The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Canadian Senior Mathematics Contest

Wednesday, November 16, 2022
(in North America and South America)

Thursday, November 17, 2022
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 2 hours

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Do not open this booklet until instructed to do so.
There are two parts to this paper. The questions in each part are arranged roughly in order of increasing difficulty. The early problems in Part B are likely easier than the later problems in Part A.

PART A

1. This part consists of six questions, each worth 5 marks.
2. **Enter the answer in the appropriate box in the answer booklet.**
   For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

PART B

1. This part consists of three questions, each worth 10 marks.
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, they will be supplied by your supervising teacher. Insert these pages into your answer booklet. Write your name, school name, and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

At the completion of the contest, insert your student information form inside your answer booklet.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on the website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some students may be shared with other mathematical organizations for other recognition opportunities.

Canadian Senior Mathematics Contest

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write solutions in the answer booklet provided.
3. Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.
4. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
5. Diagrams are not drawn to scale. They are intended as aids only.
6. No student may write both the Canadian Senior Mathematics Contest and the Canadian Intermediate Mathematics Contest in the same year.

Useful Fact:
It may be helpful to know that $\sin 2\theta = 2 \sin\theta \cos\theta$ for every angle θ.

PART A

For each question in Part A, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded only if relevant work is shown in the space provided in the answer booklet.

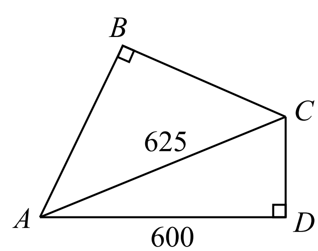
1. If $2^r = 16$ and $5^s = 25$, what is the value of $r + s$?

2. If $\dfrac{x+y}{2} = 5$ and $\dfrac{x-y}{2} = 2$, what is the value of $x^2 - y^2$?

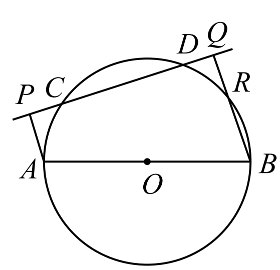
3. The sum of two positive integers is 60 and their least common multiple is 273. What are the two integers?

   (The *least common multiple* of two positive integers is the smallest positive integer which is a multiple of these two integers.)

4. In the diagram, AB is perpendicular to BC, and CD is perpendicular to AD. Also, $AC = 625$ and $AD = 600$. If $\angle BAC = 2\angle DAC$, what is the length of BC?



5. A circle has centre O and diameter $AB = 2\sqrt{19}$. Points C and D are on the upper half of the circle. A line is drawn through C and D, as shown. Points P and Q are on the line so that AP and BQ are both perpendicular to PQ. QB intersects the circle at R. If $CP = DQ = 1$ and $2AP = BQ$, what is the length of AP?



6. A bag contains exactly 15 marbles of which 3 are red, 5 are blue, and 7 are green. The marbles are chosen at random and removed one at a time from the bag until all of the marbles are removed. One colour of marble is the first to have 0 remaining in the bag. What is the probability that this colour is red?

PART B

For each question in Part B, your solution must be well-organized and contain words of explanation or justification. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

Useful Fact for Part B:

The sum of the first k perfect squares is equal to $\dfrac{k(k+1)(2k+1)}{6}$.

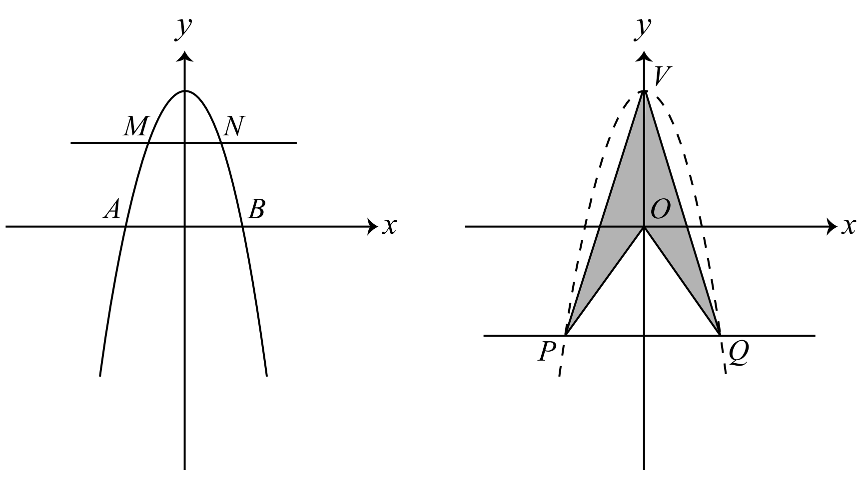
That is, $1^2 + 2^2 + 3^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$.

1. The parabola with equation $y = -x^2 + 16$ intersects the x-axis at points A and B and the horizontal line with equation $y = 7$ at points M and N, as shown.

   (a) Determine the coordinates of A and B.

   (b) Determine the area of the trapezoid $MNBA$.

   (c) Suppose that O is the origin and V is the vertex of the parabola. The line $y = -33$ intersects the parabola at points P and Q. Determine the area of quadrilateral $VPOQ$, which is shaded in the diagram below.



2. (a) Determine all real numbers $a > 0$ for which $\sqrt{a^2 + a} = \frac{2}{3}$.

   (b) For each positive integer m, determine the difference between $(m+\frac{1}{2})^2+(m+\frac{1}{2})$ and the nearest perfect square.

   (c) For every positive integer n, prove that the number of positive integers c with $n < \sqrt{c + \sqrt{c}} < n + 1$ is even.

3. For each positive integer n, let S_n be the set that contains the integers from 1 to n, inclusive; that is, $S_n = \{1, 2, 3, \ldots, n\}$.
For each positive integer $n \geq 4$, let $f(n)$ be the number of quadruples (a, b, c, d) of distinct integers from S_n for which $a - b = c - d$.
For example, $f(4) = 8$ because the possibilities for (a, b, c, d) are

$$(1,2,3,4),(1,3,2,4),(2,1,4,3),(2,4,1,3),(3,1,4,2),(3,4,1,2),(4,2,3,1),(4,3,2,1)$$

   (a) Determine the value of $f(6)$.
   (b) Determine the value of $f(40)$.
   (c) Determine two even positive integers $n < 2022$ for which 2022 is a divisor of $f(n)$.