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2021 Canadian Team Mathematics Contest

Answer Key for Team Problems

Question	Answer
1	32
2	53
3	4041
4	240 cm
5	$\blacksquare$
6	254
7	141
8	$12, \frac{20}{3}$
9	127.5
10	$\frac{5}{6}$
11	17
12	14
13	65
14	30
15	156
16	220
17	$\frac{8}{3}$
18	9
19	690
20	$\frac{7}{5}$
21	1
22	32
23	$15^\circ, 135^\circ$
24	78 L
25	$2\sqrt{26}$

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Answer Key for Individual Problems

Question	Answer
1	999 995
2	14
3	Bernice
4	$-6, 2$
5	-6
6	$\frac{4}{\sqrt{5}}$
7	5
8	64
9	$AFCGEDB$
10	85

Answer Key for Relays

Question	Answer
0	9, 108, 36°
1	$(7, 93, \frac{21}{31})$
2	$(2, 108, -4)$
3	(52,104,53)

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May 2021

Solutions

Individual Problems

1. The five largest 6-digit numbers are 999 999, 999 998, 999 997, 999 996, and 999 995.
   In order for an integer to be divisible by 5, its units digit must be either 0 or 5.
   The largest 6-digit integer that is divisible by 5 is 999 995.

   ANSWER: 999 995

2. The area of $\triangle ABC$ is equal to $\frac{1}{2} \times AC \times BD$.
   It is given that the area is 84 and that $AC = 12$.
   Substituting, this leads to the equation $84 = \frac{1}{2} \times 12 \times BD$ or $84 = 6BD$, so $BD = 14$.

   ANSWER: 14

3. Since Dara is the youngest, we can order the students by age by ordering the other four students.
   Adyant is older than Bernice and Bernice is older than Ellis, so neither Bernice nor Ellis is the oldest.
   It is given that Cici is not the oldest, so Adyant must be the oldest.
   Since Cici is older than Bernice and Bernice is older than Ellis, the students in order from oldest to youngest are Adyant, Cici, Bernice, Ellis, and Dara.
   Therefore, Bernice is the third oldest.

   ANSWER: Bernice

4. We assume $12 = x\nabla 2 = x^2 + 4x$, so $x^2 + 4x = 12$.
   This can be rearranged to $x^2 + 4x - 12 = 0$, which factors as $(x + 6)(x - 2) = 0$.
   The possible values of x are $x = 2$ and $x = -6$.

   ANSWER: $-6, 2$

5. We can re-write the equation of the first line in slope-intercept form as $y = \frac{1}{2}x - \frac{3}{2}$.
   If $k = 0$, then the second line has equation $18x = 0$ or $x = 0$ which is vertical and cannot be parallel to the first line which has slope $\frac{1}{2}$.
   This means $k \neq 0$, so the equation of the second line can be rearranged to $y = \frac{18}{k^2}x - \frac{9}{k}$.
   In order for these lines to be parallel, they must have the same slope.
   Therefore, $\frac{1}{2} = \frac{18}{k^2}$, so $k^2 = 36$ or $k = \pm 6$.
   If $k = 6$, then the equation of the second line is $y = \frac{18}{36}x - \frac{9}{6}$ or $y = \frac{1}{2}x - \frac{3}{2}$, which is the same equation as that of the other line.
   Since the lines are distinct, we conclude that $k = -6$.
   Indeed, the lines with equations $y = \frac{1}{2}x - \frac{3}{2}$ and $y = \frac{1}{2}x + \frac{3}{2}$ are parallel and distinct.

   ANSWER: -6

6. Since $ABCD$ is a square, $\angle FAD + \angle FAB = 90°$.
   Since $\triangle BAE$ is right-angled, we also have that $\angle BEA + \angle EAB = 90°$.
   Since $\angle FAB = \angle EAB$, we then have $\angle FAD + \angle EAB = \angle BEA + \angle EAB$, from which it follows that $\angle FAD = \angle BEA$.
   Two right-angled triangles are similar if they have a non-right angle in common, so $\triangle FAD$ is

similar to $\triangle BEA$.
This means $\dfrac{FD}{BA} = \dfrac{AD}{EA}$ so

$$FD = \frac{BA \times AD}{EA} = \frac{2 \times 2}{EA} = \frac{4}{EA}.$$

Since E is the midpoint of BC and $BC = 2$, we have $BE = 1$.
By the Pythagorean theorem, $EA^2 = BA^2 + BE^2 = 2^2 + 1^2 = 5$.
Since $EA > 0$, we have $EA = \sqrt{5}$ and thus $FD = \dfrac{4}{\sqrt{5}}$.

Answer: $\dfrac{4}{\sqrt{5}}$

7. Since a is a positive integer, the units digit of $20a$ is 0.
In order for $20a + 21b = 2021$, the units digit of $21b$ must be 1, and this implies that the units digit of b must be 1.
If $b \geq 100$, then $21b \geq 2100 > 2021$, which would mean $20a$ is negative.
Since a must be positive, we must have $b < 100$.
We have that b is a positive integer less than 100 and that its units digit is 1. The only possibilities for b are 1, 11, 21, 31, 41, 51, 61, 71, 81, and 91.
Solving for a in the equation $20a + 21b = 2021$ gives $a = \dfrac{2021 - 21b}{20}$.
Substituting the values of b above into this equation gives

b	1	11	21	31	41	51	61	71	81	91
a	100	89.5	79	68.5	58	47.5	37	26.5	16	5.5

From the table above, we get that there are five pairs of positive integers (a, b) satisfying $20a + 21b = 2021$: $(100, 1)$, $(79, 21)$, $(58, 41)$, $(37, 61)$, and $(16, 81)$.

Answer: 5

8. The sides opposite the sides showing 4, 9, and 15 show the numbers $s - 4$, $s - 9$, and $s - 15$, respectively.
There are 6 ways that each die can land, hence there are $6 \times 6 = 36$ ways that the dice can land.
The table below contains the sum of the numbers on the top faces for each of the 36 ways that the dice can land:

	4	9	15	$s-4$	$s-9$	$s-15$
4	8	13	19	s	$s-5$	$s-11$
9	13	18	24	$s+5$	s	$s-6$
15	19	24	30	$s+11$	$s+6$	s
$s-4$	s	$s+5$	$s+11$	$2s-8$	$2s-13$	$2s-19$
$s-9$	$s-5$	s	$s+6$	$2s-13$	$2s-18$	$2s-24$
$s-15$	$s-11$	$s-6$	s	$2s-19$	$2s-24$	$2s-30$

The probability that the total is 24 is given as $\dfrac{1}{12} = \dfrac{3}{36}$, which means exactly three of the sums in the table above must equal 24.
There are at least two sums of 24, regardless of the value of s.

Therefore, for a value of s to give a probability of $\frac{1}{12}$ of rolling 24, there needs to be exactly one more sum of 24.
We will call the sums where the dice show the same value the *diagonal* sums.
These are the sums in the diagonal of the table above, or 8, 18, 30, $2s-8$, $2s-18$, and $2s-30$.
If the third sum of 24 occurs somewhere other than as a diagonal sum, then there would be at least four possible sums of 24 since the sums on either "side" of the diagonal are equal.
For example, if $s-11=24$, then $s=35$, so one of the values on each die is $35-15=20$. This would lead to a sum of 24 if a 20 and a 4 are rolled, and this can happen in two ways.
Therefore, the only way that the probability of rolling a 24 can be $\frac{1}{12}$ is if $2s-8=24$, $2s-18=24$, or $2s-30=24$.
These equations can be solved for s to get $s=16$, $s=21$, and $s=27$.
We now need to argue that none of these values of s leads to a probability higher than $\frac{1}{12}$ of rolling a 24.
The sums that depend on s that are not diagonal sums are

$$s, s+5, s+11, s-5, s+6, 2s-13, s-11, s-6, 2s-19, 2s-24.$$

Setting each of these values equal to 24 and solving gives

$$s=24, s=19, s=13, s=29, s=18, s=\frac{37}{2}, s=35, s=30, s=\frac{43}{2}, s=24$$

and none of $s=16$, $s=21$, and $s=27$ appear in the list.
The possible values of s are $s=16$, $s=21$, and $s=27$, so the answer is $16+21+27=64$.

ANSWER: 64

9. Denote by $f(A)$ the number of paths through A, $f(B)$ the number of paths through B, and so on.

<u>$f(A)$</u>: Every path must pass through either point A or the point directly below X.
No path can pass through both of these points since this would require that the path either move up or to the left.
By symmetry, the same number of paths must pass through A and the point directly below X, so $f(A)=\frac{924}{2}=462$.

<u>$f(B)$</u>: As noted in the problem statement, $f(B)=1$ since there is only one path through B.

<u>$f(C)$</u>: We can compute $f(C)$ by multiplying the number of paths from X to C by the number of paths from C to Y.
Every path from X to C takes a total of 5 steps, 3 to the right and 2 down.
Thus, we can think of a path from X to C as a sequence of three r's and two d's. There are ten such sequences:

$$rrrdd, rrdrd, rrddr, rdrrd, rdrdr, rddrr, drrrd, drrdr, drdrr, ddrrr.$$

Another way to think about counting these sequences is to count the number of ways to place the two d's in any of the five positions. This completely determines the sequence since the other three positions must be occupied by r's.

There are five positions for the first d, then four positions for the second d for a total of $5 \times 4 = 20$. However, we will have counted each sequence twice since the order of the d's does not matter.
Therefore, there are a total of $\frac{20}{2} = 10$ sequences of two d's and three r's, which means there are a total of 10 paths from X to C.
To count the number of paths from C to Y, we can similarly count the number of sequences of three r's and four d's.
This can be done by counting the number of ways to place the three r's in any of the seven possible positions.
Once again, there are 7 positions for the first r, then 6 remaining for the second r, and 5 for the third r for a total of $7 \times 6 \times 5 = 210$ sequences.
We have again overcounted, so we need to determine by how much.
If we imagine that the three r's have different "colours" (say red, blue, and green, for example), then there are 6 possible orders in which the r's can be placed.
That is, 3 possibilities for the first r, 2 possibilities for the second r, and then the third r is fixed.
The total 210 counts the number of ways to place the three coloured r's in the seven positions.
We only care about the position of the r's, not the colour.
Each possible positioning of the r's will be included 6 times in the total 210, so the number of ways to place three r's in seven positions is $\frac{201}{6} = 35$.
Thus, the number of paths from C to Y is $\frac{210}{6} = 35$.
We can now compute $f(C) = 10 \times 35 = 350$.

The sort of reasoning used to compute $f(C)$ can be used to count paths from X to Y through the remaining points.

$\underline{f(D)}$: We first count the number of paths from X to D.
This is the same as counting the number of sequences of five r's and two d's.
Similar to the reasoning used already, this is the same as counting the number of ways to place two d's in 7 positions.
The number of such sequences is $\frac{7 \times 6}{2} = 21$.
The number of paths from D to Y is equal to 5. This is equal to the number of sequences of one r and four d's (there are five possible positions for the r and the d's are fixed after placing the r).
This means $f(D) = 21 \times 5 = 105$.

$\underline{f(E)}$: The number of paths from X to E is 4. The number of paths from E to Y is equal to the number of ways to place three d's in 8 positions, which is $\frac{8 \times 7 \times 6}{6} = 56$.
Therefore, $f(E) = 4 \times 56 = 224$.

$\underline{f(F)}$: The number of paths from X to F is equal to the number of paths from F to Y by symmetry.
The number of paths from X to F is equal to the number of ways to place three r's in six positions (or three d's in 6 positions), which is $\frac{6 \times 5 \times 4}{6} = 20$.

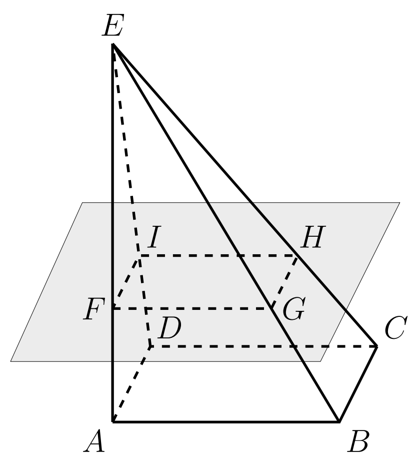
This means $f(F) = 20 \times 20 = 400$.

$\underline{f(G)}$: Again by symmetry, the number of paths from X to G is equal to the number of paths from G to Y.
The number of paths from X to G is equal to $\dfrac{6 \times 5}{2} = 15$, so $f(G) = 15 \times 15 = 225$.

We have $f(A) = 462$, $f(B) = 1$, $f(C) = 350$, $f(D) = 105$, $f(E) = 224$, $f(F) = 400$, and $f(G) = 225$. Therefore, the points listed in order from most paths to fewest paths is $AFCGEDB$.

Answer: $AFCGEDB$

10. We first find an expression for the volume of the top pyramid in terms of h.
The volume of any pyramid is equal to $\dfrac{1}{3} \times$ (area of base) $\times$ (height).
The height of the upper pyramid is $1024 - h$.
Suppose the plane intersects EA at F and EB at G.



Since the plane is parallel to the base, we have that FG is parallel to AB.
This means $\triangle EFG$ is similar to $\triangle EAB$, which means $\dfrac{FG}{AB} = \dfrac{EF}{EA}$. We know that $AB = 640$, $EA = 1024$, and $EF = 1024 - h$, so substitution and rearranging leads to

$$FG = \frac{(1024 - h)(640)}{1024} = \frac{5(1024 - h)}{8}.$$

Next, we show that the base of the new pyramid is a square.
Label by H the point at which the plane intersects EC and I the point at which the plane intersects ED.
We have already observed that FG is parallel to AB. Similarly, GH is parallel to BC, HI is parallel to CD, and IF is parallel to DA.
This means FG is parallel to HI and perpendicular to IF, which is parallel to GH, so $FGHI$ is a rectangle.
To see that $FGHI$ is a square, we will show that $IF = FG$.
We have already observed that $\triangle EFG$ is similar to $\triangle EAB$. For similar reasons, $\triangle EFI$ is similar to $\triangle EAD$.

Since E is directly above A, we have that $\angle EAB = \angle EAD = 90^\circ$.
As well, $AB = AD$, so $\triangle EAB$ is congruent to $\triangle EAD$ by side-angle-side congruence.
It follows that $\triangle EFI$ is similar to $\triangle EFG$, and since they share the side EF, $\triangle EFI$ is congruent to $\triangle EFG$.
Therefore, $FI = FG$, so $FGHI$ is a square.

We can now compute the volume of the square-based pyramid $FGHIE$.
The area of the base is

$$FG^2 = \left(\frac{5(1024-h)}{8}\right)^2$$

and the height is $1024 - h$, so the volume is

$$\frac{1}{3}(1024-h)\left(\frac{5(1024-h)}{8}\right)^2 = \frac{(1024-h)^3 5^2}{3 \times 2^6}.$$

We now need to count the positive integers h with the property that $\dfrac{(1024-h)^3 5^2}{3 \times 2^6}$ is an integer.
This is the same as counting the integers h with the property that 3×2^6 is a divisor of $(1024-h)^3 5^2$.
Since 5 is prime, 5^2 has no factor of 2 and no factor of 3. Therefore, we need to count integers h with the property that $(1024-h)^3$ is a multiple of 3×2^6.
Since 3 is prime, $(1024-h)^3$ is a multiple of 3 exactly when $1024 - h$ is a multiple of 3.
Suppose $1024 - h$ is a multiple of $2^2 = 4$. Then $(1024-h)^3$ is a multiple 2^6.
If $(1024-h)^3$ is a multiple of 2^6, then $1024 - h$ must be even.
However, if $1024 - h$ is a multiple of 2 but not a multiple of 4, then $(1024-h)^3$ cannot have 6 copies of 2 in its prime factorization.
Therefore, $(1024-h)^6$ is a multiple of 2^6 exactly when $(1024-h)$ is a multiple of 4.
We now have shown that the volume of the top pyramid is a positive integer exactly when $1024 - h$ is a multiple of both 3 and 4.
This means that the volume of the top pyramid is a positive integer exactly when $1024 - h$ is a multiple of 12.
When $h = 4$, we have $1024 - h = 1024 - 4 = 1020 = 12 \times 85$.
If $h = 4 + 12$, then $1024 - h = 12 \times 84$.
This continues to give the possibilities $h = 4$, $h = 4 + 12$, $h = 4 + 2 \times 12$, $h = 4 + 3 \times 12$, and so on.
The largest value of h is when $1024 - h = 12$ or $h = 1012 = 4 + 84 \times 12$.
Thus, the values are $h = 4 + k \times 12$ where k ranges over the integers 0 through 84.
There are 85 integers h for which the volume of the upper pyramid is an integer.

ANSWER: 85

Team Problems

1. Since $1^1 = 1$, $2^2 = 4$, and $3^3 = 27$, then $1^1 + 2^2 + 3^3 = 1 + 4 + 27 = 32$.

   Answer: 32

2. We can factor $51 = 3 \times 17$ and $52 = 2 \times 26$, so neither of these integers are prime. However, 53 is prime, so the answer is 53.

   Answer: 53

3. By regrouping terms, we have

$$\begin{aligned}
&(1 + 2 + 3 + \cdots + 2020 + 2021) - (1 + 2 + 3 + \cdots + 2018 + 2019) \\
&= (1 + 2 + 3 + \cdots + 2018 + 2019) + 2020 + 2021 - (1 + 2 + 3 + \cdots + 2018 + 2019) \\
&= 2020 + 2021 \\
&= 4041.
\end{aligned}$$

   Answer: 4041

4. Since each square shares a side with the hexagon, the side lengths of the squares are $20\,\text{cm}$.
   Since each equilateral triangle shares a side with a square (in fact, with two squares), the side lengths of the triangles are $20\,\text{cm}$.
   The perimeter of the mosaic is made up of one side from each of the six squares and one side from each of the six triangles.
   Therefore, the perimeter of the mosaic is $12 \times 20\,\text{cm} = 240\,\text{cm}$.

   Answer: $240\,\text{cm}$

5. Suppose the mass of ▲ is $x\,\text{kg}$, the mass of ● is $y\,\text{kg}$, and the mass of ■ is $z\,\text{kg}$.
   From the first scale, we have that $3y = 2x$ or $y = \frac{2}{3}x$.
   From the third scale, we have that $2y = x + 1$, so we can substitute $y = \frac{2}{3}x$ into this equation to get $\frac{4}{3}x = x + 1$. This can be solved for x to get $x = 3$.
   Substituting $x = 3$ into $3y = 2x$ gives $3y = 6$ so $y = 2$.
   From the third scale, we have that $5z = x + y = 5$, and substituting $x = 3$ and $y = 2$ gives $5z = 5$ or $z = 1$.

   Answer: ■

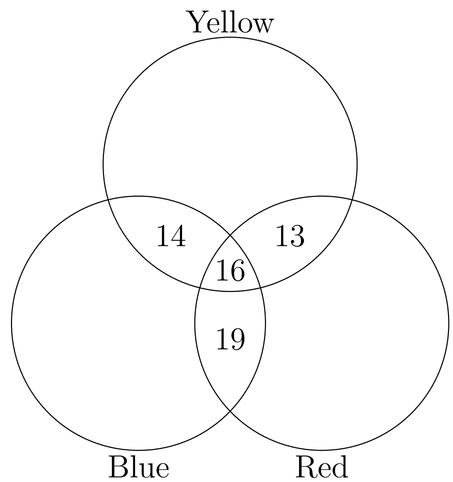
6. The number of students contacted in level 1 is 0.
   The number of students contacted in level 2 is $2 = 2^1$.
   The number of students contacted in level 3 is $4 = 2^2$.
   The number of students contacted in level n is twice the number of students that were contacted in the previous level.
   This means the number of students contacted in level 4 is $2(2^2) = 2^3$, the number of students contacted in level 5 is $2(2^3) = 2^4$, and so on.
   The number of students contacted after 8 levels is therefore

$$\begin{aligned}
2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 &= 2 + 4 + 8 + 16 + 32 + 64 + 128 \\
&= 254.
\end{aligned}$$

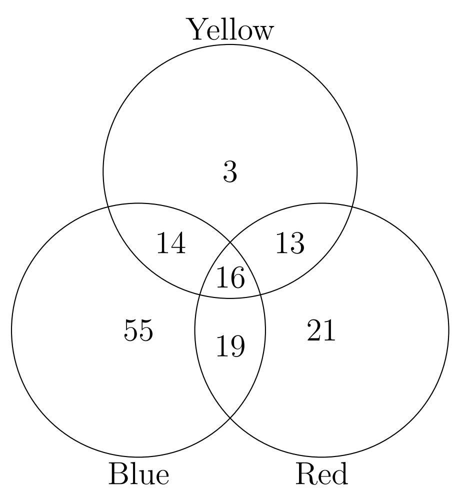
   Answer: 254

7. *Solution 1*

There are seven possibilities for the colours used by any individual student: they used exactly one colour (three possibilities), they used exactly two colours (three possibilities), and they used all three colours (one possibility).
We are given the number of students who used each combination of two colours and we are given the number of students who used all three colours.
To find the total, we need to determine how many students used only yellow, only red, and only blue. The partially-completed Venn Diagram below includes the given information:



A total of 46 students used the yellow paint. Of these 46, 14 also used blue (and not red), 13 also used red (and not blue), and 16 also used both blue and red.
Therefore, the number of students who used only yellow paint is $46 - 14 - 13 - 16 = 3$.
By similar reasoning, the number of students who used only red paint is $69 - 13 - 19 - 16 = 21$ and the number of students who used only blue paint is $104 - 14 - 19 - 16 = 55$. The completed Venn diagram is below:



The total number of students is equal to the sum of the numbers in the seven regions of the Venn diagram, or

$$3 + 21 + 55 + 14 + 13 + 19 + 16 = 141.$$

Solution 2

Since every student used at least one colour, the total $46 + 69 + 104 = 219$ includes every

student, but it counts some students multiple times.
This total includes twice the number of students who used exactly two colours and three times the number of students who used all three colours.
If we subtract from 219 the number of students who used exactly two colours and twice the number of students who used all three colours, we will get the total number of students.
The number of students is $219 - 14 - 13 - 19 - 2(16) = 141$.

Answer: 141

8. Multiplying through by $240x$, the equation becomes $240 + 3x^2 = 56x$.
This can be rearranged to $3x^2 - 56x + 240 = 0$ which factors as $(3x - 20)(x - 12) = 0$.
Therefore, the possible values of x are $x = 12$ and $x = \frac{20}{3}$. Note that these values could also be obtained using the quadratic formula.
One can check that $x = 12$ and $x = \frac{20}{3}$ indeed satisfy the equation.

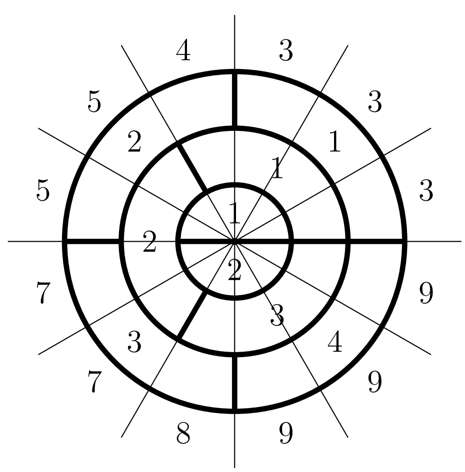
Answer: $12, \frac{20}{3}$

9. *Solution 1*
Since the battery loses $100\% - 68\% = 32\%$ every 60 minutes, the percentage of battery life after t minutes is $P = 100 - \frac{32}{60}t$.
We can solve this equation for $P = 0$ to get $t = \frac{100 \times 60}{32} = \frac{375}{2}$ minutes.
Since we are asked how long after the first hour (60 minutes) the battery will be at 0%, the answer is $\frac{375}{2} - 60 = 127.5$ minutes.

Solution 2
The battery loses its charge at a rate of $100\% - 68\% = 32\%$ every 60 minutes.
For the remaining 68% to disappear, it will take $\frac{68\%}{32\%} \times 60$ minutes, or $\frac{255}{2} = 127.5$ minutes.

Answer: 127.5

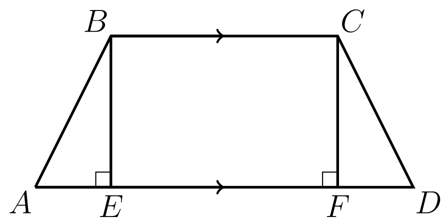
10. We divide the spinner into 12 equal sectors as shown. The score obtained if the arrow lands in each sector is written just outside the arc forming the outer boundary of the sector.



The spinner lands in each of these 12 sectors with equal probability, so the probability that the spinner lands in any one of the 12 sectors is $\frac{1}{12}$.
Of the 12 sectors, 10 have an odd score, so the probability of spinning an odd score is $\frac{10}{12} = \frac{5}{6}$.

ANSWER: $\frac{5}{6}$

11. Let E and F be points on AD so that BE and CF are perpendicular to AD.



Since BC and AD are parallel, so are BE and CF.
This means $BCFE$ a parallelogram, so $BE = CF$.
It is given that $AB = DC$, so $\triangle AEB$ and $\triangle DFC$ are right triangles with an equal hypotenuse and an equal leg.
Therefore, $\triangle AEB$ and $\triangle DFC$ are congruent.
This means $AE = DF$ and since $BC = EF$, we get $AD = AE + EF + FD = 2AE + BC$.
Substituting $BC = 9$ and $AD = 21$, we have $21 = 2AE + 9$ or $2AE = 12$ so $AE = 6$.
Since $AB = CD$ and the perimeter of the trapezoid is

$$AB + BC + CD + AD = 2CD + BC + AD = 50,$$

we get $CD = \frac{1}{2}(50 - 9 - 21) = 10$.
By the Pythagorean theorem, $CF^2 = CD^2 - FD^2$, and since $CD = 10$ and $FD = AE = 6$, we have $CF = \sqrt{100 - 36} = 8$.
The length AC is the hypotenuse of $\triangle AFC$, and we now know that $AF = AE+EF = 6+9 = 15$ and $CF = 8$.
By the Pythagorean theorem, $AC^2 = AF^2 + FC^2 = 15^2 + 8^2 = 225 + 64 = 289$.
Since $AC > 0$ and $AC^2 = 289$, $AC = \sqrt{289} = 17$.

ANSWER: 17

12. We first find the point of intersection of the two lines.
To do this, rearrange the equation of the second line to $y = \frac{1}{2}x - 3$.
Setting the y-values equal to each other, we obtain $-x + 3 = \frac{1}{2}x - 3$ and multiplying through by 2 gives $-2x + 6 = x - 6$, so $3x = 12$, or $x = 4$.
To find the y-coordinate of the point of intersection, we can substitute $x = 4$ into the first equation to get $y = -4 + 3 = -1$.
Therefore, the parabola passes through the point $(4, -1)$.
To find the x-intercept of the first line, we set $y = 0$ to get $0 = -x+3$ so $x = 3$. The x-intercept of the first line is 3.
To find the x-intercept of the second line, we set $y = 0$ to get $x - 2(0) - 6 = 0$, so $x = 6$. The

x-intercept of the second line is 6.
The parabola passes though $(3,0)$ and $(6,0)$, so its equation takes the form $y = a(x-3)(x-6)$.
To solve for a, we substitute $(x,y) = (4,-1)$ to get $-1 = a(4-3)(4-6) = -2a$ or $-1 = -2a$, so $a = \frac{1}{2}$.
The equation of the parabola is $y = \frac{1}{2}(x-3)(x-6)$.
Since the parabola passes through the point $(10,k)$, we can substitute to get $k = \frac{1}{2}(10-3)(10-6) = \frac{28}{2} = 14$. Therefore, $k = 14$.

ANSWER: 14

13. *Solution 1*
Suppose Ann, Bill, and Carol begin with A, B, and C tokens, respectively. The table below shows how many tokens they each have after each exchange:

	Ann	Bill	Carol
Initially	A	B	C
After first exchange	$A-B$	$2B$	C
After second exchange	$A-B-2C$	$2B$	$3C$
After third exchange	$2(A-B-2C)$	$2B-(A-B-2C)$	$3C$
	$2A-2B-4C$	$-A+3B+2C$	$3C$
After fourth exchange	$2A-2B-4C$	$-A+3B+2C-6C$	$3C+6C$
	$2A-2B-4C$	$-A+3B-4C$	$9C$

We know that after the four exchanges are complete, Ann, Bill, and Carol each have 36 tokens.
Therefore, $9C = 36$, so $C = 4$. Substituting this into $2A-2B-4C = 36$ and $-A+3B-4C = 36$ gives $2A-2B = 52$ and $-A+3B = 52$.
Dividing the equation $2A-2B = 52$ by 2 gives $A-B = 26$ and adding this to $-A+3B = 52$ gives $2B = 78$ so $B = 39$.
Substituting this into $A-B = 26$ gives $A = 65$.

Solution 2
The number of tokens that Ann has before the fourth exchange is the same as the number she has after the exchange.
During the exchange, the number of tokens that Carol has triples, so before the fourth exchange, Carol had $\frac{36}{3} = 12$, which means Bill gives Carol $36-12 = 24$ tokens in the fourth exchange.
Thus, in the fourth exchange, the number of tokens that Ann has does not change, the number of tokens that Bill has decreases by 24, and the number of tokens that Carol has increases by 24.
By similar reasoning, during the third exchange, the number of tokens that Carol has does not change, the number of tokens that Ann has increases by 18, and the number of tokens that Bill has decreases by 18.
During the second exchange, the number of tokens that Bill has does not change, the number of tokens that Carol has increases by 8, and the number of tokens that Ann has decreases by 8.
During the fist exchange, the number of tokens that Carol has does not change, the number of tokens that Bill has increases by 39, and the number of tokens that Ann has decreases by 39.
These results are summarized in the table below:

Exchanges	Ann		Bill		Carol	
	Before	After	Before	After	Before	After
Fourth	36	36	60	36	12	36
Third	18	36	78	60	12	12
Second	26	18	78	78	4	12
First	65	26	39	78	4	4

Before any tokens are exchanged, Ann has 65 tokens.

ANSWER: 65

14. There are seven dots in total. The number of ways there are to choose three distinct dots is $\binom{7}{3} = \frac{7 \times 6 \times 5}{6} = 35$.
If you are unfamiliar with binomial coefficients, you might deduce that there are 35 ways to choose three dots this way:

> There are 7 ways to choose a dot. There are 6 ways to choose a second dot from those remaining, and 5 ways to choose a third dot from those remaining after choosing the first two.
> This gives $7 \times 6 \times 5 = 210$ ways of choosing the dots. However, the way we have chosen the dots has imposed an order. That is, there is a "first" dot, a "second" dot, and a "third" dot.
> There are six orders in which three distinct objects can be placed, which means we have overcounted by a factor of 6.
> Therefore, there are $\frac{210}{6} = 35$ ways to choose three distinct dots out of the seven.

Three dots will form the vertices of a triangle exactly when they are not all on the same line. We will count the number of sets of three dots that are on a line and subtract this number from 35 to get the answer.
The only lines that contain at least three of the points are the horizontal line through the top three dots and the horizontal line through the bottom four dots.
There is exactly one set of three dots on the first of these two horizontal lines.
There are four points in total on the second horizontal line, so there are a total of four sets of three points on this line (we can exclude any of the four dots).
Therefore, there are a total of $1 + 4 = 5$ sets of three points that lie on a line.
The number of triangles that Jiawei can draw is $35 - 5 = 30$.

ANSWER: 30

15. The prime digits are 2, 3, 5, and 7, and the non-prime digits are 0, 1, 4, 6, 8, and 9.
If the first digit is 2, then the other two digits cannot be prime.
Since 6 of the digits are non-prime, there are $6 \times 6 = 36$ three-digit integers with first digit equal to 2 that have exactly one digit that is prime.
Similarly, with the first digit equal to each of 3 and 5, there are 36 three-digit integers with exactly one digit that is prime.
If the first digit is 4, then exactly one of the other two digits must be prime.
There are $4 \times 6 = 24$ ways to choose a prime digit and a non-prime digit.
There are 2 choices for where to put the prime digit (the tens position or the units position).
Therefore, there are $2 \times 24 = 48$ three digit integers starting with 4 that have exactly one digit that is prime.
The answer is therefore $36 + 36 + 36 + 48 = 156$.

ANSWER: 156

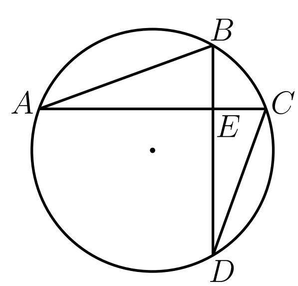
16. With $n = 1$, we have $f(2 \times 1) + 1 \times f(2) = f(2 \times 1 + 2)$ which simplifies to $f(2) + f(2) = f(4)$, so $f(4) = 20 + 20 = 40$.
With $n = 2$, we have $f(2 \times 2) + 2 \times f(2) = f(2 \times 2 + 2)$ which simplifies to $f(4) + 2f(2) = f(6)$. Since $f(4) = 40$ and $f(2) = 20$, we have $f(6) = 40 + 2(20) = 80$.
Continuing with $n = 3$, we have $f(6) + 3f(2) = f(8)$, so $f(8) = 80 + 3(20) = 140$.
With $n = 4$, we have $f(8) + 4f(2) = f(10)$, so $f(10) = 140 + 4(20) = 220$.

ANSWER: 220

17. Since $f(0) = -8$, we have $-8 = (0 - a)(0 - c) = ac$.
Since $g(0) = -8$, we have $-8 = (0 - a)(0 - b)(0 - c) = -abc$ or $abc = 8$.
Substituting $ac = -8$ into $abc = 8$ and solving for b, we get $b = -1$.
We also are given that $g(-a) = 8$. Since $b = -1$, this means $8 = (-a - a)(-a + 1)(-a - c)$ which can be simplified to $8 = -(-2a)(1 - a)(a + c)$ and then to $a(a + c)(1 - a) = 4$.
Expanding the above equation, we get $4 = a(a - a^2 + c - ac) = a^2 - a^3 + ac - a(ac)$.
From earlier, we have that $ac = -8$, so this equation simplifies to $4 = a^2 - a^3 - 8 - a(-8)$ which can be rearranged to $a^3 - a^2 - 8a + 12 = 0$.
We can factor $(a - 2)$ out of the expression on the left to get $(a - 2)(a^2 + a - 6) = 0$ which further factors as $(a - 2)(a - 2)(a + 3) = 0$.
This means that either $a = 2$ or $a = -3$. Since $a < 0$, $a \neq 2$, so $a = -3$.
Since $ac = -8$, this gives $c = \frac{8}{3}$.

ANSWER: $\frac{8}{3}$

18. If AB and DC are drawn, then $\angle ABD = \angle ACD$ since they are subtended by the same arc.



As well, we are given that $\angle AEB = \angle DEC = 90°$, so $\triangle AEB$ and $\triangle DEC$ are similar since they have two equal angles.
This means $\frac{CE}{BE} = \frac{DE}{AE}$ or $DE = \frac{AE \times CE}{BE} = \frac{6 \times 2}{3} = 4$.
Since $\angle AEB = \angle BEC = \angle CED = \angle DEA = 90°$, we can apply the Pythagorean theorem four times to compute the perimeter of $ABCD$ as

$$\begin{aligned} AB + BC + CD + DA &= \sqrt{AE^2 + BE^2} + \sqrt{BE^2 + CE^2} + \sqrt{CE^2 + DE^2} + \sqrt{DE^2 + AE^2} \\ &= \sqrt{6^2 + 3^2} + \sqrt{3^2 + 2^2} + \sqrt{2^2 + 4^2} + \sqrt{4^2 + 6^2} \\ &= \sqrt{45} + \sqrt{13} + \sqrt{20} + \sqrt{52} \\ &= 3\sqrt{5} + \sqrt{13} + 2\sqrt{5} + 2\sqrt{13} \\ &= 5\sqrt{5} + 3\sqrt{13} \end{aligned}$$

Neither 5 nor 13 has a perfect square factor other than 1, and $13 > 5$, so we must have $n = 5$ and $q = 13$, from which it follows that $m = 5$ and $p = 3$.
Therefore, $\sqrt{mn} + \sqrt{p+q} = \sqrt{25} + \sqrt{16} = 9$.

Answer: 9

19. Suppose the common ratio is r and the common difference in the first row is some integer $d > 0$. This means the entries in the first row will be 5, $5 + d$, and $5 + 2d$ and the entries in the first column will be 5, $5r$, and $5r^2$.
We know that d must be an integer since all entries in the table are integers.
The entry in the top-right cell is equal to $\frac{900}{r^2}$ which must be an integer, so r is an integer with the property that r^2 is a divisor of 900.
The prime factorization of 900 is $2^2 3^2 5^2$, from which it is possible to deduce that the perfect-square divisors of 900 are 1^2, 2^2, 3^2, 5^2, 6^2, 10^2, 15^2, and 30^2.
We have now limited the possibilities of r to 1, 2, 3, 5, 6, 10, 15, and 30.
However, the geometric sequences are increasing, so $r = 1$ is not possible.
We also cannot have $r = 30$ since this would lead to $\frac{900}{30^2} = 1$ in the top-right cell, which is less than 5.
Similarly, if $r = 15$, then the entry in the top right cell would be $\frac{900}{15^2} = 4$, which is less than 5.
The entry in the top-right cell is $5 + 2d$, which must be odd since d is an integer.
The numbers $\frac{900}{3^2} = 100$ and $\frac{900}{5^2} = 36$ are even, so we cannot have $r = 3$ or $r = 5$.
We are now down to the values $r = 2$, $r = 6$, and $r = 10$ as possibilities.
Suppose $r = 2$. Then $b = \frac{900}{r} = \frac{900}{2} = 450$ and the entry in the top-right cell is $\frac{450}{2} = 225$.
Since $5 + 2d = 225$, $d = 110$. The entries in the first column are $5 \times 2 = 10$ and $10 \times 2 = 20$, so the common difference in the middle row is $\frac{450 - 10}{2} = 220$ and the common difference in the bottom row is $\frac{900 - 20}{2} = 440$. Thus, we get the following table with $r = 2$:

5	115	225
10	230	450
20	460	900

which satisfies all of the properties, so $b = 450$ is a possibility.
With $r = 6$ and $r = 10$, we get the following tables:

5	15	25
30	90	150
180	540	900

5	7	9
50	70	90
500	700	900

which both satisfy the required properties. Therefore, the possible values of b are $b = 450$, $b = 150$, and $b = 90$, so the answer is $450 + 150 + 90 = 690$.

Answer: 690

20. Applying f to both sides of $f^{-1}(g(2)) = 7$, we get

$$f(7) = f(f^{-1}(g(2))) = g(2)$$

so we have that $g(2) = f(7) = \dfrac{2(7)+1}{7-2} = \dfrac{15}{5} = 3$.

Similarly, we can apply g to both sides of $g^{-1}(f(1)) = \dfrac{4}{5}$ to get

$$g\left(\frac{4}{5}\right) = g(g^{-1}(f(1))) = f(1)$$

so we have that $g\left(\dfrac{4}{5}\right) = f(1) = \dfrac{2(1)+1}{1-2} = -3$.

Since $g(2) = 3$ and $g\left(\dfrac{4}{5}\right) = -3$, the x-intercept is halfway between $\dfrac{4}{5}$ and 2.

The x-intercept is $\dfrac{1}{2}\left(2 + \dfrac{4}{5}\right) = \dfrac{7}{5}$.

ANSWER: $\dfrac{7}{5}$

21. By the change of base formula for logarithms, if x and y are positive real numbers, then

$$\frac{1}{\log_x y} = \frac{1}{\frac{\log_{10} y}{\log_{10} x}} = \frac{\log_{10} x}{\log_{10} y} = \log_y x.$$

By this and the identity $\log x + \log y = \log xy$, we have

$$\begin{aligned}
\frac{1}{\log_2 100!} + \frac{1}{\log_3 100!} + \cdots + \frac{1}{\log_{100} 100!} &= \log_{100!} 2 + \log_{100!} 3 + \cdots + \log_{100!} 99 + \log_{100!} 100 \\
&= \log_{100!}(2 \times 3 \times \cdots \times 99 \times 100) \\
&= \log_{100!}(1 \times 2 \times 3 \times \cdots \times 99 \times 100) \\
&= \log_{100!}(100!) \\
&= 1
\end{aligned}$$

ANSWER: 1

22. Notice that $2(1)^3 - 12(1)^2 - 2(1) + 12 = 0$ and $(1)^2 + 5(1) - 6 = 0$, so 1 is a root of both the numerator and denominator.
This means the expression is not defined for $n = 1$, but we can factor and divide both the numerator and denominator by $n - 1$.
Doing this, we get

$$\begin{aligned}
\frac{2n^3 - 12n^2 - 2n + 12}{n^2 + 5n - 6} &= \frac{(n-1)(2n^2 - 10n - 12)}{(n-1)(n+6)} \\
&= \frac{2n^2 - 10n - 12}{n+6}
\end{aligned}$$

where the final equality holds as long as $n \neq 1$.
It is easily checked that -6 is not a root of the numerator, so no further simplification by cancellation is possible.
However, we can manipulate the expression to make it easier to understand for which n the expression is equal to an integer.
We already know for integers $n \neq 1$ that the expression is equal to $\dfrac{2n^2 - 10n - 12}{n+6}$.

This expression is not defined for $n = -6$, so we can further assume that $n \neq -6$ and rearrange to

$$\begin{aligned}\frac{2n^2 - 10n - 12}{n+6} &= \frac{(2n^2 + 12n) - 22n - 12}{n+6} \\ &= \frac{2n(n+6)}{n+6} - \frac{22n+12}{n+6} \\ &= 2n - \frac{(22n+132) - 120}{n+6} \\ &= 2n - \frac{22(n+6)}{n+6} + \frac{120}{n+6} \\ &= 2n - 22 + \frac{120}{n+6}\end{aligned}$$

The quantity $2n - 22$ is an integer for all integers n, so the original expression is an integer exactly when $\frac{120}{n+6}$ is an integer.
This means we need to count the integers n for which $n + 6$ is a divisor of 120.
The divisors of 120 are

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 8, \pm 10, \pm 12, \pm 15, \pm 20, \pm 24, \pm 30, \pm 40, \pm 60, \pm 120$$

of which there are 32.
For each of these 32 divisors, we can solve for n. For example, if $n + 6 = -10$, then we have $n = -16$, so $n = -16$ is an integer for which the expression in the problem statement evaluates to an integer.
We also know that $n = 1$ makes the expression undefined, so we need to make sure that solving for n will never lead to $n = 1$.
Notice that when $n = 1$, $n + 6 = 7$, and 7 is not a divisor of 120.
Therefore, every divisor above gives an integer n for which the expression is defined and equal to an integer, so the answer is 32.

ANSWER: 32

23. Using the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$, we can rearrange $\sqrt{2}\cos 2\theta = \cos\theta + \sin\theta$ to an equivalent equation.

$$\begin{aligned}\sqrt{2}\cos 2\theta &= \cos\theta + \sin\theta \\ \sqrt{2}(\cos^2\theta - \sin^2\theta) &= \cos\theta + \sin\theta \\ \sqrt{2}(\cos\theta + \sin\theta)(\cos\theta - \sin\theta) &= \cos\theta + \sin\theta \\ (\cos\theta + \sin\theta)\big(\sqrt{2}(\cos\theta - \sin\theta) - 1\big) &= 0.\end{aligned}$$

It follows that θ satisfies the equation if and only if either $\cos\theta + \sin\theta = 0$ or $\cos\theta - \sin\theta = \frac{1}{\sqrt{2}}$.
$\cos\theta + \sin\theta = 0$ implies $\tan\theta = -1$, and the only angle θ satisfying $\tan\theta = -1$ and $0 \leq \theta \leq 180°$ is $\theta = 135°$.
If $\cos\theta - \sin\theta = \frac{1}{\sqrt{2}}$, then squaring both sides gives $\cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta = \frac{1}{2}$.
Using the identities $\cos^2\theta + \sin^2\theta = 1$ and $2\sin\theta\cos\theta = \sin 2\theta$, this can be simplified to $\sin 2\theta = \frac{1}{2}$.
We need $0 \leq \theta \leq 180°$, which is equivalent to $0 \leq 2\theta \leq 360°$.

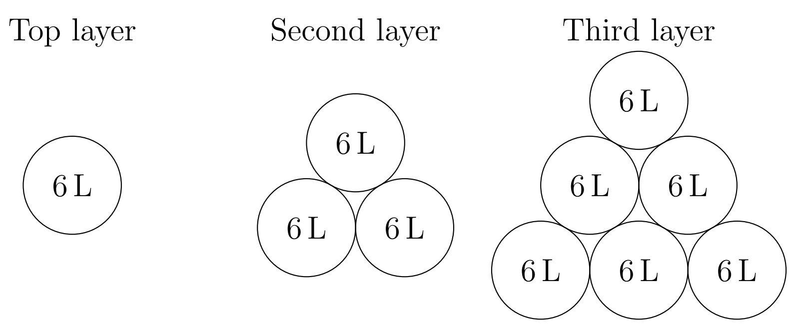
Since $\sin 2\theta = \dfrac{1}{2}$, this means $2\theta = 30^\circ$ or $2\theta = 150^\circ$.
Thus, we get two additional solutions, $\theta = 15^\circ$ and $\theta = 75^\circ$.
Notice that $45^\circ < 75^\circ < 90^\circ$, so $\cos 75^\circ < \sin 75^\circ$. This means $\cos 75^\circ - \sin 75^\circ < 0$ and cannot equal $\dfrac{1}{\sqrt{2}}$.
Thus, we reject the answer 75°. It can be shown that the other two solutions are indeed solutions.

ANSWER: $15^\circ, 135^\circ$

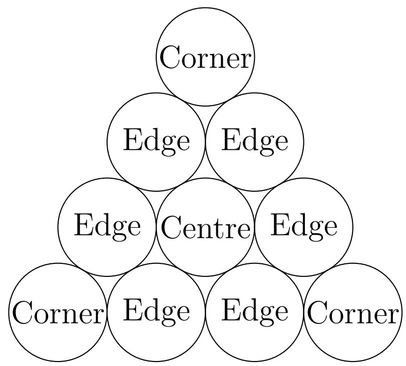
24. By symmetry, the buckets in the second layer from the top will become full at the same time. Thus, there will be a point in time at which the four buckets in the top two layers will be full and all other buckets will be empty.
Each bucket holds 6 L, so this means that after $4 \times 6\,\mathrm{L} = 24\,\mathrm{L}$ are poured, the top four buckets will be full and the other 16 will be empty.
There are six buckets in the third row from the top. We will divide them into three "corner" buckets and three "edge" buckets.
The corner buckets each receive water from exactly one bucket in the second layer.
The edge buckets each receive water from exactly two buckets in the second layer.
Every litre poured into the top bucket after the 24^{th} litre ends up in the third layer.
The corner buckets in the third layer from the top will receive one third of the water that spills into one bucket of the second layer.
Thus, for every litre poured into the top bucket after the 24^{th} litre, the corner buckets in the third layer receive $\dfrac{1}{9}$ L. The edge buckets each receive $\dfrac{2}{9}$ L. Notice that there are three corner buckets and three edge buckets, and $3\left(\dfrac{1}{9}\right) + 3\left(\dfrac{2}{3}\right) = 1$, so all of the water is accounted for.
This means the three edge buckets in the third layer will fill twice as fast as the corner buckets. Since they receive $\dfrac{2}{9}$ L for every 1 L poured into the top bucket and their volume is 6 L, the edge buckets will fill after $6 \div \dfrac{2}{9} = 6 \times \dfrac{9}{2} = 27$ additional litres have been poured into the top bucket.
This means that after $24 + 27 = 51$ litres are poured into the top bucket, we will have

- all buckets in the top two layers full,
- the three edge buckets in the third layer will be full,
- the three corner buckets in the third layer will have $27 \times \dfrac{1}{9}\,\mathrm{L} = 3\,\mathrm{L}$ in them, and
- all buckets in the bottom layer will be empty.

The diagram below shows the buckets in each of the top three layers with the amount of water in each bucket.



After another 27 L are poured into the top bucket, the three corner buckets in the third layer will be full for the first time.
As well, $27\,\mathrm{L} - 3(3\,\mathrm{L}) = 18\,\mathrm{L}$ will have spilled out of the edge buckets in the third layer and into buckets in the bottom layer.
We will categorize the ten buckets in the bottom layer as three corner buckets, one centre bucket, and six edge buckets. The diagram below shows this.



For every litre poured into the top bucket after the 51st, we know that $\frac{2}{9}$ L ends up in each edge bucket of the third layer.
Of this $\frac{2}{9}$ L, $\frac{1}{3} \times \frac{2}{9}\,\mathrm{L} = \frac{2}{27}$ L ends up in each of two edge buckets of the fourth layer, and the remaining $\frac{2}{27}$ L ends up in the centre bucket.
Since there are three edge buckets in the third layer, this means that of each litre poured into the top bucket after the 51st (up to the $51 + 27 = 78^{\text{th}}$), $3 \times \frac{2}{27}\,\mathrm{L} = \frac{2}{9}$ L ends up in the centre bucket.
No buckets other than the edge buckets in the third layer spill into the centre bucket on the bottom layer.
For each litre poured *after* the first 51 litres are poured, we know that $\frac{2}{9}$ L ends up in each edge bucket in the third layer.
One third of the water spilling from each edge bucket in the third layer ends up in the centre bucket in the bottom layer.
Since there are three edge buckets in the third layer, this implies $3 \times \frac{1}{3} \times \frac{2}{9}\,\mathrm{L} = \frac{2}{9}$ L ends up in the centre bucket for every litre poured after the first 51 litres.
Therefore, it takes at least $6 \div \frac{2}{9} = 27$ litres poured into the top bucket *after* the first 51 litres are poured to fill the centre bucket in the bottom layer.
This means it takes at least 78 L poured into the top bucket to fill the centre bucket in the bottom layer, and that the centre bucket in the bottom layer will be full after 78 L have been poured into the top bucket.
To finish the argument, we must show that no other bucket in the bottom layer fills up before the centre bucket.
One way to argue this is to notice that after 78 L have been poured into the top bucket, all ten buckets in the top three layers will be full as well as the centre bucket in the bottom layer.
This accounts for $11 \times 6\,\mathrm{L} = 66\,\mathrm{L}$ of the 78 L.
We also know that each edge bucket in the bottom layer will have received $\frac{2}{27}$ L from an edge bucket in the third layer for each of the 27 L poured into the top bucket after the 51st litre.

This accounts for an additional $6 \times 27 \times \dfrac{2}{27}\,\text{L} = 12\,\text{L}$.
Since $66\,\text{L} + 12\,\text{L} = 78\,\text{L}$, this accounts for all $78\,\text{L}$.
Therefore, the centre bucket is the first in the bottom layer to fill, and this happens after $78\,\text{L}$ of water are poured into the top bucket.

ANSWER: $78\,\text{L}$

25. If we set $ME = x$, then the side length of the cube is $2x$.
The side MH of $\triangle MNH$ is the hypotenuse of right-angled $\triangle EMH$.
By the Pythagorean theorem, $MH^2 = ME^2 + EH^2 = x^2 + (2x)^2 = 5x^2$, and since $MH > 0$ and $x > 0$, we have $MH = \sqrt{5}x$.
The diagonals AC and BD each have length $\sqrt{(2x)^2 + (2x)^2} = 2\sqrt{2}x$ by the Pythagorean theorem.
Since $ABCD$ is a square, the point N is the intersection of AC and BD, and is the midpoint of each line.
Therefore, $AN = DN = \sqrt{2}x$.
The side MN of $\triangle MNH$ is the hypotenuse of right-angled $\triangle NAM$, so

$$MN^2 = AN^2 + AM^2 = (\sqrt{2}x)^2 + x^2 = 3x^2.$$

Since $MN > 0$, we have $MN = \sqrt{3}x$.
The side NH of $\triangle MNH$ is the hypotenuse of $\triangle DNH$, so

$$NH^2 = DN^2 + DH^2 = (\sqrt{2}x)^2 + (2x)^2 = 6x^2.$$

Since $NH > 0$, we have $NH = \sqrt{6}x$.
We now have that the side lengths of $\triangle MNH$ are $\sqrt{5}x$, $\sqrt{3}x$, and $\sqrt{6}x$, and that its area is $13\sqrt{14}$.
There are several ways to compute the area of a triangle in terms of its side lengths.
We will use the cosine law to compute $\cos\angle MNH$, then use the Pythagorean identity to compute $\sin\angle MNH$.
Once $\sin\angle MNH$ is known, the area can be computed as $\triangle MNH$ as $\dfrac{1}{2}(MN)(NH)\sin\angle MNH$.
By the cosine law, we have

$$MH^2 = MN^2 + NH^2 - 2(MN)(NH)\cos\angle MNH.$$

Substituting the values in terms of x of MH, MN, and NH, we have

$$5x^2 = 3x^2 + 6x^2 - 2(\sqrt{3}x)(\sqrt{6}x)\cos\angle MNH$$

which can be rearranged to get

$$\cos\angle MNH = \frac{4x^2}{2\sqrt{18}x^2} = \frac{4}{6\sqrt{2}} = \frac{\sqrt{2}}{3}.$$

Since $\angle MNH$ is an angle of a triangle, we have $\sin\angle MNH > 0$.
By the Pythagorean identity, we have

$$\sin\angle MNH = \sqrt{1 - \cos^2\angle MNH} = \sqrt{1 - \frac{2}{9}} = \frac{\sqrt{7}}{3}.$$

Therefore, the area of $\triangle MNH$ is

$$\frac{1}{2}(\sqrt{3}x)(\sqrt{6}x)\frac{\sqrt{7}}{3} = \sqrt{\frac{7}{2}}x^2.$$

We are given that the area of $\triangle MNH$ is $13\sqrt{14}$, which means $13\sqrt{14} = \sqrt{\frac{7}{2}}x^2$.

Rearranging this equation gives $13\sqrt{2}\sqrt{7} = \frac{\sqrt{7}}{\sqrt{2}}x^2$ and then $x^2 = 26$.

Therefore, $x = \sqrt{26}$, so the side length is $2x = 2\sqrt{26}$.

ANSWER: $2\sqrt{26}$

Relay Problems

(Note: Where possible, the solutions to parts (b) and (c) of each Relay are written as if the value of t is not initially known, and then t is substituted at the end.)

0. (a) Evaluating, $\dfrac{2+5\times 5}{3} = \dfrac{2+25}{3} = \dfrac{27}{3} = 9$.

   (b) The area of a triangle with base $2t$ and height $2t-6$ is $\frac{1}{2}(2t)(2t-6)$ or $t(2t-6)$.
   The answer to (a) is 9, so $t=9$ which means $t(2t-6) = 9(12) = 108$.

   (c) Since $\triangle ABC$ is isosceles with $AB = BC$, it is also true that $\angle BCA = \angle BAC$.
   The angles in a triangle add to 180°, so

$$\begin{aligned} 180^\circ &= \angle ABC + \angle BAC + \angle BCA \\ &= \angle ABC + 2\angle BAC \\ &= t^\circ + 2\angle BAC \end{aligned}$$

   The answer to (b) is 108, so $t = 108$. Therefore,

$$\angle BAC = \frac{1}{2}(180^\circ - t^\circ) = \frac{1}{2}(180^\circ - 108^\circ) = \frac{1}{2}(72^\circ) = 36^\circ.$$

ANSWER: 9, 108, 36°

1. (a) Expanding both sides, we have $y^2 - 10y + 25 = y^2 - 18y + 81$.
   Rearranging, we have $8y = 56$, so $y = 7$.

   (b) The triangle with an angle of $6t^\circ$ is isosceles, so each of the other angles measures

$$\frac{1}{2}(180^\circ - 6t^\circ) = 90^\circ - 3t^\circ.$$

   Since AB is parallel to DC, we have that $\angle ABD = 90^\circ - 3t^\circ$.
   The angles in $\triangle ABD$ sum to 180°, so $x + 18 + (90-3t) = 180$ which can be solved for x to get $x = 180 - 90 - 18 + 3t = 72 + 3t$.
   Since $t = 7$, we can substitute to get $x = 72 + 3(7) = 93$.

   (c) There are four possibilities for the picture printed on each card: a blue dinosaur, a blue robot, a green dinosaur, or a green robot.
   Let x denote the number of green dinosaurs.
   From the information given, we have $14 + 16 + 36 + x = t$ or $x = t - 66$.
   The number of cards with a blue robot printed on them is 36, so the probability we seek is $\dfrac{x+36}{t} = \dfrac{t-66+36}{t} = \dfrac{t-30}{t}$.
   Since $t = 93$, the probability is $\dfrac{93-30}{93} = \dfrac{63}{93} = \dfrac{21}{31}$.

ANSWER: $(7, 93, \frac{21}{31})$

2. (a) The slope of the line is $\dfrac{-7-k}{13-(-5)} = -\dfrac{7+k}{18}$.
   Therefore, $-\dfrac{7+k}{18} = -\dfrac{1}{2} = -\dfrac{9}{18}$, so $7 + k = 9$ from which it follows that $k = 2$.

(b) Let a be the number of litres of water in bucket A, b be the number of litres in bucket B, and c be the number of litres in bucket C.
The information given translates in to the equtions $a = \frac{1}{2}c+6$, $b = \frac{a+c}{2}$, and $c = 18t+8$.
Substituting $c = 18t+8$ into $a = \frac{1}{2}c+6$ gives $a = \frac{1}{2}(18t+8)+6 = 9t+10$.
Substituting the values of a and c in terms of t into $b = \frac{a+c}{2}$, we have

$$b = \frac{(9t+10)+(18t+8)}{2} = \frac{27t}{2}+9.$$

Therefore,

$$a+b+c = (9t+10)+\left(\frac{27t}{2}+9\right)+(18t+8) = \frac{81t}{2}+27.$$

Using that $t = 2$, we have $a+b+c = 81+27 = 108$.

(c) We can factor ax^2+6ax to get $y = ax(x+6)$, so the x-intercepts are 0 and -6.
This means the vertex is at $x = \frac{0+(-6)}{2} = -3$.
Thus, the y-value of the vertex is $y = a(-3)(-3+6) = -9a$.
The height of the triangle is $-9a$ (remember, a is negative), and the base has length $0-(-6) = 6$.
Therefore, the area of the triangle is $\frac{1}{2}(6)(-9a) = -27a$.
We are given that the area is t, so $t = -27a$.
With $t = 108$, we get that $a = -4$.

Answer: $(2, 108, -4)$

3. (a) By the Pythagorean theorem in $\triangle ABD$, $AD^2+20^2 = 29^2$, so $AD^2 = 841-400 = 441$.
Since $AD > 0$, $AD = \sqrt{441} = 21$.
This means $DC = AC - AD = 69-21 = 48$.
Notice that $5^2+12^2 = 25+144 = 169 = 13^2$. Therefore, $16(5^2)+16(12)^2 = 16(13)^2$ or $(4\times 5)^2+(4\times 12)^2 = (4\times 13)^2$, so $20^2+48^2 = 52^2$.
Applying the Pythagorean theorem to $\triangle CDB$, we have

$$CB^2 = BD^2+CD^2 = 20^2+48^2 = 52^2.$$

Since $CB > 0$, $CB = 52$.

(b) Observe that the time it takes to run a given distance (in minutes) is the product of the distance (in kilometres) and the rate (in minutes per kilometre).
The time it took Lawrence to complete the run was $8\times\frac{d}{2} = 4d$ minutes.
The time it took George to complete the run was $12\times\frac{d}{2} = 6d$ minutes.
Therefore, it took George $6d-4d = 2d$ minutes longer than Lawrence to complete the run.
Substituting $d = 52$, it took George 104 minutes longer than Lawrence to complete the run.

(c) Let the numbers be a and b.
We have that $a+b = t$ and $a^2-b^2 = 208$.

Factoring the difference of squares, we have $(a-b)(a+b) = 208$, so $t(a-b) = 208$, which can be solved for $a-b$ to get $a-b = \dfrac{208}{t}$.
Adding this equation to $a+b=t$, we get $2a = t + \dfrac{208}{t}$ so $a = \dfrac{t}{2} + \dfrac{104}{t}$.
Substituting $t = 104$, we get $a = \dfrac{104}{2} + \dfrac{104}{104} = 53$. Observe that since $(a-b)(a+b) = 208$ and $a+b$ and 208 are positive, we have that $a-b > 0$ so $a > b$.
Therefore, a is the largest of the two numbers so the answer is 53.

Answer: (52,104,53)