The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Canadian Senior Mathematics Contest

Wednesday, November 15, 2023
(in North America and South America)

Thursday, November 16, 2023
(outside of North America and South America)

Time: 2 hours

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Do not open this booklet until instructed to do so.
There are two parts to this paper. The questions in each part are arranged roughly in order of increasing difficulty. The early problems in Part B are likely easier than the later problems in Part A.

PART A

1. This part consists of six questions, each worth 5 marks.
2. **Enter the answer in the appropriate box in the answer booklet.** For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

PART B

1. This part consists of three questions, each worth 10 marks.
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, they will be supplied by your supervising teacher. Insert these pages into your answer booklet. Write your name, school name, and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

At the completion of the contest, insert your student information form inside your answer booklet.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on the website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some students may be shared with other mathematical organizations for other recognition opportunities.

Canadian Senior Mathematics Contest

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write solutions in the answer booklet provided.
3. Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.
4. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
5. Diagrams are not drawn to scale. They are intended as aids only.
6. No student may write both the Canadian Senior Mathematics Contest and the Canadian Intermediate Mathematics Contest in the same year.

PART A

For each question in Part A, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded only if relevant work is shown in the space provided in the answer booklet.

1. Two prime numbers p and q satisfy the equation $p + q = 31$. What is pq?

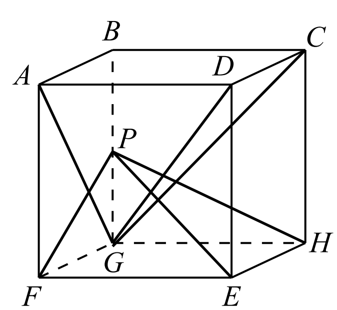
2. The integer 203 has an odd ones digit, an even tens digit, and an even hundreds digit. How many integers between 100 and 999 have an odd ones digit, an even tens digit, and an even hundreds digit?

3. The distance from point $P(x, y)$ to the origin $O(0, 0)$ is 17.
   The distance from point $P(x, y)$ to $A(16, 0)$ is also 17.
   What are the two possible pairs of coordinates (x, y) for P?

4. A store sells shirts, water bottles, and chocolate bars. Shirts cost \$10 each, water bottles cost \$5 each, and chocolate bars cost \$1 each. On one particular day, the store sold x shirts, y water bottles, and z chocolate bars. The total revenue from these sales was \$120. If x, y and z are integers with $x > 0$, $y > 0$ and $z > 0$, how many possibilities are there for the ordered triple (x, y, z)?

5. What are all pairs of integers (r, p) for which $r^2 - r(p + 6) + p^2 + 5p + 6 = 0$?

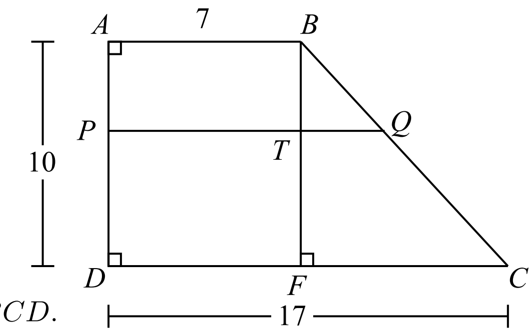
6. Cube $ABCDEFGH$ has edge length 6, and has P on BG so that $BP = PG$. What is the volume of the three-dimensional region that lies inside both square-based pyramid $EFGHP$ and square-based pyramid $ABCDG$?



PART B

For each question in Part B, your solution must be well-organized and contain words of explanation or justification. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

1. In the diagram, $ABCD$ is a trapezoid with $\angle DAB = \angle ADC = 90^\circ$. Also, $AB = 7$, $DC = 17$, and $AD = 10$. Point P is on AD and point Q is on BC so that PQ is parallel to AB and to DC. Point F is on DC so that BF is perpendicular to DC. BF intersects PQ at T.



   (a) Determine the area of trapezoid $ABCD$.

   (b) Determine the measure of $\angle BQP$.

   (c) If $PQ = x$, determine the length of AP in terms of x.

   (d) Determine the length of PQ for which the areas of trapezoids $ABQP$ and $PQCD$ are equal.

2. The rectangular region A is in the first quadrant. The bottom side and top side of A are formed by the lines with equations $y = 0.5$ and $y = 99.5$, respectively. The left side and right side of A are formed by the lines with equations $x = 0.5$ and $x = 99.5$, respectively.

   (a) Determine the number of lattice points that are on the line with equation $y = 2x + 5$ and are inside the region A. (A point with coordinates (r, s) is called a *lattice point* if r and s are both integers.)

   (b) For some integer b, the number of lattice points on the line with equation $y = \frac{5}{3}x + b$ and inside the region A is at least 15. Determine the greatest possible value of b.

   (c) For some real numbers m, there are no lattice points that lie on the line with equation $y = mx + 1$ and inside the region A. Determine the greatest possible real number n that has the property that, for all real numbers m with $\frac{2}{7} < m < n$, there are no lattice points on the line with equation $y = mx + 1$ and inside the region A.

3. In this problem, you may choose to consider angles in either degrees or in radians.

   (a) Determine an angle x for which $\sin\left(\frac{x}{5}\right) = \sin\left(\frac{x}{9}\right) = 1$.

   (b) There are sequences of 100 distinct positive integers $n_1, n_2, \ldots, n_{100}$ with the property that, for all integers i and j with $1 \leq i < j \leq 100$ and for all angles x, we have $\sin\left(\frac{x}{n_i}\right) + \sin\left(\frac{x}{n_j}\right) \neq 2$. Determine such a sequence $n_1, n_2, \ldots, n_{100}$ and prove that it has this property.

   (c) Suppose that $m_1, m_2, \ldots, m_{100}$ is a list of 100 distinct positive integers with the property that, for each integer $i = 1, 2, \ldots, 99$, there is an angle x_i with $\sin\left(\frac{x_i}{m_i}\right) + \sin\left(\frac{x_i}{m_{i+1}}\right) = 2$. Prove that, for every such sequence $m_1, m_2, \ldots, m_{100}$ with $m_1 = 6$, there exists an angle t for which

$$\sin\left(\frac{t}{m_1}\right) + \sin\left(\frac{t}{m_2}\right) + \cdots + \sin\left(\frac{t}{m_{100}}\right) = 100$$

   (The sum on the left side consists of 100 terms.)