The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

cemc.uwaterloo.ca

2022 Canadian Team Mathematics Contest

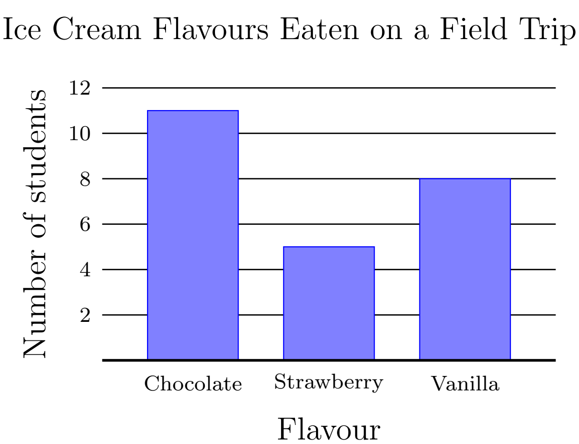
Individual Problems (45 minutes)

IMPORTANT NOTES:

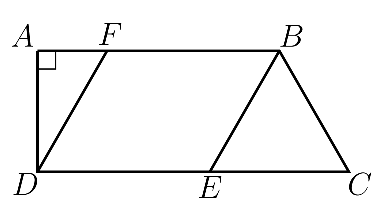
- Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) previously stored information such as formulas, programs, notes, etc., (iv) a computer algebra system, (v) dynamic geometry software.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

PROBLEMS:

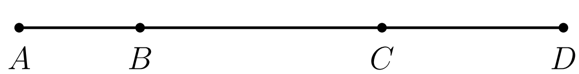
1. The bar graph below shows how many students chose each flavour of ice cream on a recent field trip. What fraction of the students chose chocolate ice cream?



2. In trapezoid $ABCD$, AB is parallel to DC and $\angle DAF = 90^\circ$. Point E is on DC so that $EB = BC = CE$. Point F is on AB so that DF is parallel to EB. In degrees, what is the measure of $\angle FDA$?



3. Line segment AD is divided into three segments by points B and C, so that $AB : BC = 1 : 2$ and $BC : CD = 6 : 5$. The length of AD is 56 units. What is the length of AB?



4. The series below includes the consecutive even integers from 2 to 2022 inclusive, where the signs of the terms alternate between positive and negative:

$$S = 2 - 4 + 6 - 8 + 10 - \cdots - 2016 + 2018 - 2020 + 2022$$

What is the value of S?

5. What is the largest integer n with the properties that $200 < n < 250$ and that $12n$ is a perfect square?

6. Each of A, B, C, and D is a positive two-digit integer. These integers satisfy each of the equations

$$\begin{aligned} B &= 3C \\ D &= 2B - C \\ A &= B + D \end{aligned}$$

What is the largest possible value of $A + B + C + D$?

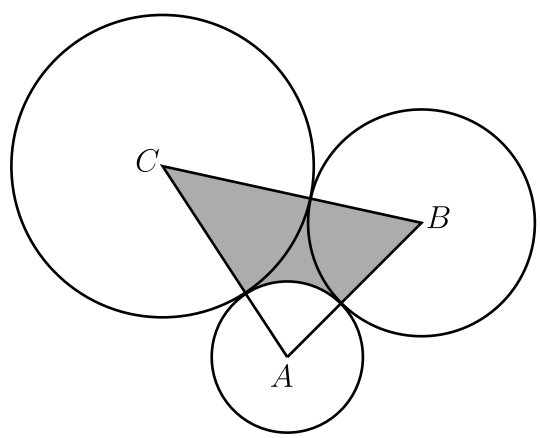
7. What is the sum of the digits of the integer equal to $3 \times 10^{500} - 2022 \times 10^{497} - 2022$?

8. The integers a and b have the property that the expression

$$\frac{2n^3 + 3n^2 + an + b}{n^2 + 1}$$

is an integer for every integer n. What is the value of the expression above when $n = 4$?

9. Three circles with centres A, B and C have radii $\sqrt{3} - 1$, $3 - \sqrt{3}$ and $1 + \sqrt{3}$ respectively. Each circle is externally tangent to the other two as shown. The area of the shaded region is of the form $a\sqrt{3} + b\pi + c\pi\sqrt{3}$ for some rational numbers a, b and c. What is the value of $a + b + c$?



10. Starting with a four-digit integer that is not a multiple of 1000, an integer with fewer digits can be obtained by removing the leading digit and ignoring leading zeros. For example, removing the leading digit from 1023 gives the integer 23, and removing the leading digit from 2165 gives 165. How many integers from 1001 to 4999, inclusive, other than multiples of 1000, have the property that the integer obtained by removing the leading digit is a factor of the original integer?

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Relay Problem #1 (Seat a)

What is the largest integer that can be placed in the box so that $\frac{\square}{11} < \frac{2}{3}$?

Relay Problem #1 (Seat b)

Let t be TNYWR.
If $6x + t = 4x - 9$, what is the value of $x + 4$?

Relay Problem #1 (Seat c)

Let t be TNYWR.
What is the area of the triangle enclosed by the x-axis, the y-axis, and the line with equation $y = tx + 6$?

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Relay Problem #2 (Seat a)

Let x be the number of prime numbers between 10 and 30.
What is the number equal to $\dfrac{x^2-4}{x+2}$?

Relay Problem #2 (Seat b)

Let t be TNYWR.
Alida, Bono, and Cate each have some jelly beans.
The number of jelly beans that Alida and Bono have combined is $6t+3$.
The number of jelly beans that Alida and Cate have combined is $4t+5$.
The number of jelly beans that Bono and Cate have combined is $6t$.
How many jelly beans does Bono have?

Relay Problem #2 (Seat c)

Let t be TNYWR.
There is exactly one real number x with the property that both $x^2-tx+36=0$ and $x^2-8x+t=0$.
What is the value of x?

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Relay Problem #3 (Seat a)

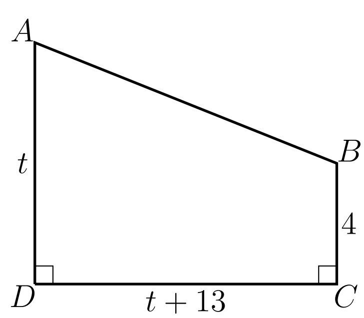
A cube has edge length x. The surface area of the cube is 1014. What is the value of x?

Relay Problem #3 (Seat b)

Let t be TNYWR.
If $\dfrac{5+x}{t+x} = \dfrac{2}{3}$, what is the value of x?

Relay Problem #3 (Seat c)

Let t be TNYWR.
Trapezoid $ABCD$ has $\angle ADC = \angle BCD = 90^\circ$, $AD = t$, $BC = 4$, and $CD = t + 13$.
What is the length of AB?



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Team Problems (45 minutes)

IMPORTANT NOTES:

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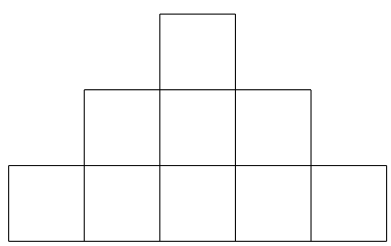
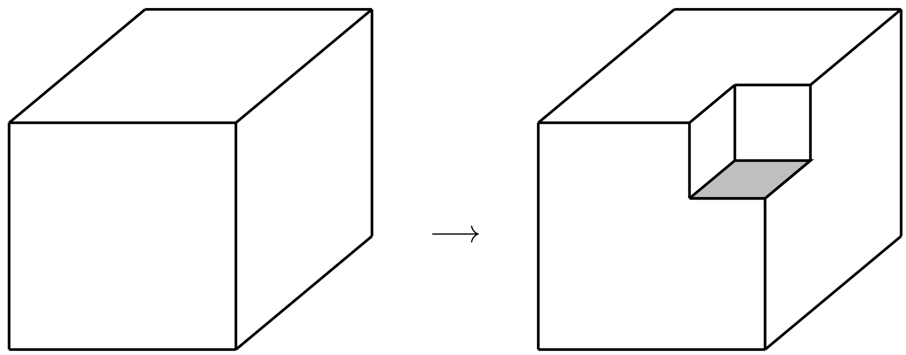
PROBLEMS:

1. If $x = 2z$, $y = 3z - 1$, and $x = 40$, what is the value of y?
2. What is the smallest two-digit positive integer that is a multiple of 6 but is not a multiple of 4?
3. The integer 2022 is positive and has four digits. Three of its digits are 2 and one of its digits is 0. What is the difference between the largest and smallest four-digit integers that can be made using three 2's and one 0 as digits?
4. A total of n people were asked a question in a survey. Exactly 76% of the n people responded "yes" and exactly 24% of the n people responded "no". What is the smallest possible value of n?
5. In the diagram, there are exactly nine 1×1 squares. What is the largest number of 1×1 squares that can be shaded so that no two shaded squares share a side?



6. If $\frac{1}{x} + \frac{1}{2x} + \frac{1}{3x} = \frac{1}{12}$, what is the value of x?
7. There is exactly one isosceles triangle that has a side of length 10 and a side of length 22. What is the perimeter of this triangle?
8. Consider the lines with equations $y = mx$ and $y = -2x + 28$ where m is some real number. The area enclosed by the two lines and the x-axis in the first quadrant is equal to 98. What is the value of m?

9. A solid cube has a volume of $1000\,\text{cm}^3$. A cube with volume $64\,\text{cm}^3$ is removed from one corner of the cube. The resulting solid has a total of nine faces: three large squares that were faces of the original cube, three of the original faces with a square removed from one corner, and three smaller squares. One of the smaller square faces is shaded. The ratio of the area of the shaded face to the surface area of the new solid is of the form $2 : x$. What is the value of x?



10. For some integers n, the expression $\dfrac{8(n-1)}{(n-1)(n-2)}$ is equal to an integer M. What is the sum of all possible values of M?

11. A total of \$425 was invested in three different accounts, Account A, Account B and Account C. After one year, the amount in Account A had increased by 5%, the amount in Account B had increased by 8%, and the amount in Account C had increased by 10%. The increase in dollars was the same in each of the three accounts. How much money was originally invested in Account C?

12. All the points on the line with equation $y = 3x + 6$ are translated up 3 units, then translated left 4 units, and then reflected in the line with equation $y = x$. Determine the y-intercept of the resulting line.

13. A list of integers consists of $(m+1)$ ones, $(m+2)$ twos, $(m+3)$ threes, $(m+4)$ fours, and $(m+5)$ fives. The average (mean) of the list of integers is $\dfrac{19}{6}$. What is m?

14. Ann, Bohan, Che, Devi, and Eden each ordered a different meal at a restaurant and asked for their own bill. Although the correct five bills arrived at the table, none of them were given to the correct person. If the following four statements are true, who was given the bill for Bohan's meal?

    - Che and Eden ordered meals with the same cost.
    - Devi's bill was given to the same person whose bill was given to Devi.
    - Ann, Bohan, and Eden were each given a bill for less money than the meal they ordered.
    - Ann's bill was given to Che.

15. The points $P(2,0)$, $Q(11,-3)$ and $R(x,3)$ are the vertices of a triangle with $\angle PQR = 90^\circ$. What is the value of x?

16. The equation $x^2 - 7x + k = 0$ has solutions $x = 3$ and $x = a$. The equation $x^2 - 8x + k + 1 = 0$ has solutions $x = b$ and $x = c$. What is the value of $a + bc$?

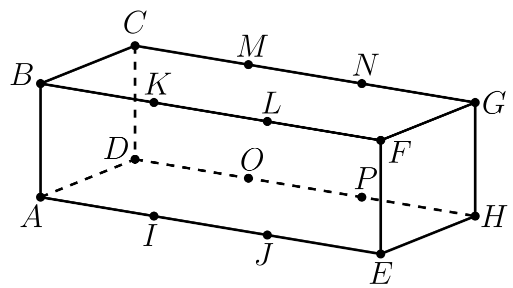
17. The functions $f(x)$ and $g(x)$ are defined by $f(x) = 9^x$ and $g(x) = \log_3(9x)$. The real number x satisfies $g(f(x)) = f(g(2))$. What is the value of x?

18. The real number θ is an angle measure in degrees that satisfies $0^\circ < \theta < 360^\circ$ and

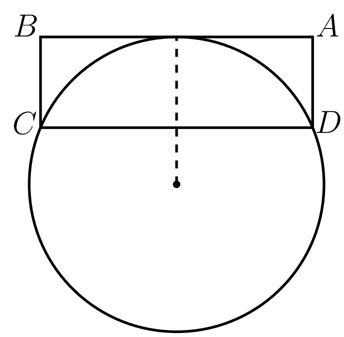
$$2022^{2\sin^2\theta - 3\sin\theta + 1} = 1.$$

The sum of the possible values of θ is k°. What is the value of k?

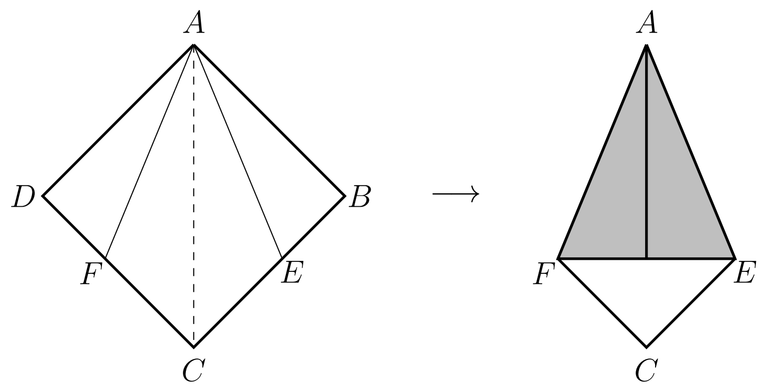
19. The vertices of a $3 \times 1 \times 1$ rectangular prism are A, B, C, D, E, F, G, and H so that AE, BF, CG, and DH are edges of length 3. Point I and point J are on AE so that $AI = IJ = JE = 1$. Similarly, points K and L are on BF so that $BK = KL = LF = 1$, points M and N are on CG so that $CM = MN = NG = 1$, and points O and P are on DH so that $DO = OP = PH = 1$. For every pair of the 16 points A through P, Maria computes the distance between them and lists the 120 distances. How many of these 120 distances are equal to $\sqrt{2}$?



20. In the diagram, the circle has radius $\sqrt{5}$. Rectangle $ABCD$ has C and D on the circle, A and B outside the circle, and AB tangent to the circle. What is the area of $ABCD$ if $AB = 4AD$?



21. A square piece of paper $ABCD$ is white on one side and grey on the other side. Initially, the paper is flat on a table with the grey side down. Point E is on BC so when the paper is folded along AE, B lands on diagonal AC. Similarly, point F is on DC so that when the paper is folded along AF, D lands on AC. After these folds, the resulting shape is kite $AECF$. What fraction of the area of $AECF$ is grey?



22. The integers x, y, z, and p satisfy the equation $x^2 + xz - xy - yz = -p$. Given that p is prime, what is the value of $|y + z|$ in terms of p? Your answer should refer to the variable p so that it works for *every* prime number p.

23. Riley has 64 cubes with dimensions $1 \times 1 \times 1$. Each cube has its six faces labelled with a 2 on two opposite faces and a 1 on each of its other four faces. The 64 cubes are arranged to build a $4 \times 4 \times 4$ cube. Riley determines the total of the numbers on the outside of the $4 \times 4 \times 4$ cube. How many different possibilities are there for this total?

24. Jane places the integers 1 through 9 in the nine cells of a 3×3 grid. The sum of the three integers in a row is called a "row sum" and the product of the three integers in a column is called a "column product". After Jane arranges the integers, the following properties hold.

- Each integer is used exactly once.
- The integer 1 is in the leftmost cell of the second row.
- The integer 8 is in the centre cell.
- The three row sums are equal.
- The largest column product and the smallest column product differ by at most 40.

Determine all possibilities for the largest column product.

Question 25 is on the next page

25. Tetrahedron $ABCD$ has base $\triangle ABC$. Point E is the midpoint of AB. Point F is on AD so that $FD = 2AF$, point G is on BD so that $GD = 2BG$, and point H is on CD so that $HD = 2CH$. Point M is the midpoint of FG and point P is the point of intersection of the line segments EH and CM. What is the ratio of the volume of tetrahedron $EBCP$ to the volume of tetrahedron $ABCD$?