The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2023
Canadian Team Mathematics Contest

April 2023

Solutions

Individual Problems

1. Working backwards, since Jin had 5 chocolates left after eating half of them, she had $2 \times 5 = 10$ chocolates after giving chocolates to Brian.
   Since Jin gave 8 chocolates to Brian and had 10 chocolates after doing this, Jin was given $10 + 8 = 18$ chocolates by Ingrid.
   Ingrid gave one third of her chocolates to Jin, which means Ingrid started with $n = 3 \times 18 = 54$ chocolates.

   ANSWER: 54

2. Since 20% of 30 is $0.2 \times 30 = 6$, we want to find k so that $k\%$ of 25 equals 6.
   This means $\frac{k}{100} \times 25 = 6$ or $\frac{k}{4} = 6$ and so $k = 24$.

   ANSWER: 24

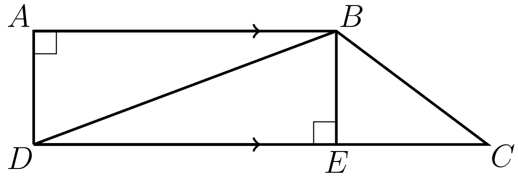
3. Note that $2023 = 33 \times 60 + 43$, and since there are 60 minutes in an hour, this means that 2023 minutes is equal to 33 hours and 43 minutes.
   For 33 hours and 43 minutes to pass, 24 hours will pass and then 9 hours and 43 minutes will pass.
   In 24 hours it will be 1:00 a.m. again.
   In 9 hours and 43 minutes after 1:00 a.m., it will be 10:43 a.m.

   ANSWER: 10:43 a.m.

4. As soon as two lockers are painted blue in the top row, the other two lockers in the top row must be painted red.
   Once the top row is painted, the colours of the lockers in the bottom row are determined.
   If the lockers in the top row are numbered 1, 2, 3, and 4, then there are six possibilities for the two blue lockers.
   They are 1 and 2, 1 and 3, 1 and 4, 2 and 3, 2 and 4, and 3 and 4.

   ANSWER: 6

5. Let E be on CD such that BE is perpendicular to CD as shown.



   Since AB and DE are parallel and $\angle DAB$ and $\angle DEB$ are right angles, $\angle ADE$ and $\angle ABE$ are right angles as well, and so $ABED$ is a rectangle.
   Using $AB = 4$ and the fact that $ABED$ is a rectangle, we get $DE = 4$.
   Using $CD = CE + DE$, $DE = 4$, and $CD = 6$, we get $CE = 6 - 4 = 2$.
   In $\triangle BEC$, $\angle BEC = 90^\circ$ and $\angle BCE = \angle BCD = 45^\circ$, and so $\angle EBC = 180^\circ - 90^\circ - 45^\circ = 45^\circ$.
   This means $\triangle BEC$ is isosceles, and so $BE = CE = 2$.
   By the Pythagorean theorem applied to right-angled $\triangle BED$, we have $DE^2 + BE^2 = BD^2$, and so $BD^2 = 4^2 + 2^2 = 20$.
   Since $BD > 0$, $BD = \sqrt{20} = 2\sqrt{5}$.

   ANSWER: $2\sqrt{5}$

6. Let t be the amount of time in hours that the train will take to make the trip if it is on time and let d be the distance in kilometres between City A and City B.

   If the train is 24 minutes late, then it is $\frac{24}{60}$ hours late, which means it takes $t + \frac{24}{60} = t + \frac{2}{5}$ hours to reach City B.

   Similarly, if the train is 32 minutes early, then it takes $t - \frac{32}{60} = t - \frac{8}{15}$ hours to reach City B.

   From the given assumptions, this means $80 = \frac{d}{t+\frac{2}{5}}$ and $90 = \frac{d}{t-\frac{8}{15}}$.

   Cross multiplying these equations, we get $80\left(t+\frac{2}{5}\right) = d$ and $90\left(t-\frac{8}{15}\right) = d$.

   Equating the values of d and solving for t, we get

$$\begin{aligned}80\left(t+\frac{2}{5}\right) &= 90\left(t-\frac{8}{15}\right)\\ 80t+32 &= 90t-48\\ 80 &= 10t\\ 8 &= t\end{aligned}$$

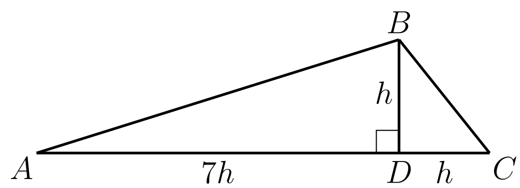
   Substituting $t=8$ into $d = 80\left(t+\frac{2}{5}\right)$ gives $d = 80\left(8+\frac{2}{5}\right) = 672$.

   The speed in km/h that the train should travel in order to arrive on time is $\frac{672}{8} = 84$.

   Answer: 84

7. We are given that $\tan\angle BAC = \frac{1}{7}$, but $\angle BAC = \angle BAD$ and $\triangle BAD$ has a right angle at D, so $\tan\angle BAD = \frac{BD}{AD} = \frac{1}{7}$. Since $BD = h$, $AD = 7\cdot BD = 7h$.

   Using that $\tan\angle BCD = \tan\angle BCA = 1$, we get $CD = h$ by similar reasoning. Therefore, $AC = AD + CD = 7h + h = 8h$.



   Applying the Pythagorean theorem to $\triangle ABD$, we get that $AB = \sqrt{(7h)^2+h^2} = \sqrt{50h^2}$.
   Applying the Pythagorean theorem to $\triangle BCD$, we get that $BC = \sqrt{h^2+h^2} = \sqrt{2h^2}$.
   The perimeter of $\triangle ABC$ is

$$\begin{aligned}AB+BC+AC &= \sqrt{50h^2}+\sqrt{2h^2}+8h\\ &= \sqrt{2\times 5^2\times h^2}+\sqrt{2h^2}+8h\\ &= 5h\sqrt{2}+h\sqrt{2}+8h && \text{(since } h>0\text{)}\\ &= h(8+6\sqrt{2})\end{aligned}$$

   Since the perimeter of $\triangle ABC$ is given to be $24+18\sqrt{2} = 3(8+6\sqrt{2})$, we conclude that $h = 3$.

   Answer: 3

8. Suppose $ABCDE$ is a Tim number. That is, A, B, C, D, and E are the digits of the number and since it has five digits, $A \neq 0$.
Since $ABCDE$ is a multiple of 15, it must be a multiple of both 3 and 5.
Since $ABCDE$ is a multiple of 5, either $E = 0$ or $E = 5$.
Since $ABCDE$ is a multiple of 3, $A + B + C + D + E$ is a multiple of 3.
We are also given that $C = 3$ and $D = A + B + C = A + B + 3$.
Combining $C = 3$ and $D = A + B + 3$ with the fact that $A + B + C + D + E$ is a multiple of 3, we have that $A + B + 3 + A + B + 3 + E = 2A + 2B + E + 6$ is a multiple of 3.

Since 6 is a multiple of 3, this means $2A + 2B + E$ is a multiple of 3.
We are looking for triples (A, B, E) with the property that $E = 0$ or $E = 5$ and $2A + 2B + E$ is a multiple of 3. However, we must keep in mind that $A + B + 3 = D$ is a digit, which means $A + B + 3 \leq 9$ or $A + B \leq 6$.
We will consider two cases: $E = 0$ and $E = 5$. We note that once we choose values for A and B, the value of D will be determined.

$\underline{E = 0}$: In this case, we need $2A + 2B$ to be a multiple of 3, so $A + B$ is a multiple of 3. Combined with the requirement that $A + B \leq 6$, this means $A + B = 3$ or $A + B = 6$. We also need to remember that $A \neq 0$, which means (A, B) is one of $(1, 2)$, $(2, 1)$, $(3, 0)$, $(1, 5)$, $(2, 4)$, $(3, 3)$, $(4, 2)$, $(5, 1)$, and $(6, 0)$ for a total of nine possibilities.

$\underline{E = 5}$: In this case, we need $2A + 2B + 5$ to be a multiple of 3. For the same reason as in the previous case, $A + B \leq 6$.
If $A + B = 1$, then $2A + 2B + 5 = 7$ which is not a multiple of 3.
If $A + B = 2$, then $2A + 2B + 5 = 9$ which is a multiple of 3.
If $A + B = 3$, then $2A + 2B + 5 = 11$ which is not a multiple of 3.
If $A + B = 4$, then $2A + 2B + 5 = 13$ which is not a multiple of 3.
If $A + B = 5$, then $2A + 2B + 5 = 15$ which is a multiple of 3.
If $A + B = 6$, then $2A + 2B + 5 = 17$ which is not a multiple of 3.

Therefore, we must have that $A+B = 2$ or $A+B = 5$. Keeping in mind that $A \neq 0$, we get that (A, B) can be $(1, 1)$, $(2, 0)$, $(1, 4)$, $(2, 3)$, $(3, 2)$, $(4, 1)$, or $(5, 0)$ for a total of seven possibilities.

This gives a total of $9 + 7 = 16$ Tim numbers, which are listed below. One can check that all 16 of these integers are indeed Tim numbers.

$$12360, 15390, 21360, 24390, 30360, 33390, 42390, 51390,$$

$$60390, 11355, 14385, 20355, 23385, 32385, 41385, 50385$$

Answer: 16

9. *Solution 1*
Multiplying $4x + 7y + z = 11$ by 2 gives $8x + 14y + 2z = 22$ and multiplying $3x + y + 5z = 15$ by 3 gives $9x + 3y + 15z = 45$.
Adding $8x + 14y + 2z = 22$ and $9x + 3y + 15z = 45$ gives $17x + 17y + 17z = 67$, and after dividing through by 17, we get $x + y + z = \dfrac{67}{17}$.
The fraction $\dfrac{67}{17}$ is in lowest terms, which means $p = 67$ and $q = 17$, so $p - q = 67 - 17 = 50$.

Solution 2
We will try to solve the system of equations

$$\begin{aligned} 4x + 7y + z &= 11 \\ 3x + y + 5z &= 15 \\ x + y + z &= \frac{p}{q} \end{aligned}$$

for x, y, and z in terms of p and q and see what happens. For this particular system, it is impossible to express x, y, and z in terms of only p and q. However, a relationship between p and q will be revealed by the process of attempting to solve the system.

Subtracting the second equation from the first, we get $x + 6y - 4z = -4$, and subtracting $x + y + z = \frac{p}{q}$ from this equation gives $5y - 5z = -4 - \frac{p}{q}$.
After dividing by 5, we get $y - z = -\frac{4}{5} - \frac{p}{5q}$.
Next, we subtract 4 times the second equation from 3 times the first equation to get

$$\begin{aligned} 3(4x + 7y + z) - 4(3x + y + 5z) &= 3(11) - 4(15) \\ 12x + 21y + 3z - 12x - 4y - 20z &= 33 - 60 \\ 17y - 17z &= -27 \\ y - z &= -\frac{27}{17} \end{aligned}$$

We now have two expressions for $y - z$, and if we set them equal, we get an equation involving p and q. We can manipulate this equation as follows.

$$\begin{aligned} -\frac{4}{5} - \frac{p}{5q} &= -\frac{27}{17} \\ -\frac{4}{5} + \frac{27}{17} &= \frac{p}{5q} \\ -4 + \frac{5 \times 27}{17} &= \frac{p}{q} \\ \frac{-68 + 135}{17} &= \frac{p}{q} \\ \frac{67}{17} &= \frac{p}{q} \end{aligned}$$

and since $\frac{67}{17}$ is in lowest terms, $p = 67$ and $q = 17$, so $p - q = 67 - 17 = 50$.

ANSWER: 50

10. Fix a positive integer k and let $n = 2k$ (we are interested in even n).
The two-element subsets of S_{2k} consisting of two integers that have a sum of $2k+1$ are $\{1, 2k\}$, $\{2, 2k-1\}$, $\{3, 2k-2\}$, and so on up to $\{k, k+1\}$. There are k subsets in the list.
A subset X of S_{2k} contains two integers with a sum of $2k+1$ exactly when it contains both of the integers from at least one of the k subsets above.

We will now count the number of subsets X of S_{2k} with the property that no two integers in the subset have a sum of $2k+1$.
We will do this by counting the number of subsets X of S_{2k} that contain at most one of the

two integers from each of the k two-element subsets listed earlier.
Suppose $\{a, b\}$ is one of these k subsets. The set X could contain neither a nor b, it could contain a and not b, and it could contain b and not a.
This gives 3 possibilities for each of the k subsets.
Every integer in S_{2k} appears in exactly one of these k subsets, so a choice of one of these three possibilities for each of the k subsets $\{a, b\}$ completely determines X.
These choices are independent, which means the number of subsets X with the desired property is exactly 3^k.

Since there are 2^{2k} subsets in total, we have

$$p(2k) = \frac{3^k}{2^{2k}} = \left(\frac{3}{4}\right)^k$$

We want to find values of k that satisfy $p(2k) < \frac{1}{4}$. Using the formula above, this is equivalent to $\left(\frac{3}{4}\right)^k < \frac{1}{4}$, which is equivalent to $3^k < 4^{k-1}$.
The table below has positive integer values of k in the left column, the value of 3^k in the middle column, and 4^{k-1} in the right column.

k	3^k	4^{k-1}
1	3	1
2	9	4
3	27	16
4	81	64
5	243	256

We see that $k = 5$ is the smallest even positive integer for which $3^k < 4^{k-1}$, so the answer is $n = 2 \times 5 = 10$.

Answer: 10

Team Problems

1. The cod accounts for $100\% - 40\% - 40\% = 20\%$ of the total pieces of fish sold. Therefore, the number of pieces of cod sold was $0.2 \times 220 = 44$.

ANSWER: 44

2. Rearranging the given equation, we get $\frac{x}{2} = 14$ which implies that $x = 28 = 2^2 \times 7$.
Therefore, $\sqrt{7x} = \sqrt{7 \times 2^2 \times 7} = \sqrt{2^2 \times 7^2} = 2 \times 7 = 14$.

ANSWER: 14

3. The slope of AB is $\frac{-43 - (-23)}{-33 - (-13)} = \frac{-20}{-20} = 1$.
Since AB is neither horizontal nor vertical, the slope of a line perpendicular to it is the negative of the reciprocal of 1, or $\frac{-1}{1} = -1$.

ANSWER: -1

4. Factoring $119^2 - 17^2$ as a difference of squares, we have $119^2 - 17^2 = (119 - 17)(119 + 17)$. Then

$$\sqrt{\frac{119^2 - 17^2}{119 - 17} - 10^2} = \sqrt{\frac{(119 - 17)(119 + 17)}{119 - 17} - 10^2} = \sqrt{119 + 17 - 100} = \sqrt{36} = 6$$

ANSWER: 6

5. *Solution 1*
Substituting $p = 2q$ into $p + q + r = 70$ gives $2q + q + r = 70$ or $3q + r = 70$.
Substituting $q = 3r$ into $3q + r = 70$ gives $3(3r) + r = 70$ or $10r = 70$ which means $r = 7$.
Substituting $r = 7$ into $q = 3r$ gives $q = 21$, and substituting $q = 21$ into $p = 2q$ gives $p = 42$.

Solution 2
The equation $p = 2q$ is equivalent to $q = \frac{1}{2}p$.
The equation $q = 3r$ is equivalent to $r = \frac{1}{3}q$.
Substituting $q = \frac{1}{2}p$ into $r = \frac{1}{3}q$ gives $r = \frac{1}{6}p$.
Substituting these values of q and r into $p + q + r = 70$ gives $p + \frac{1}{2}p + \frac{1}{6}p = 70$.
Therefore,

$$p = \frac{70}{1 + \frac{1}{2} + \frac{1}{6}} = \frac{70}{\frac{10}{6}} = 42$$

ANSWER: 42

6. If we let $d = b - a$, then we have $b = a + d$, and since $c - b = b - a = d$, we have $c = b + d = a + 2d$. Thus, $(a, b, c) = (a, a + d, a + 2d)$.
Since $b > a$, $d > 0$, and so we can enumerate the triples by considering possible values of d starting at $d = 1$.
When $d = 1$, we get the triples $(1, 2, 3)$, $(2, 3, 4)$, $(3, 4, 5)$, $(4, 5, 6)$, $(5, 6, 7)$, $(6, 7, 8)$, $(7, 8, 9)$, and $(8, 9, 10)$ for a total of 8 triples.
When $d = 2$, we get the triples $(1, 3, 5)$, $(2, 4, 6)$, $(3, 5, 7)$, $(4, 6, 8)$, $(5, 7, 9)$, and $(6, 8, 10)$ for a total of 6 triples.

When $d = 3$, we get the triples $(1, 4, 7)$, $(2, 5, 8)$, $(3, 6, 9)$, and $(4, 7, 10)$ for a total of 4 triples.
When $d = 4$, we get the triples $(1, 5, 9)$ and $(2, 6, 10)$ for a total of 2 triples.
If $d \geq 5$, then since a is at least 1, $c = a + 2d \geq 1 + 2 \times 5 = 11$. This means d cannot be any larger than 4, so we have found all triples.
In total, there are $8 + 6 + 4 + 2 = 20$ triples.

Answer: 20

7. The positive integer 4446 is even and the sum of its digits is $4 + 4 + 4 + 6 = 18$ which is a multiple of 9, so 4446 itself is divisible by both 2 and 9. (Similar to the well-known divisibility rule for 3, a positive integer is divisible by 9 exactly when the sum of its digits is divisible by 9.)
Factoring, we have $4446 = 2 \times 3^2 \times 247$, and it can be checked that $247 = 13 \times 19$.
Therefore, $4446 = 2 \times 3^2 \times 13 \times 19$, so the distinct prime factors of 4446 are 2, 3, 13, and 19, the sum of which is 37.

Answer: 37

8. Since every line segment must be used exactly once, the integer will have seven digits with each digit from 1 to 7 used exactly once.
Observe that the path that goes from E to B to C to E to D to B to A to C creates the integer 7645123.

We will now argue that this is the largest integer that can be created.
Since every digit from 1 through 7 is used exactly once, if we start with any digits other than 7 and 6, in that order, the integer formed must be smaller than 7645123.
This means that the path forming the largest possible integer must start with E, then go to B, and then C.
From C, the options are to go to A or E. If the path continues to A, then the integer would start with 763, which is guaranteed to be smaller than 7645123, regardless of how the path is completed.
The path must start with E, B, C, E, so the first three digits of the largest inter are 764.
The line segments connecting E to B and E to C have already been used, which means that the path must continue to D since otherwise we would have to reuse a line segment. We now have shown that the first four digits of the largest integer are 7645.
Since the path arrived at D from E, there is nowhere to go from D but to B, and then by similar reasoning, it must continue to A and finally to C, so the largest integer must be 7645123.

Answer: 7645123

9. *Solution 1*
Let V equal the volume of the pool in litres and let x be the rate at which each hose outputs water in litres per hour.
With three hoses, the pool will fill in 12 hours at a rate of $3x$ litres per hour. This means $3x = \frac{V}{12}$ or $x = \frac{V}{36}$.
One of the hoses stops working after 5 of the 12 hours, which means that the hose stops working when there are $\frac{5}{12}V$ litres in the pool, or $\frac{7}{12}V$ more litres to be added to the pool.

Starting at 11:00 a.m., the new rate at which water is entering the pool is $2x$, and $\frac{7}{12}V$ litres need to be added to the pool.
If we let t be the amount of time in hours remaining to fill the pool with two hoses, then we

have $2x = \dfrac{\frac{7}{12}V}{t}$ and so $t = \dfrac{7V}{24x}$.
Substituting $x = \dfrac{V}{36}$, we get

$$t = \frac{7V}{24\frac{V}{36}} = \frac{21}{2} = 10.5$$

Therefore, another 10.5 hours are required, so the pool will be full 10.5 hours after 11:00 a.m., or at 9:30 p.m.

Solution 2
With three hoses working, the pool will fill in 12 hours, which means that one hose could fill the pool in 36 hours. We will say that the pool takes 36 hose-hours to fill.

At the time when one of the hoses stops working, three hoses have been working for 5 hours. In other words, which is the same as if one hose had been working for $3 \times 5 = 15$ hours. In other words, 15 hose hours have been spent putting water in the pool.
This means $36 - 15 = 21$ hose hours are still required. There are two hoses still working, so it will take and additional $\dfrac{21}{2} = 10.5$ hours to fill the pool.
Since 10.5 hours after 11:00 a.m. is 9:30 p.m., the pool will be filled at 9:30 pm.

ANSWER: 9:30 p.m.

10. Substituting $x = 2$ and $y = -3$ into the two equations gives

$$\begin{aligned} 2(a^2+1) - 2b(-3) &= 4 \\ 2(1-a) + b(-3) &= 9 \end{aligned}$$

which can be simplified to get

$$\begin{aligned} 2a^2 + 6b &= 2 \\ 2a + 3b &= -7 \end{aligned}$$

Dividing the first equation by 2 gives $a^2 + 3b = 1$ from which we can subtract $2a + 3b = -7$ to get $a^2 - 2a = 8$.
Rearranging gives $a^2 - 2a - 8 = 0$ which can be factored to get $(a-4)(a+2) = 0$.
Therefore, there are two possible values of a, which are $a = 4$ and $a = -2$. Substituting these values into $2a + 3b = -7$ and solving for b gives $b = -5$ and $b = -1$, respectively.
The ordered pairs $(a, b) = (4, -5)$ and $(a, b) = (-2, -1)$ are the two possibilities. It can be checked that the two systems of equations

$$\begin{aligned} 17x + 10y &= 4 \\ -3x - 5y &= 9 \end{aligned} \qquad\qquad \begin{aligned} 5x + 2y &= 4 \\ 3x - y &= 9 \end{aligned}$$

both have the unique solution $(x, y) = (2, -3)$.

ANSWER: $(4, -5)$ and $(-2, -1)$

11. If we set $x = 2023$, then

$$2023^4 - (2022)(2024)(1 + 2023^2) = x^4 - (x-1)(x+1)(1+x^2)$$

Expanding, we have $(x-1)(x+1)(1+x^2) = (x^2-1)(x^2+1) = x^4 - 1$.
This means $x^4 - (x-1)(x+1)(1+x^2) = x^4 - (x^4 - 1) = 1$, so the answer is 1.

ANSWER: 1

12. For the square root in the numerator to be defined, we need $75 - x \geq 0$ or $x \leq 75$, and for the square root in the denominator to be defined, we need $x - 25 \geq 0$ or $25 \leq x$. Therefore, for the expression to be an integer, we must have $25 \leq x \leq 75$.

Suppose $\dfrac{\sqrt{75-x}}{\sqrt{x-25}}$ is equal to some integer n. Then $\sqrt{75-x} = n\sqrt{x-25}$.
Squaring both sides, we get $75 - x = n^2(x - 25)$ or $75 - x = xn^2 - 25n^2$. This equation can be rearranged to get $25n^2 + 75 = x(n^2 + 1)$ and then solved for x to get $x = \dfrac{25n^2+75}{n^2+1}$.
Rearranging, we have

$$x = \frac{25n^2+75}{n^2+1} = \frac{25n^2+25+50}{n^2+1} = \frac{25(n^2+1)}{n^2+1} + \frac{50}{n^2+1} = 25 + \frac{50}{n^2+1}$$

Observe that $n^2 + 1 > 0$ for all n, so the cancellation above is justified.
Since x and 25 are both integers, $\dfrac{50}{n^2+1}$ must also be an integer, so 50 is a multiple of $n^2 + 1$.
Since $n^2 + 1$ is a divisor of 50 and it must be positive, we have that $n^2 + 1$ must be one of 1, 2, 5, 10, 25, or 50.

Setting $n^2 + 1$ equal to each of these values and solving gives

$$\begin{aligned}
n^2+1 &= 1 && \Rightarrow n = 0 \\
n^2+1 &= 2 && \Rightarrow n = \pm 1 \\
n^2+1 &= 5 && \Rightarrow n = \pm 2 \\
n^2+1 &= 10 && \Rightarrow n = \pm 3 \\
n^2+1 &= 25 && \Rightarrow n = \pm\sqrt{24} \\
n^2+1 &= 50 && \Rightarrow n = \pm 7
\end{aligned}$$

Note that n is the ratio of two square roots, which means it must be nonnegative. As well, $\sqrt{24}$ is not an integer, so this means $n = 0$, $n = 1$, $n = 2$, $n = 3$, or $n = 7$.
Substituting these values of n into $x = \dfrac{25n^2+75}{n^2+1}$, we get that the possible values of x are 75, 50, 35, 30, and 26, of which there are 5.

ANSWER: 5

13. The two-digit perfect squares are 16, 25, 36, 49, 64, and 81.
Notice that no two of these perfect squares have a common first digit, which means that in a mystical integer, either a digit is the rightmost digit or there is only one possibility for the digit to its right.
This means, for each possible first digit, the largest mystical integer with that first digit is the mystical integer with as many digits as possible.
To make a mystical integer starting with 1 have as many digits as possible, we must follow the 1 by a 6, which must be followed by 4 and then 9. No two-digit perfect square starts with 9, so 1649 is the longest, and hence, the largest mystical integer with its first digit equal to 1.
The only mystical integer with its first digit equal to 2 is 25 since no two-digit perfect square has a first digit of 5.
The longest mystical integer starting with 3 is 3649.
The longest mystical integer starting with 4 is 49.
There are no mystical integers starting with 5, 7, or 9.
The longest mystical integer with a first digit of 6 is 649.
The longest mystical integer starting with 8 is 81649. This is the only five-digit mystical integer, so it must be the largest.

ANSWER: 81649

14. Suppose a, b, c, and d are the digits 1 through 4 in some order and are placed in the top-left subgrid as shown below.

a	b	
c	d	

In order for the bottom-left subgrid to contain the integers from 1 through 4, the leftmost two cells in the bottom row must contain c and d in some order. Similarly, in order for the top-right subgrid to contain the digits from 1 through 4, the third cells in the first two rows must contain a and c in some order.
Therefore, there are four possible ways to fill in the rest of the cells, excluding the bottom-right cell. They are shown below.

a	b	a
c	d	c
a	b	

a	b	c
c	d	a
a	b	

a	b	a
c	d	c
b	a	

a	b	c
c	d	a
b	a	

In each of these four grids, the top-left, top-right, and bottom-left subgrids all satisfy the condition that each digit occurs exactly once in it.
In the first three of four, the three digits already in the bottom-right subgrid are different, so there is exactly one way to place a digit in the bottom right cell to make the grid satisfy the condition.
For the first of the four grids above, we need to place a in the bottom-right cell. For the second, we need to place c. For the third, we need to place b.
The fourth of these grids lacks b and c in the bottom-right subgrid. With only one cell left to fill, there is no way to place a digit in it so that every digit occurs in the subgrid.

There are 24 ways to order the digits from 1 through 4 as a, b, c, and d. We have shown that there are three ways to fill the grid for each of these 24 orderings, so the answer is $3 \times 24 = 72$.

ANSWER: 72

15. Two numbers are reciprocals if their product is 1. Therefore, we are looking for real numbers x for which $\left(x - \frac{5}{x}\right)(x-4) = 1$.
Multiplying through by x gives $(x^2-5)(x-4) = x$. Expanding gives $x^3 - 4x^2 - 5x + 20 = x$ which can be rearranged to get $x^3 - 4x^2 - 6x + 20 = 0$.
Given that the problem states that there are exactly three real numbers with the given condition, these three real numbers must be the solutions to the cubic equation above. A cubic equation has at most three real roots, which means this cubic must have exactly three real roots. The sum of the roots of a cubic is the negative of the coefficient of x^2, which means the answer is 4.
However, in the interest of presenting a solution that does not assume the question "works", we will factor this cubic to find the three real numbers.

A bit of guessing and checking reveals that $2^3 - 4(2^2) - 6(2) + 20 = 0$, which means $x - 2$ is a factor of the cubic. We can factor to get

$$(x-2)(x^2 - 2x - 10) = 0$$

By the quadratic formula, the other two roots of the cubic are

$$\frac{2 \pm \sqrt{2^2 - 4(-10)}}{2} = 1 \pm \sqrt{11}$$

The three real numbers are 2, $1 - \sqrt{11}$, and $1 + \sqrt{11}$ which have a sum of

$$2 + (1 - \sqrt{11}) + (1 + \sqrt{11}) = 4$$

Answer: 4

16. From the similarity of $\triangle PBA$ and $\triangle ABC$, we get $\dfrac{PB}{AB} = \dfrac{AB}{CB}$.
Substituting, we get $\dfrac{PB}{8} = \dfrac{8}{11}$ which can be rearranged to get $PB = \dfrac{64}{11}$.
From the similarity of $\triangle QAC$ and $\triangle ABC$, we get $\dfrac{QC}{AC} = \dfrac{AC}{BC}$.
Substituting, we get $\dfrac{QC}{6} = \dfrac{6}{11}$ which can be rearranged to get $QC = \dfrac{36}{11}$.
We now know the lengths of PB, QC, and BC. Since $PQ = BC - PB - QC$, we get

$$PQ = 11 - \frac{64}{11} - \frac{36}{11} = \frac{121 - 64 - 36}{11} = \frac{21}{11}$$

Answer: $\frac{21}{11}$

17. Let t be the x-coordinate of B and note that $t > 0$. The parabola has reflective symmetry across the y-axis, which implies $ABCD$ has reflective symmetry across the y-axis. Thus, the x-coordinate of A is $-t$.
Therefore, $AB = 2t$, and so the area of $ABCD$ is $(2t)^2 = 4t^2$.
The point C lies on the parabola with equation $y = x^2 - 4$ and has the same x-coordinate as B. Therefore, the coordinates of C are $(t, t^2 - 4)$.
Since BC is vertical, its length is the difference between the y-coordinates of B and C.
It is given that C is below the x-axis, so its y-coordinate is negative. The y-coordinate of B is 0, and so we have that $BC = 0 - (t^2 - 4) = 4 - t^2$.
Since $ABCD$ is a square, $AB = BC$, and so $2t = 4 - t^2$.
Rearranging this equation gives $t^2 + 2t - 4 = 0$, and using the quadratic formula, we get

$$t = \frac{-2 \pm \sqrt{2^2 - 4(-4)}}{2} = -1 \pm \sqrt{5}$$

The quantity $-1 - \sqrt{5}$ is negative, and we are assuming t is positive, which means $t = -1 + \sqrt{5}$.
We know that the area of $ABCD$ is $4t^2$, so the area is $4(-1 + \sqrt{5})^2 = 4(6 - 2\sqrt{5}) = 24 - 8\sqrt{5}$.

Answer: $24 - 8\sqrt{5}$

18. In this solution, "log" (without explicitly indicating the base) means "$\log_{10}$".
Using logarithm rules, we have

$$\begin{aligned} ab &= (\log_4 9)(108\log_3 8) \\ &= 108\frac{\log 9}{\log 4} \times \frac{\log 8}{\log 3} \\ &= 108\frac{\log 3^2}{\log 2^2} \times \frac{\log 2^3}{\log 3} \\ &= \frac{108 \times 2 \times 3 \times (\log 3) \times (\log 2)}{2 \times (\log 2) \times (\log 3)} \\ &= 108 \times 3 \\ &= 324 \end{aligned}$$

Since $ab = 324 = 18^2$ and $\sqrt{ab} > 0$, we have $\sqrt{ab} = 18$.

ANSWER: 18

19. The multiples of 3 from 1 through 25 are 3, 6, 9, 12, 15, 18, 21, and 24.
The only multiple of 5 in this list is 15.

Jolene chooses the red ball numbered 15 with probability $\frac{1}{5}$ since there are exactly five red balls.
If Jolene chooses the red ball numbered with 15, then there will be 7 green balls remaining with a multiple of 3 on them. The probability that Tia chooses a ball numbered with a multiple of 3 in this situation is $\frac{7}{24}$.
The probability that Jolene and Tia win the game if Jolene chooses the ball numbered with 15 is $\frac{1}{5} \times \frac{7}{24} = \frac{7}{120}$.

Jolene chooses a red ball other than the one numbered with 15 with probability $\frac{4}{5}$.
In this situation, there are 8 green balls numbered with a multiple of 3, so the probability that Tia chooses a green ball numbered with a multiple of 3 is $\frac{8}{24}$.
The probability that Jolene and Tia win the game if Jolene does not choose the red ball numbered by 15 is $\frac{4}{5} \times \frac{8}{24} = \frac{32}{120}$.
Since exactly one of the events "Jolene chooses the ball numbered with 15" and "Jolene does not choose the ball numbered with 15" must occur, the probability that Jolene and Tia win the game is equal to the sum of the probabilities that they win in each of the two situations.
Therefore, the probability that they win the game is

$$\frac{7}{120} + \frac{32}{120} = \frac{39}{120} = \frac{13}{40}$$

ANSWER: $\frac{13}{40}$

20. The remainder when 468 is divided by d is r. This means that $0 \le r < d$ and there is a unique integer q_1 such that $468 = q_1 d + r$.
Similarly, there are unique integers q_2 and q_3 such that $636 = q_2 d + r$ and $867 = q_3 d + r$.
This gives three equations

$$\begin{aligned} 468 &= q_1 d + r \\ 636 &= q_2 d + r \\ 867 &= q_3 d + r \end{aligned}$$

Subtracting the first equation from the second, the first from the third, and the second from the third gives the three equations

$$168 = (q_2 - q_1)d$$
$$399 = (q_3 - q_1)d$$
$$231 = (q_3 - q_2)d$$

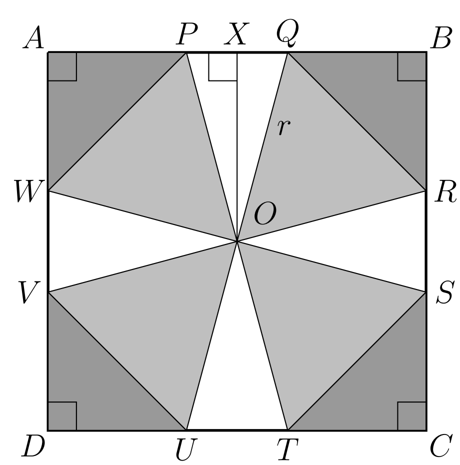
Since q_1, q_2, and q_3 are integers, so are $q_2 - q_1$, $q_3 - q_1$, and $q_3 - q_2$. Therefore, d must be a divisor of 168, 399, and 231.

Since d is a divisor of 168 and 231, it is a factor of $231 - 168 = 63$.
Since d is a divisor of 63 and 168, it is a factor of $168 - 2 \times 63 = 42$.
Since d is a divisor of 63 and 42, it is a factor of $63 - 42 = 21$.
Now note that $168 = 8 \times 21$, $399 = 19 \times 21$, and $231 = 11 \times 21$, so 21 is a divisorof all three integers.
We have shown that d must be a divisor of 21 and that 21 is a divisor of all three integers. This implies that the possible values of d are the positive divisors of 21, which are 1, 3, 7, and 21.

If $d = 1$, then the remainder is 0 when each of 468, 636, and 867 is divided by d. In this case, $d + r = 1 + 0 = 1$.
If $d = 3$, then the remainder is 0 when each of 468, 636, and 867 is divided by d. In this case, $d + r = 3 + 0 = 3$.
If $d = 7$, then the remainder is 6 when each of 468, 636, and 867 is divided by d. In this case, $d + r = 7 + 6 = 13$.
If $d = 21$, then the remainder is 6 when each of 468, 636, and 867 is divided by d. In this case, $d + r = 21 + 6 = 27$.

ANSWER: 27

21. *Solution 1*
Let X be the foot of the altitude of $\triangle PQO$ from O to PQ, and let $r = OQ$.



It is given that $\triangle RBQ$ is isosceles and since $\angle QBR = 90°$, we must have that $\angle BQR = 45°$.
As well, $\triangle ROQ$ is equilateral so $\angle RQO = 60°$.
Points X, Q, and B are on a line, so $\angle XQO = 180° - 60° - 45° = 75°$.
By very similar reasoning, $\angle XPO = 75°$, which means $\triangle OPQ$ is isosceles with $OP = OQ$.

Since $\angle OQX = 75^\circ$, we have $\sin 75^\circ = \dfrac{OX}{OQ} = \dfrac{OX}{r}$ and so $OX = r\sin 75^\circ$.
Similarly, $QX = r\cos 75^\circ$, and since $\triangle OPQ$ is isosceles, OX bisects PQ, which means $PQ = 2r\cos 75^\circ$.

From the double-angle formula, $\sin 150^\circ = 2\sin 75^\circ \cos 75^\circ$. Using this fact and the fact that $\sin 150^\circ = \dfrac{1}{2}$, we can compute the area of $\triangle OPQ$ as

$$\begin{aligned}\frac{1}{2} \times PQ \times OX &= \frac{1}{2}(2r\cos 75^\circ)(r\sin 75^\circ)\\ &= \frac{r^2}{2}(2\sin 75^\circ \cos 75^\circ)\\ &= \frac{r^2}{2}\sin 150^\circ\\ &= \frac{r^2}{2} \times \frac{1}{2}\\ &= \frac{r^2}{4}\end{aligned}$$

Since $\triangle ROQ$ is equilateral, $QR = OQ = r$, and since $\angle RQB = 45^\circ$, $\cos 45^\circ = \dfrac{BQ}{QR} = \dfrac{BQ}{r}$.
Rearranging and using that $\cos 45^\circ = \dfrac{1}{\sqrt{2}}$, we get $BQ = \dfrac{r}{\sqrt{2}}$.
Since $BQ = BR$, we get $BR = \dfrac{r}{\sqrt{2}}$ as well.
Therefore, the area of $\triangle BRQ$ is

$$\frac{1}{2} \times BQ \times BR = \frac{1}{2} \times \frac{r}{\sqrt{2}} \times \frac{r}{\sqrt{2}} = \frac{r^2}{4}$$

The areas of $\triangle PQO$ and $\triangle BRQ$ are equal, so the ratio is $1:1$.

Solution 2
Let $OQ = r$, which implies $QR = r$ as well since $\triangle ROQ$ is equilateral.
Since $\triangle BRQ$ has a right angle at B and is isosceles, the Pythagorean theorem implies that $BQ^2 + BR^2 = r^2$ or $2BQ^2 = r^2$, so $BQ = \dfrac{r}{\sqrt{2}}$.
Therefore, the area of $\triangle BRQ$ is $\dfrac{1}{2} \times \dfrac{r}{\sqrt{2}} \times \dfrac{r}{\sqrt{2}} = \dfrac{r^2}{4}$.
Since $\triangle BRQ$ is isosceles and right angled at B, $\angle RQB = 45^\circ$. As well, $\triangle ROQ$ is equilateral, which means $\angle ROQ = 60^\circ$. Using that P, Q, and B lie on the same line, we can compute $\angle PQO = 180^\circ - 60^\circ - 45^\circ = 75^\circ$.
By similar reasoning, $\angle QPO = 75^\circ$, which implies that $\triangle PQO$ is isosceles with $OP = OQ = r$.
Using the two known angles in $\triangle PQO$, we get $\angle POQ$ as $\angle POQ = 180^\circ - 75^\circ - 75^\circ = 30^\circ$.
This means the area of $\triangle OPQ$ is $\dfrac{1}{2} \times OP \times OQ \times \sin 30^\circ = \dfrac{r^2}{4}$.
The areas of $\triangle PQO$ and $\triangle BRQ$ are equal, so the ratio is $1:1$.

ANSWER: $1:1$

22. Since $a_1, a_2, a_3, \ldots$ has $a_1 = 1$ and a common difference of 3, we have that $a_n = 1 + 3(n-1)$ for all $n \geq 1$. By similar reasoning, we get that $b_m = 2 + 10(m-1)$ for all $m \geq 1$.

For fixed m, the term b_m appears in both sequences exactly when there is some n such that $a_n = b_m$.
Using the closed forms established earlier, the term b_m appears in both sequences exactly when there exists n such that $1 + 3(n-1) = 2 + 10(m-1)$, which is equivalent to $n = \dfrac{10m-6}{3}$.

In other words, b_m is in both sequences exactly when the expression $\dfrac{10m-6}{3}$ is a positive integer, which happens exactly when $10m - 6$ is a positive multiple of 3.

Note that m is a positive integer, so $10m - 6$ is at least 4, which means $10m - 6$ is positive for all m.
Since 6 is a multiple of 3, $10m - 6$ is a multiple of 3 exactly when $10m$ is a multiple of 3, but 3 and 10 have no prime factors in common, $10m$ is a multiple of 3 exactly when m is a multiple of 3.
We have now shown that b_m is in both sequences exactly when m is a multiple of 3.

Let $m = 3k$ for some $k \geq 1$. Then $b_m = 2 + 10(3k-1) = 30k - 8$ and we wish to find the smallest possible k for which $30k - 8 > 2023$.

This inequality is equivalent to $30k > 2031$ which is equivalent to $k > \dfrac{2031}{30} = 67 + \dfrac{21}{30}$. The smallest positive integer k with this property is $k = 68$. Therefore,

$$b_{3\times 68} = b_{204} = 2 + 10(204-1) = 2 + 10(203) = 2032$$

is the smallest integer that is larger than 2023 and is in both sequences.

Answer: 2032

23. For every real number x, $-1 \leq \sin x \leq 1$, which means that $\sin 3A$ and $\sin C$ are both at most 1.
Therefore, $3\sin C$ has a maximum possible value of 3, so $\sin 3A + 3\sin C \leq 4$.
The only way that $\sin 3A + 3\sin C = 4$ is if $\sin 3A = 1$ and $\sin C = 1$.
Since C corresponds to an angle in a triangle, $0° < C < 180°$.
This and $\sin C = 1$ together imply that $C = 90°$. We conclude that $\triangle ABC$ has a right angle at C.

Since A is an angle in a triangle, we have that $0° < A < 90°$. However, we have already shown that this triangle has a right angle not at A, which means $0° < A < 90°$.
Multiplying this inequality through by 3 gives $0° < 3A < 270°$.
This and $\sin 3A = 1$ together imply that $3A = 90°$, which means $A = 30°$.
It follows that $\triangle ABC$ is a $30°$ - $60°$ - $90°$ triangle with hypotenuse $AB = 10$.
The side BC is opposite the $30°$ angle, so its length is half that of the hypotenuse, or $BC = 5$.
The side AC is opposite the $60°$ angle, so its length is $AC = \dfrac{\sqrt{3}}{2} \times AB = 5\sqrt{3}$.
The area of $\triangle ABC$ is $\dfrac{1}{2} \times AC \times BC = \dfrac{1}{2} \times 5 \times 5\sqrt{3} = \dfrac{25\sqrt{3}}{2}$.
Suppose the length of the altitude from C to AB is h. Then the area of $\triangle ABC$ is also equal to $\dfrac{1}{2} \times AB \times h = 5h$.
Therefore, $5h = \dfrac{25\sqrt{3}}{2}$, and so $h = \dfrac{5\sqrt{3}}{2}$.

Answer: $\frac{5\sqrt{3}}{2}$

24. Squaring both sides of the equation $x+y+z=2$ gives $x^2+y^2+z^2+2(xy+yz+zx)=4$. Substituting $xy+yz+zx=0$ into the above equation gives $x^2+y^2+z^2=4$.
From $x+y+z=2$, we get $x+y=2-z$, and from $xy+yz+xz=0$, we get $xy=-z(x+y)$. Substituting $x+y=2-z$ into $xy=-z(x+y)$, we get $xy=z(z-2)$.

Regardless of the values of x and y, $(x-y)^2 \geq 0$. Using this observation as well as $x^2+y^2+z^2=4$ and $xy=z(z-2)$, we get the following equivalent inequalities.

$$\begin{aligned} 0 &\leq (x-y)^2 \\ 0 &\leq x^2-2xy+y^2 \\ 2xy &\leq x^2+y^2 \\ 2xy+z^2 &\leq x^2+y^2+z^2 \\ 2z(z-2)+z^2 &\leq 4 \\ 3z^2-4z-4 &\leq 0 \\ (3z+2)(z-2) &\leq 0 \end{aligned}$$

If $z<-\frac{2}{3}$, then both $3z+2$ and $z-2$ are negative, so their product is positive.
If $-\frac{2}{3} \leq z \leq 2$, then $3z+2 \geq 0$ and $z-2 \leq 0$, so their product is non-positive.
If $z>2$, then $3z+2$ and $z-2$ are both positive, so their product is positive.
Therefore, in order for $(3z+2)(z-2) \leq 0$, we need that $-\frac{2}{3} \leq z \leq 2$.

We have shown that z is at least $-\frac{2}{3}$ and at most 2, but to be sure these are the actual minimum and maximum possible values of z, we must show that there are solutions to the system of equations that have z taking each of these two values.

If $z=2$, the equations become $x+y+2=2$ or $x+y=0$ and $xy+2y+2x=0$ which can be factored as $xy+2(x+y)=0$. Substituting $x+y=0$ gives $xy=0$, so $x=0$ or $y=0$.
Since $x+y=0$ and at least one of x and y is 0, it must be the case that they are both 0. This gives the solution $(x,y,z)=(0,0,2)$.

If $z=-\frac{2}{3}$, then the first equation gvies $x+y-\frac{2}{3}=2$ or $x+y=\frac{8}{3}$. The second equation gives $xy-\frac{2}{3}(x+y)=0$. Substituting $x+y=\frac{8}{3}$ into this equation and rearranging gives $xy=\frac{16}{9}$.
Multiplying $x+y=\frac{8}{3}$ through by x gives $x^2+xy=\frac{8}{3}x$, and after substituting $xy=\frac{16}{9}$ gives $x^2+\frac{16}{9}=\frac{8}{3}x$ which is equivalent to $9x^2-24x+16=0$.
Factoring, we get $(3x-4)^2=0$ so $3x=4$, which means $x=\frac{4}{3}$. Substituting this into $x+y=\frac{4}{3}$ and solving leads to $y=\frac{4}{3}$.
Therefore, $(x,y,z)=\left(\frac{4}{3},\frac{4}{3},-\frac{2}{3}\right)$ is a soluton.

We have shown that in any solution, we must have $-\frac{2}{3} \leq z \leq 2$ and that there are solutions with $z=-\frac{2}{3}$ and $z=2$. Thus, $b=2$ and $a=-\frac{2}{3}$, so $b-a=2-\left(\frac{2}{3}\right)=\frac{8}{3}$.

Answer: $\frac{8}{3}$

25. With $x = 1$, $\dfrac{x-1}{3x-2} = 0$, and so the given identity implies $f(1) + f(0) = 1$. Since we need to find $f(0) + f(1) + f(2)$, it remains to find the value of $f(2)$.

With $x = 2$, $\dfrac{x-1}{3x-2} = \dfrac{2-1}{6-2} = \dfrac{1}{4}$. Applying the identity gives $f(2) + f\left(\dfrac{1}{4}\right) = 2$.

With $x = \dfrac{1}{4}$, $\dfrac{x-1}{3x-2} = \dfrac{\frac{1}{4}-1}{\frac{3}{4}-2} = \dfrac{1-4}{3-8} = \dfrac{3}{5}$. Applying the identity gives $f\left(\dfrac{1}{4}\right) + f\left(\dfrac{3}{5}\right) = \dfrac{1}{4}$.

With $x = \dfrac{3}{5}$, $\dfrac{x-1}{3x-2} = \dfrac{\frac{3}{5}-1}{\frac{9}{5}-2} = \dfrac{3-5}{9-10} = 2$. Applying the identity gives $f\left(\dfrac{3}{5}\right) + f(2) = \dfrac{3}{5}$.

Let $a = f(2)$, $b = f\left(\dfrac{1}{4}\right)$, and $c = f\left(\dfrac{3}{5}\right)$. We have shown that the following three equations hold:

$$\begin{aligned} a + b &= 2 \\ b + c &= \frac{1}{4} \\ a + c &= \frac{3}{5} \end{aligned}$$

Adding the first and third equations gives $2a + (b + c) = 2 + \dfrac{3}{5} = \dfrac{13}{5}$.

Since $b + c = \dfrac{1}{4}$, this means $2a = \dfrac{13}{5} - \dfrac{1}{4} = \dfrac{47}{20}$, and so $f(2) = a = \dfrac{47}{40}$.

We already know that $f(0) + f(1) = 1$, so $f(0) + f(1) + f(2) = 1 + \dfrac{47}{40} = \dfrac{87}{40}$.

Remark: It can be shown that the given condition implies that $f(x) = \dfrac{-9x^3 + 6x^2 + x - 1}{2(-3x+1)(3x-2)}$ for all x except $x = \dfrac{1}{3}$ and $= \dfrac{2}{3}$ and that $f\left(\dfrac{1}{3}\right)$ and $f\left(\dfrac{2}{3}\right)$ are cannot be uniquely determined. However, the values of $f(0)$, $f(1)$, and $f(2)$ are uniquely determined, which is all that was needed for this problem.

Answer: $\frac{87}{40}$

Relay Problems

(Note: Where possible, the solutions to parts (b) and (c) of each relay are written as if the value of t is not initially known, and then t is substituted at the end.)

0. (a) Evaluating, $\dfrac{6 \times 5 - 2}{4} = \dfrac{30 - 2}{4} = \dfrac{28}{4} = 7$.

   (b) The area of a triangle with base $2t$ and height $2t - 4$ is $\frac{1}{2}(2t)(2t-4)$ or $t(2t-4)$.
   Substituting $t = 7$ gives an area of $t(2t-4) = 7(10) = 70$.

   (c) Since $\triangle ABC$ is isosceles with $AB = BC$, it is also true that $\angle BCA = \angle BAC$.
   The angles in a triangle have a sum of 180°, so

$$\begin{aligned} 180^\circ &= \angle ABC + \angle BAC + \angle BCA \\ &= \angle ABC + 2\angle BAC \\ &= t^\circ + 2\angle BAC \end{aligned}$$

   Substituting $t = 70$, we get $180^\circ = 70^\circ + 2\angle BAC$ and so $\angle BAC = \dfrac{180^\circ - 70^\circ}{2} = 55^\circ$.

   ANSWER: $(7,\ 70,\ 55^\circ)$

1. (a) Since $9 = 3^2$, the area of the garden is $3^2\,\mathrm{m}^2$ or $(3\,\mathrm{m}) \times (3\,\mathrm{m})$.
   Therefore, the side-length of the garden is $3\,\mathrm{m}$.
   The perimeter is $4 \times 3\,\mathrm{m} = 12\,\mathrm{m}$, so $N = 12$.

   (b) Each of the nine small squares is divided into two congruent triangular sections, so the square is divided into $9 \times 2 = 18$ triangular sections of equal area.
   Of the 18 sections, 8 are shaded, so $\dfrac{8}{18} = \dfrac{4}{9}$ of $ABCD$ is shaded.
   The area of $ABCD$ is t^2, so the shaded area is $\dfrac{4t^2}{9}$.
   Substituting $t = 12$, we get that the area of the shaded region is $\dfrac{4 \times 12^2}{9} = 64$.

   (c) Expanding, $n(n-1)(n+1) + n = n(n^2-1) + n = n^3 - n + n = n^3$.
   This means $t = n^3$ so $n = \sqrt[3]{t}$.
   Substituting $t = 64$ gives $n = \sqrt[3]{64} = 4$.

   ANSWER: $(12, 64, 4)$

2. (a) Multiplying through by $29(n+1)$ gives $29 > 4(n+1)$ which can be rearranged to get $25 > 4n$ or $n < \dfrac{25}{4} = 6.25$.
   The positive integers that satisfy the inequality are $n = 1$, $n = 2$, $n = 3$, $n = 4$, $n = 5$, and $n = 6$, so the answer is 6.

   (b) Let V be the volume of the tank.
   Using the second given piece of information, we know that when $\dfrac{t^2}{4}$ litres are added, the amount of water goes from $\dfrac{2V}{10}$ to $\dfrac{5V}{10}$, which means $\dfrac{t^2}{4}$ is the difference between these two amounts of water.
   Therefore, $\dfrac{t^2}{4} = \dfrac{5V - 2V}{10} = \dfrac{3V}{10}$.

Solving for V gives $V = \dfrac{10}{3} \times \dfrac{t^2}{4} = \dfrac{10t^2}{12} = \dfrac{5t^2}{6}$.

Since adding $\dfrac{t}{2}$ litres to the initial x litres leads to the tank having $\dfrac{2V}{10}$ litres in it, $x = \dfrac{2V}{10} - \dfrac{t}{2}$.

Substituting $V = \dfrac{5t^2}{6}$ gives $x = \dfrac{2}{10} \times \dfrac{5t^2}{6} - \dfrac{t}{2} = \dfrac{t^2}{6} - \dfrac{t}{2}$.

Substituting $t = 6$, we get $x = \dfrac{6^2}{6} - 3 = 3$.

(c) The line segment OP passes through the origin and P, so its slope is equal to $\dfrac{b}{a}$.

Since the slope is $\dfrac{12}{5}$ and the line passes through the origin, there is some k for which $b = 12k$ and $a = 5k$.

By the Pythagorean theorem, $a^2+b^2 = OP^2$. Substituting $b = 12k$, $a = 5k$, and $OP = 13t$, we get the following equivalent equations.

$$\begin{aligned} a^2 + b^2 &= OP^2 \\ (5k)^2 + (12k)^2 &= (13t)^2 \\ 25k^2 + 144k^2 &= 169t^2 \\ 169k^2 &= 169t^2 \\ k^2 &= t^2 \end{aligned}$$

Since $P(a, b)$ is in the first quadrant, $a = 5k$ is positive, so k is positive. As well, since $13t$ is given to be a length, we can assume t is positive, which means $k = t$, so $a = 5t$.

By reasoning similar to earlier, there is some m so that $c = 4m$ and $d = 3m$.

By the Pythagorean theorem, $c^2+d^2 = OQ^2$, so we get the following equivalent equations.

$$\begin{aligned} c^2 + d^2 &= OQ^2 \\ (4m)^2 + (3m)^2 &= (10t)^2 \\ 16m^2 + 9m^2 &= 100t^2 \\ 25m^2 &= 100t^2 \\ m^2 &= 4t^2 \end{aligned}$$

Since c and t are both positive, $m = \sqrt{4t^2} = 2t$, so $c = 8t$.

Therefore, $a + c = 5t + 8t = 13t$.

Substituting $t = 3$, we get $a + c = 13(3) = 39$.

Answer: $(6, 3, 39)$

3. (a) The positive divisors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24.

Their sum is

$$1 + 2 + 3 + 4 + 6 + 8 + 12 + 24 = 60$$

(b) Subtracting the second equation from the first gives

$$\left(a - \frac{t}{6}b\right) - \left(a - \frac{t}{5}b\right) = 20 - (-10)$$

which simplifies to $\left(\dfrac{t}{5} - \dfrac{t}{6}\right) b = 30$ or $\dfrac{t}{30}b = 30$.

Solving for b gives $b = \dfrac{900}{t}$. Substituting $t = 60$ gives $b = \dfrac{900}{60} = 15$.

(c) Using that the parabola passes through $(0, 60)$ gives $60 = a(0)^2 + b(0) + c$ or $c = 60$.

The roots of $ax^2 + bx + c$ are $x = 4$ and $x = \frac{t}{3}$. [We are implicitly assuming that $t \neq 12$ since otherwise the question would not have a unique answer.]

The product of the roots of $ax^2 + bx + c$ is $\frac{c}{a} = \frac{60}{a}$.

Therefore, $\frac{60}{a} = 4 \times \frac{t}{3}$ which can be rearranged to get $a = \frac{3 \times 60}{4t} = \frac{45}{t}$. Substituting $t = 15$ gives $a = 3$.

ANSWER: $(60, 15, 3)$