The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2022 Hypatia Contest

Tuesday, April 12, 2022
(in North America and South America)

Wednesday, April 13, 2022
(outside of North America and South America)

Solutions

1. (a) Regular hexagon $ABCDEF$ has side length $2x$, and so $AB = 2x$.
   Since $\triangle OAB$ is equilateral, then $OA = OB = AB = 2x$.
   The radius of the circle is equal to OA and thus is $2x$.

   (b) Since M is the midpoint of AB and $OA = OB$, then
   OM is perpendicular to AB.
   Since M is the midpoint of AB, then $AM = \frac{1}{2}AB = x$.
   Using the Pythagorean Theorem in right-angled
   $\triangle OAM$, we get $OA^2 = OM^2 + AM^2$ or
   $(2x)^2 = OM^2 + x^2$, and so $OM^2 = 3x^2$ or
   $OM = \sqrt{3}x$ (since $OM > 0$).
   Alternatively, notice that $\triangle OAM$ is a 30°-60°-90° trian-
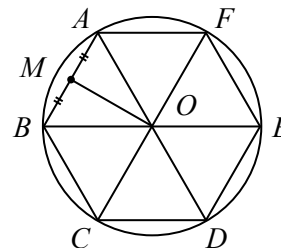   gle, and so $AM : OA : OM = 1 : 2 : \sqrt{3} = x : 2x : \sqrt{3}x$.

   Figure 2

   (c) The diagonals AD, BE and CF divide $ABCDEF$ into six congruent equilateral triangles.
   Thus the area of $ABCDEF$ is six times the area of $\triangle OAB$ (having base AB and height
   OM), or
   $$6 \times \tfrac{1}{2} \times AB \times OM = 3 \times 2x \times \sqrt{3}x = 6\sqrt{3}x^2$$

   (d) The area of the shaded region is determined by subtracting the area of $ABCDEF$ from
   the area of the circle with centre O and radius $2x$.
   Thus, the area of the shaded region is
   $$\pi(2x)^2 - 6\sqrt{3}x^2 = 4\pi x^2 - 6\sqrt{3}x^2 = (4\pi - 6\sqrt{3})x^2$$

   The area of this shaded region is 123, and so $(4\pi - 6\sqrt{3})x^2 = 123$ or $x^2 = \dfrac{123}{4\pi - 6\sqrt{3}}$.

   Since $x > 0$, we get $x = \sqrt{\dfrac{123}{4\pi - 6\sqrt{3}}}$ and so $x = 7.5$ when rounded to the nearest tenth.

2. (a) With 1 kg of muffin batter, exactly 24 mini muffins and 2 large muffins can be made. Thus
   with 2 kg of muffin batter, exactly $2 \times 24 = 48$ mini muffins and $2 \times 2 = 4$ large muffins
   can be made, and so $n = 4$.

   (b) *Solution 1*
   With 2 kg of muffin batter, exactly 36 mini muffins and 6 large muffins can be made.
   With 2 kg of muffin batter, exactly 48 mini muffins and 4 large muffins can be made.
   Adding these, we get that with 2 kg $+$ 2 kg $=$ 4 kg of muffin batter, exactly $36 + 48 = 84$
   mini muffins and $6 + 4 = 10$ large muffins can be made, and so $x = 4$.

   Solution 2
   Let m be the number of kilograms of muffin batter needed to make 1 mini muffin.
   Let ℓ be the number of kilograms of muffin batter needed to make 1 large muffin.
   Since 1 kg of muffin batter makes exactly 24 mini muffins and 2 large muffins, then
   $1 = 24m + 2\ell$.
   Since 2 kg of muffin batter makes exactly 36 mini muffins and 6 large muffins, then
   $2 = 36m + 6\ell$.
   Subtracting the second equation from 3 times the first equation, we get
   $3 \times 1 - 2 = 3 \times (24m + 2\ell) - (36m + 6\ell)$ or $1 = 36m$, and so $m = \dfrac{1}{36}$.

   Substituting $m = \dfrac{1}{36}$ into the second equation, we get $2 = 36\left(\dfrac{1}{36}\right) + 6\ell$ or $1 = 6\ell$, and

so $\ell = \dfrac{1}{6}$.

Since $\dfrac{1}{36}$ kg of muffin batter is needed to make 1 mini muffin, then $84 \cdot \dfrac{1}{36} = \dfrac{7}{3}$ kg of muffin batter is needed to make 84 mini muffins.

Since $\dfrac{1}{6}$ kg of muffin batter is needed to make 1 large muffin, then $10 \cdot \dfrac{1}{6} = \dfrac{5}{3}$ kg of muffin batter is needed to make 10 large muffins.

Thus, exactly 84 mini muffins and 10 large muffins can be made with $\dfrac{7}{3} + \dfrac{5}{3} = \dfrac{12}{3} = 4$ kg of muffin batter, and so $x = 4$.

(c) In part (b) *Solution 2*, it was determined that $\frac{1}{36}$ kg of muffin batter is needed to make 1 mini muffin, and $\frac{1}{6}$ kg of muffin batter is needed to make 1 large muffin.

Therefore, the number of kilograms of muffin batter needed to make 1 large muffin is 6 times the number of kilograms of batter needed to make 1 mini muffin $\left(6 \times \frac{1}{36} = \frac{1}{6}\right)$.

In other words, the batter needed to make 1 large muffin can make 6 mini muffins. Then, using the amount of batter that makes 7 large muffins, $6 \times 7 = 42$ mini muffins can be made.

3. (a) If the first number in a sequence is 3 and the sequence is generated by the function $x^2 - 3x + 1$, then the second number in the sequence is

$$3^2 - 3(3) + 1 = 1,$$

and the third number in the sequence is

$$1^2 - 3(1) + 1 = -1,$$

and the fourth number in the sequence is

$$(-1)^2 - 3(-1) + 1 = 5.$$

The first four numbers in the sequence are $3, 1, -1, 5$.

(b) Let the first and second numbers in the sequence generated by the function $x^2 - 4x + 7$ be f and s, respectively.

Then, the first three numbers in the sequence are $f, s, 7$.

Since the third number in the sequence is 7, then $s^2 - 4s + 7 = 7$.

Solving this equation, we get $s^2 - 4s = 0$ or $s(s - 4) = 0$, which has solutions $s = 0$ and $s = 4$, and so the first three numbers in the sequence could be $f, 0, 7$ or $f, 4, 7$.

If the second number in the sequence is 0, then $f^2 - 4f + 7 = 0$.

The discriminant of this equation is $(-4)^2 - 4(1)(7) = -12$ (less than zero) and so there are no real solutions.

Thus, there is no first number in this sequence for which the second number is 0.

If the second number in the sequence is 4, then $f^2 - 4f + 7 = 4$, or $f^2 - 4f + 3 = 0$ and so $(f - 1)(f - 3) = 0$, which has solutions $f = 1$ and $f = 3$.

Therefore, if 7 is the third number in a sequence generated by the function $x^2 - 4x + 7$, then the first three numbers in the sequence could be $1, 4, 7$ or $3, 4, 7$, and so the possible first numbers in the sequence are 1 and 3.

(c) The first two numbers in the sequence are c, c, and so $c^2 - 7c - 48 = c$.

Solving this equation, we get $c^2 - 8c - 48 = 0$ or $(c + 4)(c - 12) = 0$, which has solutions $c = -4$ and $c = 12$.

(d) The first number in the sequence is a and the second number is b, and so

$$a^2 - 12a + 39 = b.$$

The second number in the sequence is b and the third number is a, and so

$$b^2 - 12b + 39 = a.$$

Subtracting the second equation from the first and simplifying, we get

$$\begin{aligned}
(a^2 - 12a + 39) - (b^2 - 12b + 39) &= b - a \\
a^2 - b^2 - 12a + 12b &= b - a \\
a^2 - b^2 - 11a + 11b &= 0 \\
(a - b)(a + b) - 11(a - b) &= 0 \\
(a - b)(a + b - 11) &= 0
\end{aligned}$$

Since $a \neq b$, then $a - b \neq 0$ and so $a + b - 11 = 0$ or $b = 11 - a$.
Substituting into the first equation, we get $a^2 - 12a + 39 = 11 - a$ or $a^2 - 11a + 28 = 0$.
Factoring gives $(a - 4)(a - 7) = 0$ and so the possible values of a are 4 and 7.
(Note that the two possible sequences are $4, 7, 4, \ldots$ and $7, 4, 7, \ldots$.)

4. (a) Written as a product of its prime factors, $240 = 2^4 3^1 5^1$ and so

$$f(240) = (1 + 4)(1 + 1)(1 + 1) = (5)(2)(2) = 20$$

(b) Suppose $f(N) = 6$ and N is refactorable. Then 6 divides N.
N is divisible by 6 exactly when its prime factors include at least one 2 and at least one 3 (since $6 = 2 \times 3$).
Assume that N contains an additional prime factor p, distinct from 2 and 3.
In this case, the divisors of N are $1, 2, 3, 6, p, 2p, 3p$, and $6p$, which is too many divisors.
Thus, it must be that N is a positive integer of the form $N = 2^u 3^v$, where u and v are positive integers, and so $f(N) = (1 + u)(1 + v) = 6$.
Since u and v are positive integers, then $1 + u \geq 2$ and $1 + v \geq 2$ and so there are exactly two possibilities: $u = 1$ and $v = 2$ or $u = 2$ and $v = 1$.
When $u = 1$ and $v = 2$, $N = 2^1 3^2 = 18$ and when $u = 2$ and $v = 1$, $N = 2^2 3^1 = 12$.
The refactorable numbers N with $f(N) = 6$ are 12 and 18.

(c) Since N is refactorable, then $f(N) = 256 = 2^8$ divides N.
Thus for some integer $w \geq 8$, N is of the form $N = 2^w p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $2 < p_1 < p_2 < \cdots < p_k$ are prime numbers, and $a_1, a_2, \cdots, a_k$ are positive integers.
In this case, $f(N) = (1 + w)(1 + a_1)(1 + a_2) \cdots (1 + a_k) = 2^8$, and so each of the factors $1 + w, 1 + a_1, 1 + a_2, \cdots, 1 + a_k$ is a power of 2 (since their product is 2^8).
This means that each of the exponents $w, a_1, a_2, \cdots, a_k$ is one less than a power of two.
Since $w \geq 8$, then $1 + w \geq 9$ and so $1 + w \geq 2^4$ (the smallest power of 2 greater than 8).
Since $f(N) = (1 + w)(1 + a_1)(1 + a_2) \cdots (1 + a_k) = 2^8$, then $2^4 \leq 1 + w \leq 2^8$ and $1 \leq (1 + a_1)(1 + a_2) \cdots (1 + a_k) \leq 2^4$, and so w must be equal to $15, 31, 63, 127$, or 255, and each of $a_1, a_2, \cdots, a_k$ must be equal to $1, 3, 7$, or 15.
For example, if $1 + w = 2^8$, then $w = 256 - 1 = 255$ and $N = 2^{255}$.
If $1 + w = 2^4$, then $w = 15$ and N is of the form $N = 2^{15} p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$.
Thus, the smallest N having the greatest number of distinct prime factors is $N = 2^{15} 3^1 5^1 7^1 11^1$, and we may confirm that $f(N) = (1 + 15)(1 + 1)(1 + 1)(1 + 1)(1 + 1) = 2^4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^8$,

as required.

Comparing these first two values of N, we recognize that $2^{15}3^{1}5^{1}7^{1}11^{1}$ is significantly less than 2^{255}.

Further, we recognize that

- all remaining possible values of N have exactly 2, 3 or 4 distinct prime factors,
- where exponents are equal, smaller prime factors give smaller values of N,
- the greatest exponents must occur on the smallest prime factors, and
- we recall that each exponent is one less than a power of 2.

In the table below, we use the above information to determine the smallest possible values of N having 1, 2, 3, 4, and 5 distinct prime factors.

Further, we compare the size of each of these values of N within each of the 5 groups.

Number of distinct prime factors of N	Values of N
1	2^{255}
2	$2^{15}3^{15} < 2^{31}3^{7} < 2^{63}3^{3} < 2^{127}3^{1}$
3	$2^{15}3^{3}5^{3} < 2^{15}3^{7}5^{1} < 2^{31}3^{3}5^{1} < 2^{63}3^{1}5^{1}$
4	$2^{15}3^{3}5^{1}7^{1} < 2^{31}3^{1}5^{1}7^{1}$
5	$2^{15}3^{1}5^{1}7^{1}11^{1}$

Finally, we compare the smallest values of N from each of the 5 rows in the table above.

Since $2^{15}3^{3}5^{1}7^{1} < 2^{15}3^{1}5^{1}7^{1}11^{1} < 2^{15}3^{3}5^{3} < 2^{15}3^{15} < 2^{255}$, then the smallest refactorable number N with $f(N) = 256$, is $N = 2^{15}3^{3}5^{1}7^{1} = 30\,965\,760$.

(d) Let m be a positive integer of the form $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1 < p_2 < \cdots < p_k$ are prime numbers, and $a_1, a_2, \cdots, a_k$ are positive integers.

We begin by stating a value of N and then proceed to show that it works.

For each $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, choose $N = p_1^{\left(p_1^{a_1}-1\right)} p_2^{\left(p_2^{a_2}-1\right)} \cdots p_k^{\left(p_k^{a_k}-1\right)}$.

On the contest paper, the *useful fact* states that $2^n \geq n + 1$ for all positive integers n.

Since $p \geq 2$ for all prime numbers p, then $p^n \geq 2^n \geq n + 1$.

Since $p^n \geq n + 1$, then $p^n - 1 \geq n$, and so $p_i^{a_i} - 1 \geq a_i$ for all integers i where $1 \leq i \leq k$.

Thus, $p_1^{a_1}$ divides $p_1^{\left(p_1^{a_1}-1\right)}$ since each is a power with base p_1 and $p_1^{a_1} - 1 \geq a_1$.

Similarly, $p_2^{a_2}$ divides $p_2^{\left(p_2^{a_2}-1\right)}$, and so on.

Therefore, $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ divides $N = p_1^{\left(p_1^{a_1}-1\right)} p_2^{\left(p_2^{a_2}-1\right)} \cdots p_k^{\left(p_k^{a_k}-1\right)}$.

Further, $f(N) = (1 + p_1^{a_1} - 1)(1 + p_2^{a_2} - 1) \cdots (1 + p_k^{a_k} - 1) = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} = m$.

Thus for every integer $m > 1$, there exists a refactorable number N such that $f(N) = m$.