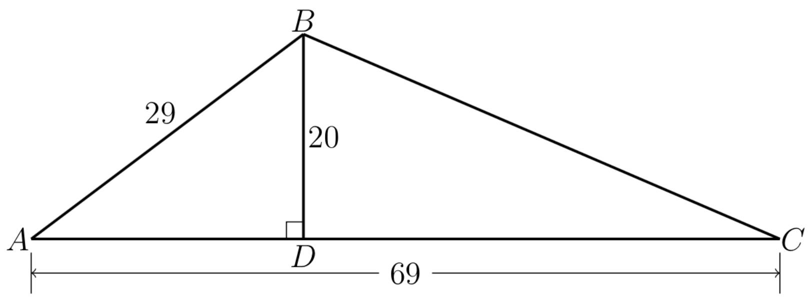
Relay Problem #3 (Seat a)

In the diagram, D is on side AC of $\triangle ABC$ so that BD is perpendicular to AC. If $AB = 29$, $AC = 69$, and $BD = 20$, what is the length of BC?



Relay Problem #3 (Seat b)

Let d be TNYWR.
Lawrence runs $\frac{d}{2}$ km at an average speed of 8 minutes per kilometre.
George runs $\frac{d}{2}$ km at an average speed of 12 minutes per kilometre.
How many minutes more did George run than Lawrence?

Relay Problem #3 (Seat c)

Let t be TNYWR.
The sum of two numbers is t and the positive difference between the squares of these two numbers is 208. What is the larger of the two numbers?