The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

cemc.uwaterloo.ca

2024 Canadian Team Mathematics Contest

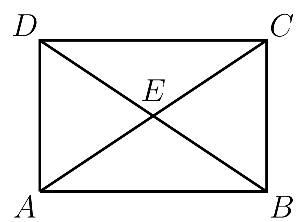
Individual Problems (45 minutes)

IMPORTANT NOTES:

- Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) previously stored information such as formulas, programs, notes, etc., (iv) a computer algebra system, (v) dynamic geometry software.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

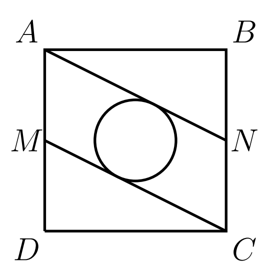
PROBLEMS:

1. If May 3 is a Thursday, what day of the week is May 29 in the same year?
2. How many integers are there between $-\sqrt{11}$ and $\sqrt{29}$?
3. In rectangle $ABCD$, the two diagonals intersect at E. The ratio of the measure of $\angle AED$ to the measure of $\angle DEC$ is $7 : 11$. In degrees, what is the measure of $\angle BDC$?



4. The sequence of real numbers 80, x, y, z, 3125 is a geometric sequence. What is the value of y?

   (A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant, called the *common ratio*. For example, 3, -6, 12, -24 are the first four terms of a geometric sequence.)
5. The ordered pair of integers (x, y) has $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$. For how many such pairs is $9 \leq x^2 + y^2 \leq 25$?
6. In square $ABCD$, M is the midpoint of AD and N is the midpoint of BC. A circle is drawn so that it is tangent to both AN and MC. If $AB = 2$, what is the area of the circle?

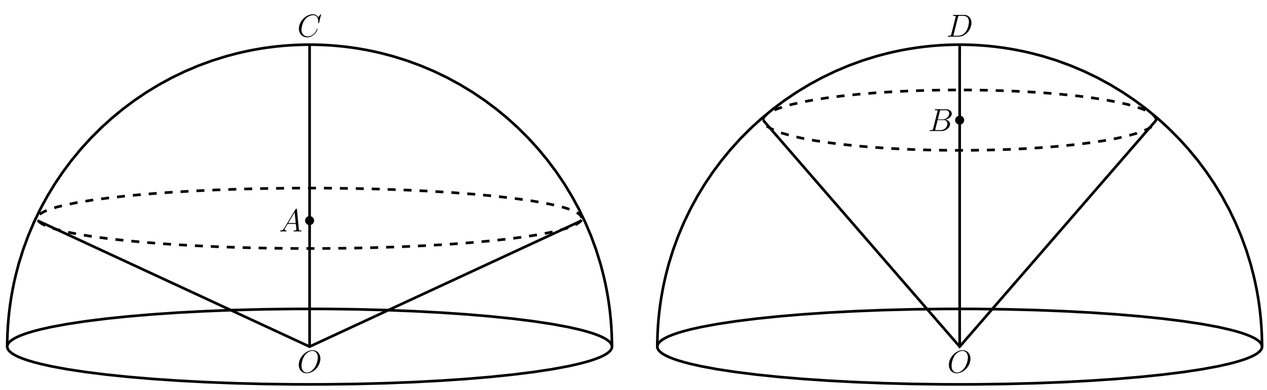


7. A nickel is a coin worth 5 cents, a dime is a coin worth 10 cents, and a quarter is a coin worth 25 cents. Rolo has x nickels, y dimes and z quarters. Pat has y nickels, z dimes and x quarters. Sarki has z nickels, x dimes and y quarters. None of them has any other coins. If the total value of all their coins is 6480 cents, how many coins does Pat have?

8. What is the sum of all positive integers less than 1000 that have only odd digits?

9. A sphere with centre O is cut into two hemispheres, each of which is placed on its circular base. Radii OC and OD are drawn in the two hemispheres, each perpendicular to the base. Point A is on OC so that $OA = \frac{1}{3}OC$ and point B is on OD so that $OB = \frac{2}{3}OD$. In each hemisphere, a plane parallel to the base cuts the sphere along a circular cross section of the hemisphere. The plane cutting the hemisphere with radius OC passes through A and the plane cutting the hemisphere with radius OD passes through B. Each cross section forms the base of a cone with its vertex at O. What is the ratio of the volume of the cone with A on its base to the volume of the cone with B on its base?

   The volume of a cone with height h and a circular base of radius r is $\frac{1}{3}\pi r^2 h$.



10. One ant is placed on each of the eight vertices of a cube. At the same time, each of the eight ants randomly chooses one of the three edges connected to its vertex and crawls along that edge to another vertex, then stops. All of the ants travel at the same speed. What is the probability that none of the ants meet another ant on an edge or at a vertex?

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Team Problems (45 minutes)

IMPORTANT NOTES:

- Calculating devices are not permitted.
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PROBLEMS:

1. A cyclist travelled 8 km in 20 minutes moving at a constant speed. At this same speed, how far would the cyclist travel in 2 hours?

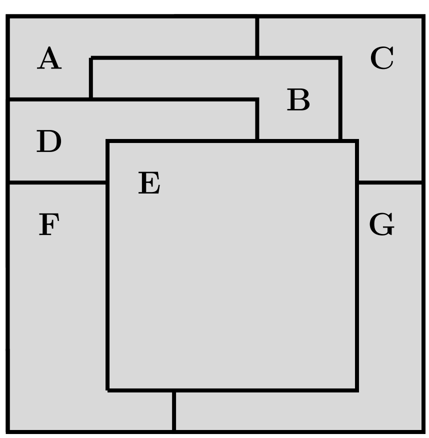
2. If $\sqrt{x+5} = 5$, what is the value of x?

3. The digits A and B make the multiplication below correct. What is $A + B$?

$$\begin{array}{r} 2\ A\ 7 \\ \times \quad\ \ 3 \\ \hline 7\ 1\ B \end{array}$$

4. Euclidville Transit charges \$3.25 for a single ride on the train. They also sell a one-month pass for \$90 that allows the pass holder to ride the train as many times as they want during that month. Marcy plans to ride the train k times in the month of May and determines that she will spend less money if she buys a \$90 pass. What is the smallest possible value of the integer k?

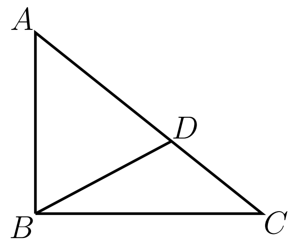
5. Seven identical square pieces of paper were labelled with the letters **A** through **G**. These papers were then placed on a table one at a time, in some order, ending with the paper labelled E. The diagram shows the view from above. In what order were the papers placed on the table?



6. Two lines with equations $2x - 6y + 42 = 0$ and $15x + ky + d = 0$ are perpendicular and intersect at a point on the y-axis. What is the value of d?

7. Ashwin chooses a two-digit integer x and reverses the digits of x to obtain a larger integer y. He notices that $y - x = 18$. How many possibilities are there for x?

8. In $\triangle ABC$, $\angle ABC = 90°$, $AB = 3$, $BC = 4$, and $AC = 5$. Point D is on AC so that $\triangle ADB$ and $\triangle CDB$ have the same area. What is the length of BD?



9. Xin rolls a fair 8-sided die with sides numbered 1 to 8 and a fair 9-sided die with sides numbered 1 to 9. What is the probability that the product of the numbers rolled is divisible by 6?

10. How many ordered pairs of integers (a, b) satisfy both $a + b > -23$ and $ab = 120$?

11. The quadratic function with equation $y = -2x^2 + 4kx - 10k$ has a maximum of 48. What is the sum of all possible values of k?

12. The integers A, B, C, and D have the following properties.

    - A is equal to $\frac{5}{2}$ times the average (mean) of A, B, C, and D.
    - B is equal to $\frac{4}{5}$ times the average (mean) of A, B, C, and D.
    - $C + D = 140$.

    What is $A + B + C + D$?

13. What is the value of n for which $\sqrt[3]{\dfrac{3^8 + 3^n}{3^2 + 3^n}} = 3$?

14. A recursive sequence is defined by $t_1 = 3$, $t_2 = 4$, and

$$t_n = \begin{cases} \frac{1}{2}t_{n-1} + t_{n-2} & \text{if } t_{n-1} \text{ is even} \\ t_{n-1} - t_{n-2} & \text{if } t_{n-1} \text{ is odd} \end{cases}$$

    for every integer $n \geq 3$. What is the sum of the first 2024 terms of this sequence?

15. Ada writes a sequence of nine letters using only the letters A, B, C, D, and E. Each letter is only allowed to be followed by some of the other letters, as summarized in the table.

Letter	Letters allowed to follow
A	B or C
B	C or E
C	D or E
D	A or B
E	A or D

    How many different possible sequences have A in the first position, D in the fifth position, and A in the ninth position?

16. A two-player game is played with 12 square tiles arranged on a board in three rows of four tiles each. The two players alternate turns during which they must remove exactly one tile.

A player loses if their move causes a remaining tile to share an edge with at least two empty squares.

The four images below, from left to right, represent each state of the board in a game that ended after three turns. In order, the first three turns were: Player 1 removed Tile F, Player 2 removed Tile L, and Player 1 removed Tile H. Player 1 lost the game because their second turn caused Tile G to share an edge with two empty squares.

A	B	C	D
E	F	G	H
I	J	K	L

A	B	C	D
E		G	H
I	J	K	L

A	B	C	D
E		G	H
I	J	K	

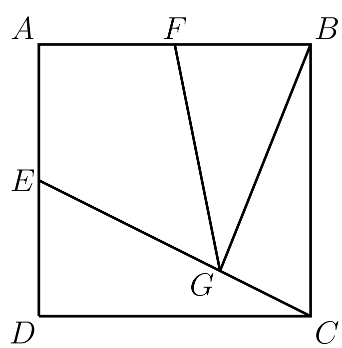
A	B	C	D
E		G	
I	J	K	

In the board shown below, Tiles C and I have been removed and it is now Ferd's turn. Which tile should Ferd remove in order to ensure that he wins the game?

A	B		D
E	F	G	H
	J	K	L

17. For how many angles x with $0° \leq x \leq 2024°$ is $\left(4^{\sin^2 x}\right)\left(2^{\cos^2 x}\right) = 2\sqrt{2}$?

18. Square $ABCD$ has sides of length 14. The midpoints of sides AD and AB are E and F, respectively. Point G is on EC such that the ratio of the area of $\triangle FGB$ to the area of $ABCD$ is $5:28$. What is the ratio $EG:GC$?



19. A function $f(x)$ has the property that

$$f\left(x - \frac{1}{x}\right) = x^3 - \frac{1}{x^3}$$

for all $x \neq 0$. What is $f(1)$?

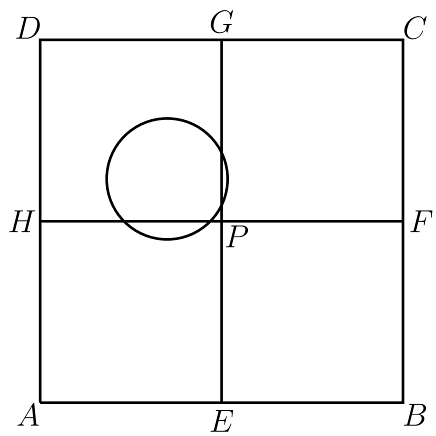
20. A computer program prints the positive integers in increasing order starting at 1. What integer is the computer printing when it prints the digit 1 for the 1000^{th} time?

21. The real number x satisfies $x^{\log_{10} 2024} + 2024^{\log_{10} x} - 4\sqrt{506} = 0$. What is the value of x?

22. Square $ABCD$ has side length 6. The midpoints of AB, BC, CD, and AD are E, F, G, and H, respectively. EG and FH intersect at P.

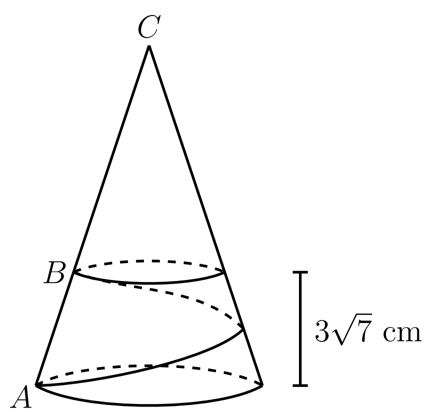
A circle of radius 1 is drawn randomly so that it is entirely inside square $ABCD$. What is the probability that the circle intersects the interior of exactly two of the line segments EP, FP, GP, and HP? (In the example below, the circle intersects the interior of GP and HP, but does not intersect the interior of FP or EP.)

(The *interior* of a line segment is the set of points on the line segment excluding the endpoints. For example, the interior of EP is the set of points on the segment EP except for E and P.)



23. A 6×6 grid has a coin in each of its 36 cells. Each coin is showing either a head or a tail. Samar notices that in each row, the number of coins showing a head is even and in each column, the number of coins showing a head is even. In how many ways can the coins be arranged so that this is true?

24. A cone with a circular base has A on the circumference of its base, C at its vertex, and B on AC (on the surface of the cone) so that its height above the base is $3\sqrt{7}$ cm. The circumference of the base of the cone is 3π cm and the circular cross section through B parallel to the base has a circumference of π cm. A path is drawn on the surface of the cone from A to B that wraps around the cone exactly once. If p is the shortest possible length of such a path, what is p^2?



25. For which positive real numbers a does the polynomial

$$p(x) = x^4 + 2x^3 + (3 - a^2)x^2 + (2 - 2a^2)x + (1 - a^2)$$

have exactly two distinct real roots?

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Relay Problem #1 (Seat a)

How many integers n with $1 \leq n \leq 20$ are divisible by 2 or 3 but not divisible by both 2 and 3?

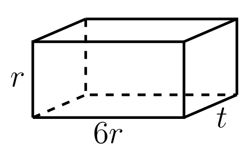
Relay Problem #1 (Seat b)

Let t be TNYWR.
If (x, y) satisfies the system of equations below, what is the value of x?

$$\begin{aligned} 4x + 3y &= 60 \\ -4x + 3y &= t + 2 \end{aligned}$$

Relay Problem #1 (Seat c)

Let t be TNYWR.
The rectangular prism in the diagram below has dimensions $6r$ by r by t. Its surface area is $18r^2$.
What is the value of the positive number r?



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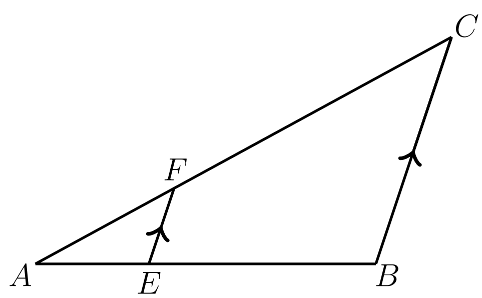
Relay Problem #2 (Seat a)

What is the slope of a line perpendicular to the line with equation $13x + \frac{13}{2}y = 9$?

Relay Problem #2 (Seat b)

Let t be TNYWR.

In $\triangle ABC$, the points E and F are on sides AB and AC, respectively, so that EF is parallel to BC and the ratio $AE : EB$ is $1 : 2$. If $AE = 4$, $FC = 10$, and $EF = 4t$, what is the perimeter of $\triangle ABC$?



Relay Problem #2 (Seat c)

Let t be TNYWR.

The y-intercept of the graph of a quadratic function equals 5. The function's minimum value is -3 and its graph passes through the points $P(4,5)$ and $Q\left(\dfrac{2t}{11}, h\right)$. What is the value of h?

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Relay Problem #3 (Seat a)

The sum of 10 integers is 83. One of the integers is 11. What is the average (mean) of the other 9 integers?

Relay Problem #3 (Seat b)

Let t be TNYWR.
The line $ax + by = t$ has y-intercept -2 and passes through the point $(16, 2)$. What is the value of $a - b$?

Relay Problem #3 (Seat c)

Let t be TNYWR.
How many squares of side length $t + 2$ are needed to fully cover a rectangle with dimensions $4t + 8$ by $t^3 + 2t^2$ without any squares overlapping?