The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2024 Euclid Contest

Wednesday, April 3, 2024
(in North America and South America)

Thursday, April 4, 2024
(outside of North America and South America)

Solutions

1. (a) For every $x \neq 0$, we note that $\dfrac{x^4 + 3x^2}{x^2} = x^2 + 3$.

Therefore, when $x = 2$, we have $\dfrac{x^4 + 3x^2}{x^2} = x^2 + 3 = 2^2 + 3 = 7$.

Alternatively, when $x = 2$, we have $\dfrac{x^4 + 3x^2}{x^2} = \dfrac{2^4 + 3 \cdot 2^2}{2^2} = \dfrac{28}{4} = 7$.

(b) By the Pythagorean Theorem in $\triangle ABC$, we have

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ (t+1)^2 &= 10^2 + (t-1)^2 \\ t^2 + 2t + 1 &= 100 + t^2 - 2t + 1 \\ 4t &= 100 \end{aligned}$$

and so $t = 25$.

Alternatively, we could remember the Pythagorean triple 5-12-13 and scale this triple by a factor of 2 to obtain the Pythagorean triple 10-24-26, noting that the difference between $t + 1$ and $t - 1$ is 2 as is the difference between 26 and 24, which gives $t + 1 = 26$ and so $t = 25$.

(c) Since $\dfrac{2}{y} + \dfrac{3}{2y} = 14$, then $\dfrac{4}{2y} + \dfrac{3}{2y} = 14$ or $\dfrac{7}{2y} = 14$.

Therefore, $2y = \dfrac{7}{14} = \dfrac{1}{2}$ and so $y = \dfrac{1}{4}$.

2. (a) Suppose that the sequence is $a, b, c, 13, e, 36$.
Since $36 = 13 + e$, then $e = 36 - 13 = 23$.
Since $e = c + 13$, then $23 = c + 13$ and so $c = 10$.
Since $13 = b + c$ and $c = 10$, then $b = 3$.
Since $c = a + b$ and $c = 10$ and $b = 3$, then $a = 10 - 3 = 7$.
Therefore, the first term is 7.

(b) From the given information, we obtain $5r^2 + 5r^3 = (5r)^2$ and so $5r^2 + 5r^3 = 25r^2$.
Since $r \neq 0$, we can divide by $5r^2$ to obtain $1 + r = 5$, which gives $r = 4$.

(c) Suppose that Jimmy's marks on his first, second, third, and fourth tests were w, x, y, and z, respectively.

Since Jimmy's average on his first, second and third tests was 65, then $\dfrac{w + x + y}{3} = 65$ or $w + x + y = 195$.

Since Jimmy's average on his second, third and fourth tests was 80, then $\dfrac{x + y + z}{3} = 80$ or $x + y + z = 240$.

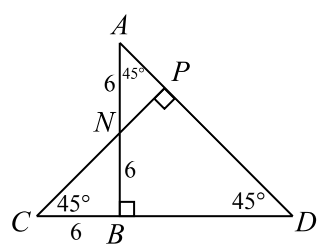
Since his mark on the fourth test was 2 times his mark on the first test, then $z = 2w$.
Thus, $w + x + y = 195$ and $x + y + 2w = 240$.
Subtracting the first equation from the second equation, we obtain $w = 45$ and so his mark on the fourth test was $z = 2w = 90$.

3. (a) Since $y = r(x-3)(x-r)$ passes through $(0, 48)$, then $48 = r(0-3)(0-r)$.
Thus, $48 = 3r^2$ and so $r^2 = 16$ or $r = \pm 4$.

(b) With 13% sales tax on an item whose price is \$$B$, the total cost is \$$(1.13B)$.
With 5% sales tax on an item whose price is \$$B$, the total cost is \$$(1.05B)$.
From the given information \$$(1.13B)$ − \$$(1.05B)$ = \$24 or $1.13B - 1.05B = 24$.
Therefore, $0.08B = 24$, which gives $B = 300$.
Alternatively, we could note that the difference in total prices is the difference in the amount of tax paid. This is the difference between 13% of the original price and 5% of the original price; this difference is equal to 8% of the original price. If 8% of the original price is equal to \$24, then 1% of the original price is equal to \$3 and so the original price is \$3 × 100 = \$300.

(c) When $n = 1$, $f(2n) = (f(n))^2$ becomes $f(2) = (f(1))^2$.
Since $f(1) = 3$, then $f(2) = 3^2 = 9$.
When $m = 1$, $f(2m+1) = 3f(2m)$ becomes $f(3) = 3f(2)$.
Since $f(2) = 9$, then $f(3) = 3 \cdot 9 = 27$.
When $n = 2$, $f(2n) = (f(n))^2$ becomes $f(4) = (f(2))^2$.
Since $f(2) = 9$, then $f(4) = 9^2 = 81$.
Therefore, $f(2) + f(3) + f(4) = 9 + 27 + 81 = 117$.

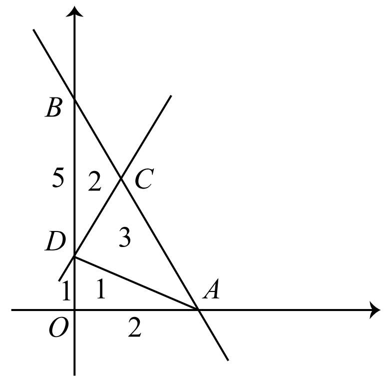
4. (a) Since $\triangle ABD$ is right-angled at B and has $\angle ADB = 45^\circ$, then $\angle BAD = 45^\circ$.
Similarly, $\triangle CPD$ is right-angled and isosceles with $\angle PCD = 45^\circ$.
Further, $\triangle APN$ and $\triangle CBN$ are also both right-angled and isosceles.
Since $CB = 6$ and $NB = CB$, then $NB = 6$.
Since $AB = 12$ and $NB = 6$, then $AN = AB - NB = 6$.



Since $\triangle APN$ is right-angled and isoceles, then its sides are in the ratio $1 : 1 : \sqrt{2}$.
Thus, $AP = PN = \frac{1}{\sqrt{2}} AN = \frac{6}{\sqrt{2}} = 3\sqrt{2}$.
Alternatively, if $AP = PN = x$, then the Pythagorean Theorem gives $AN^2 = AP^2 + PN^2$ and so $6^2 = 2x^2$ which gives $AP^2 = x^2 = 18$.
Thus, the area of $\triangle APN$ is $\frac{1}{2} \cdot AP \cdot PN = \frac{1}{2} \cdot 3\sqrt{2} \cdot 3\sqrt{2} = 9$.

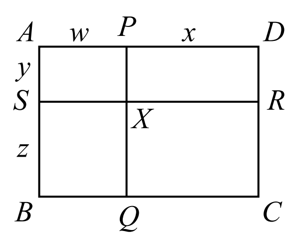
(b) The line with equation $y = -3x + 6$ has y-intercept 6, which means that $OB = 6$.
To find the x-intercept of this line, we set $y = 0$ and obtain the equation $-3x + 6 = 0$ which gives $3x = 6$ or $x = 2$. This means that $OA = 2$.
Since $\triangle ABO$ is right-angled at O, its area is $\frac{1}{2} \cdot OB \cdot OA = \frac{1}{2} \cdot 6 \cdot 2 = 6$.
Since the area of $\triangle ACD$ is $\frac{1}{2}$ of the area of $\triangle ABO$, then the area of $\triangle ACD$ is 3.
Next, we note that the line with equation $y = mx + 1$ has y-intercept 1; thus, $OD = 1$.
This means that the area of $\triangle ADO$ is $\frac{1}{2} \cdot OD \cdot OA = \frac{1}{2} \cdot 1 \cdot 2 = 1$.

We can determine the area of $\triangle BCD$ by subtracting the areas of $\triangle ACD$ and $\triangle ADO$ from that of $\triangle ABO$, which tells us that the area of $\triangle BCD$ is $6 - 3 - 1 = 2$.



Now, we can consider BD, which has length $6 - 1 = 5$, as the base of $\triangle BCD$; the corresponding height of $\triangle BCD$ is the distance from C to the y-axis, which we call h.
Thus, $\frac{1}{2} \cdot 5 \cdot h = 2$ and so $h = \frac{4}{5}$.
This means that C has x-coordinate $\frac{4}{5}$.
Since C is on the line with equation $y = -3x + 6$, we have $y = -3 \cdot \frac{4}{5} + 6 = \frac{18}{5}$.
Therefore, the coordinates of C are $(\frac{4}{5}, \frac{18}{5})$.

5. (a) Suppose that $AP = w$, $PD = x$, $AS = y$, and $SB = z$.



We use the notation $|APXS|$ to represent the area of $APXS$, and so on.
Thus, $|APXS| = wy$, $|PDRX| = xy$, $|SXBQ| = wz$, and $|XRCQ| = xz$.
Then,

$$|APXS| \cdot |XRCQ| = wy \cdot xz = xy \cdot wz = |PDRX| \cdot |SXQB|$$

If $|APXS| = 2$, $|PDRX| = 3$, and $|SWQB| = 6$, then $a = |XRQC| = \dfrac{2 \cdot 6}{3} = 4$.

If $|APXS| = 2$, $|PDRX| = 6$, and $|SWQB| = 3$, then $a = |XRQC| = \dfrac{2 \cdot 3}{6} = 1$.

If $|APXS| = 6$, $|PDRX| = 2$, and $|SWQB| = 3$, then $a = |XRQC| = \dfrac{6 \cdot 3}{2} = 9$.

Since we are told that there are three possible values for a, then these are 1, 4 and 9.
(Can you explain why there are exactly three such values?)

(b) The x-intercepts of the parabola with equation $y = x^2 - 4tx + 5t^2 - 6t$ are

$$x = \frac{4t \pm \sqrt{(-4t)^2 - 4(5t^2 - 6t)}}{2}$$

The distance, d, between these intercepts is their difference, which is

$$d = \frac{4t + \sqrt{(-4t)^2 - 4(5t^2 - 6t)}}{2} - \frac{4t - \sqrt{(-4t)^2 - 4(5t^2 - 6t)}}{2} = \sqrt{(-4t)^2 - 4(5t^2 - 6t)}$$

From this we see that d is as large as possible exactly when the discriminant is as large as possible. Here, the discriminant, Δ, is

$$\Delta = (-4t)^2 - 4(5t^2 - 6t) = 16t^2 - 20t^2 + 24t = -4t^2 + 24t$$

Completing the square,

$$\Delta = -4(t^2 - 6t) = -4(t^2 - 6t + 9 - 9) = -4(t^2 - 6t + 9) + 36 = -4(t - 3)^2 + 36$$

Since $(t-3)^2 \geq 0$, then $\Delta \leq 36$ and $\Delta = 36$ exactly when $(t-3)^2 = 0$ or $t = 3$.
Therefore, the discriminant is maximized when $t = 3$, which means that the distance between the x-intercepts is as large as possible when $t = 3$.

6. (a) Every multiple of 21 is of the form $21k$ for some integer k.
For such a multiple to be between 10 000 and 100 000, we need $10\,000 < 21k < 100\,000$ or $\frac{10\,000}{21} < k < \frac{100\,000}{21}$.
Since $\frac{10\,000}{21} \approx 476.2$ and $\frac{100\,000}{21} \approx 4761.9$ and k is an integer, then $477 \leq k \leq 4761$. (Note that k is greater than 476.2 and is an integer, so must be at least 477; similarly, k is at most 4761.)
We also want the units digit of $21k$ to be 1.
This means that the units digit of k itself is 1, since the units digit of the product of 21 and k is equal to units digit of k because the units digit of 21 is 1.
Therefore, the possible values of k are 481, 491, 501, …, 4751, 4761.
There are 429 such values. To see this, we can see that counting the integers in this list is the same as counting the integers in the list 48, 49, 50, …, 475, 476. This list is equivalent to removing the integers from 1 to 47 from the list of integers from 1 to 476, giving $476 - 47 = 429$ integers.
Thus, $M = 429$.

(b) *Solution 1*
We can partition the N students at Strickland S.S. into four groups:

- a students who are in the physics club and are in the math club
- b students who are in the physics club and are not in the math club
- c students who are not in the physics club but are in the math club
- d students who are not in the physics club and are not in the math club

	In Math Club	Not in Math Club
In Physics Club	a	b
Not in Physics Club	c	d

From the given information, there are $\frac{2}{5}N$ students in the physics club. In other words, $a+b=\frac{2}{5}N$.

Among the students in the physics club, twice as many are not in the math club as are in the math club. This means that $b=\frac{2}{3}\cdot\frac{2}{5}N=\frac{4}{15}N$ and $a=\frac{1}{3}\cdot\frac{2}{5}N=\frac{2}{15}N$.

From the given information, there are $\frac{1}{4}N$ students in the math club. In other words, $a+c=\frac{1}{4}N$.

Since $a=\frac{2}{15}N$, then $c=\frac{1}{4}N-\frac{2}{15}N=\frac{7}{60}N$.

Since $a+b+c+d=N$, then $d=N-a-b-c=N-\frac{4}{15}N-\frac{2}{15}N-\frac{7}{60}N=\frac{29}{60}N$.

Lastly, we know that $500<N<600$.

Since each of a, b, c, and d is an integer, then N must be divisible by 60.

Therefore, $N=540$ and so the number of students not in either club is $d=\frac{29}{60}\cdot 540=261$.

Solution 2

Since there are N students at Strickland S.S., then $\frac{2}{5}N$ are in the physics club and $\frac{1}{4}N$ are in the math club.

Since each of $\frac{2}{5}N$ and $\frac{1}{4}N$ must be an integer, then N must be divisible by 5 and must be divisible by 4.

Since 5 and 4 share no common divisor larger than 1, then N must be divisible by $5\cdot 4=20$.

Thus, we let $N=20m$ for some positive integer m.

In this case, $\frac{2}{5}N=8m$ students are in the physics club and $\frac{1}{4}N=5m$ students are in the math club.

Now, among the $8m$ students in the physics club, twice as many are not in the math club as are in the math club.

In other words, $\frac{1}{3}$ of the $8m$ students in the physics club are in the math club. This means that m must be divisible by 3, since 3 is a prime number and 8 is not divisible by 3.

Therefore, $m=3k$ for some positive integer k, which means that $N=20m=60k$ and $\frac{2}{5}N=8m=24k$ and $\frac{1}{4}N=5m=15k$.

Since $500<N<600$ and N is a multiple of 60, then $N=540$, which means that $k=9$.

Thus, the number of students in the physics club is $24k=216$, of whom $\frac{1}{3}\cdot 216=72$ are in the math club and $\frac{2}{3}\cdot 216=144$ are not in the math club.

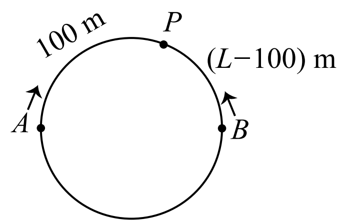
Also, the number of students in the math club is $15k=135$.

Finally, we know that

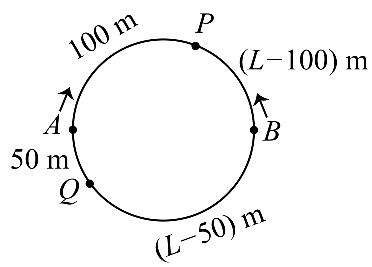
- there are 540 students at the school,
- 72 of whom are in both the physics club and the math club,
- 144 of whom are in the physics club and not in the math club, and
- $135-72=63$ are in the math club and not in the physics club.

Therefore, the number of students in neither club is $540-72-144-63=261$.

7. (a) Suppose that the length of the track is $2L$ m, that Arun's constant speed is a m/s, and that Bella's constant speed is b m/s.
When Arun and Bella run over the same interval of time, the ratio of the distances that they run is equal to the ratio of their speeds.
Consider the interval of time from the start to when they first meet. In the diagram, A is Arun's starting point, B is Bella's starting point, and P is this first meeting point.



Since Arun has run 100 m and together they have covered half of the length of the track, then Bella has run $(L - 100)$ m.
Thus, $\dfrac{a}{b} = \dfrac{100}{L-100}$.
From their first meeting point P to their second meeting point, which we label Q, Bella runs 150 m.



Over this time, Arun runs from P to B to Q.
Since Bella runs $150-100 = 50$ m past A, then $QB = (L-50)$ m (because $AB = L$ m and $AQ = 50$ m) and so Arun runs $(L-100)$ m $+ (L-50)$ m which is equal to $(2L-150)$ m.
Thus, over this second interval of time, $\dfrac{a}{b} = \dfrac{2L-150}{150}$.
Equating expressions for $\dfrac{a}{b}$ and solving,

$$\begin{aligned} \frac{100}{L-100} &= \frac{2L-150}{150} \\ 100 \cdot 150 &= (L-100)(2L-150) \\ 15\,000 &= 2L^2 - 350L + 15\,000 \\ 350L &= 2L^2 \end{aligned}$$

Since $L \neq 0$, then $2L = 350$, and so the total length of the track is 350 m.

Checking, if the length of the track is 350 m, then half of the length is 175 m.
This means that from the start to P, Arun runs 100 m and Bella runs 75 m.
Also, from P to Q, Bella runs 150 m and Arun runs 200 m.
Note that $\dfrac{100}{75} = \dfrac{200}{150}$ so these numbers are consistent with the given information.

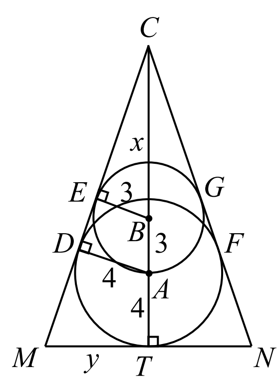
(b) Using exponent laws, the following equations are equivalent:

$$\begin{aligned} 4^{1+\cos^3\theta} &= 2^{2-\cos\theta}\cdot 8^{\cos^2\theta} \\ (2^2)^{1+\cos^3\theta} &= 2^{2-\cos\theta}\cdot (2^3)^{\cos^2\theta} \\ 2^{2+2\cos^3\theta} &= 2^{2-\cos\theta}\cdot 2^{3\cos^2\theta} \\ 2^{2+2\cos^3\theta} &= 2^{2-\cos\theta+3\cos^2\theta} \\ 2+2\cos^3\theta &= 2-\cos\theta+3\cos^2\theta \\ 2\cos^3\theta-3\cos^2\theta+\cos\theta &= 0 \\ \cos\theta(2\cos^2\theta-3\cos\theta+1) &= 0 \\ \cos\theta(2\cos\theta-1)(\cos\theta-1) &= 0 \end{aligned}$$

and so $\cos\theta = 0$ or $\cos\theta = 1$ or $\cos\theta = \frac{1}{2}$.
Since $0^\circ \leq \theta \leq 360^\circ$, the solutions are $\theta = 90^\circ, 270^\circ, 0^\circ, 360^\circ, 60^\circ, 300^\circ$.
Listing these in increasing order, the solutions to the original equation are

$$\theta = 0^\circ, 60^\circ, 90^\circ, 270^\circ, 300^\circ, 360^\circ$$

8. (a) We join B to E and A to D.
Since MC is tangent to the circles with centres A and B at D and E, respectively, then AD and BE are perpendicular to MC.
Since the radius of the circle with centre B is 3, then $AB = 3$ and $BE = 3$.
Since the radius of the circle with centre A is 4, then $AD = 4$ and $AT = 4$.
Let $CB = x$ and $MT = y$.



We note that $\triangle CEB$, $\triangle CDA$ and $\triangle CTM$ are all similar, since they are right-angled at E, D and T, respectively, and share a common angle at C.
Since $\triangle CEB$ and $\triangle CDA$ are similar, then $\dfrac{CB}{CA} = \dfrac{BE}{AD}$ and so $\dfrac{x}{x+3} = \dfrac{3}{4}$ which gives $4x = 3x + 9$ and so $x = 9$.
By the Pythagorean Theorem, $CE = \sqrt{CB^2 - BE^2} = \sqrt{9^2 - 3^2} = \sqrt{72} = 6\sqrt{2}$.
Since $\triangle CEB$ and $\triangle CTM$ are similar, then $\dfrac{BE}{CE} = \dfrac{MT}{CT}$ and so $\dfrac{3}{6\sqrt{2}} = \dfrac{y}{9+3+4}$ which gives $y = \dfrac{16\cdot 3}{6\sqrt{2}} = \dfrac{8}{\sqrt{2}} = 4\sqrt{2}$.

Finally, the area of $\triangle MNC$ is equal to $\frac{1}{2} \cdot MN \cdot CT$.

If we joined B to G, we would see that $\triangle CEB$ is congruent to $\triangle CGB$ (each is right-angled, they have a common hypotenuse, and $BE = BG$). This means that $\angle BCE = \angle BCG$, which in turn means that $MT = TN$.

Since $MT = TN$, then $MN = 2 \cdot 4\sqrt{2} = 8\sqrt{2}$ and so the area of $\triangle MNC$ is $\frac{1}{2} \cdot 8\sqrt{2} \cdot 16$ or $64\sqrt{2}$.

(b) First, we note that

$$\log_3 z = \frac{\log_{10} z}{\log_{10} 3} = \frac{2\log_{10} z}{2\log_{10} 3} = \frac{\log_{10}(z^2)}{\log_{10}(3^2)} = \frac{\log_{10}(z^2)}{\log_{10} 9} = \log_9(z^2)$$

Similarly, $\log_4 y = \log_{16}(y^2)$ and $\log_5 x = \log_{25}(x^2)$.

We also note from the original system of equations that $x > 0$ and $y > 0$ and $z > 0$.

Therefore, we can re-write the original system of equations as

$$\begin{aligned} \log_9 x + \log_9 y + \log_9(z^2) &= 2 \\ \log_{16} x + \log_{16}(y^2) + \log_{16} z &= 1 \\ \log_{25}(x^2) + \log_{25} y + \log_{25} z &= 0 \end{aligned}$$

Using logarithm rules, this is equivalent to the system

$$\begin{aligned} \log_9(xyz^2) &= 2 \\ \log_{16}(xy^2z) &= 1 \\ \log_{25}(x^2yz) &= 0 \end{aligned}$$

and to the system

$$\begin{aligned} xyz^2 &= 9^2 = 81 \\ xy^2z &= 16^1 = 16 \\ x^2yz &= 25^0 = 1 \end{aligned}$$

Multiplying these three equations together, we obtain $x^4y^4z^4 = 1296$ and so $(xyz)^4 = 6^4$.

Thus, $xyz = 6$.

Since $xyz^2 = 81$ and $xyz = 6$, then $z = \frac{xyz^2}{xyz} = \frac{81}{6} = \frac{27}{2}$.

Similarly, $y = \frac{xy^2z}{xyz} = \frac{16}{6} = \frac{8}{3}$ and $x = \frac{x^2yz}{xyz} = \frac{1}{6}$.

Therefore, $(x, y, z) = \left(\frac{1}{6}, \frac{8}{3}, \frac{27}{2}\right)$.

We can check by substitution that this triple does satisfy the original system of equations.

9. (a) Suppose that a sequence of n steps includes p steps in the positive direction and m steps in the negative direction.
This means that $n = p + m$ (total number of steps) and $d = p - m$ (final position).
Since $n = 9$ and $d = 5$, then $p + m = 9$ and $p - m = 5$ which give $p = 7$ and $m = 2$.
Thus, we need to count the number of sequences of 9 steps that include 7 steps in the positive direction and 2 steps in the negative direction.
This is the same as creating a sequence of 9 letters, 7 of which are R and 2 of which are L.
There are $\binom{9}{2} = \frac{9 \cdot 8}{2} = 36$ such sequences.
(Alternatively, we could see that there are 36 sequences by counting directly: there are 8 sequences where the first L appears in the first position, 7 where the first L appears appears in the second position, and so on.)

(b) *Solution 1*
Since $n = 9$ and $d = 3$, then $p + m = 9$ and $p - m = 3$ which gives $p = 6$ and $m = 3$.
This means that we can think about sequences of 9 letters that include 6 R's and 3 L's.
In this context, a direction change happens when the sequence changes from a block of one letter to a block of another letter (that is, when the sequence has an occurrence of "RL" or "LR").
For the sequence to change directions an even number of times, the number of blocks of letters in the sequence is odd.
For the number of blocks to be odd, the sequence must begin and end with the same letter.
If the sequence of 9 letters begins and ends with an R, the 7 letters in between include 4 R's and 3 L's; there are $\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$ such sequences.
If the sequence of 9 letters begins and ends with an L, the 7 letters in between include 6 R's and 1 L; there are $\binom{7}{1} = 7$ such sequences.
In total, there are $35 + 7 = 42$ such sequences.

Solution 2
As in Solution 1, we want to count the number of sequences of 9 letters that include 6 R's and 3 L's and that have an odd number of blocks.
Since there are 3 L's, there can be at most 3 blocks of L's, and so at most 7 blocks.

<u>Case 1: There are 3 blocks</u>
Suppose that the block order is R-L-R.
Here, there are 3 L's in the block of L's and 6 R's distributed between two blocks of R's.
There are 5 ways to distribute the R's: $1 + 5$, $2 + 4$, $3 + 3$, $4 + 2$, $5 + 1$.
This means that there are 5 such sequences.
Suppose that the block order is L-R-L.
The block of R's includes 6 R's and there are 3 L's to distribute between two blocks.
There are 2 ways to do this: $1 + 2$ or $2 + 1$.
This means that there are 2 such sequences.
In total, there are $5 + 2 = 7$ sequences in this case.

<u>Case 2: There are 7 blocks</u>
Since there cannot be 4 blocks of L's, the block order must be R-L-R-L-R-L-R.
Each block of L's includes exactly 1 L, and there are 2 additional R's to distribute after one R is placed in each block.
There are 4 ways to put these two R's in the same block, and 6 ways to put them in

separate blocks (1st and 2nd, 1st and 3rd, 1st and 4th, 2nd and 3rd, 2nd and 4th, 3rd and 4th).
Therefore, there are 10 different sequences in this case.

Case 3: There are 5 blocks
Suppose that the block order is R-L-R-L-R.
The two blocks of L's include 3 L's in total; there are 2 ways to distribute these ($1+2$ or $2+1$).
The three blocks of L's include 6 R's in total.
Starting with one R in each block, there are 3 additional R's to distribute.
There are 3 ways in which the 3 R's can go in the same block: $3+0+0$, $0+3+0$, $0+0+3$.
There are 6 ways in which the 3 R's can go in two blocks: $2+1+0$, $2+0+1$, $1+2+0$, $0+2+1$, $1+0+2$, $0+1+2$.
There is 1 way in which the 3 R's can go in three different blocks: $1+1+1$.
Thus, there are 2 ways to distribute the L's and $3+6+1=10$ ways to distribute the R's and so $2\cdot 10=20$ such sequences.
Suppose that the block order is L-R-L-R-L.
Each block of L's includes exactly 1 L, and the two blocks of R's include exactly 6 R's.
There are 5 ways to distribute these R's, as we saw in a similar situation in Case 1.
This means that there are 5 such sequences.
In total, there are $20+5=25$ sequences in this case.

Combining the cases, there are $10+7+25=42$ such sequences.

(c) Consider the sequences of length n that end at $x=d$ with $d\geq 0$.
Suppose that such a sequence includes p steps in the positive direction and m steps in the negative direction.
Since $p+m=n$ (total number of steps) and $p-m=d$ (ending position), then $2p=n+d$ (which gives $p=\dfrac{n+d}{2}$) and $2m=n-d$ (which gives $m=\dfrac{n-d}{2}$).
This means that all sequences of length n that end at $x=d$ with $d\geq 0$ correspond to the same values of p and m, and so we treat p and m as fixed in what follows.
Before proceeding to the general case, we deal with some specific small values.

When $n=1$, there is exactly 1 sequence that ends with $d\geq 0$. This sequence moves 1 step to the right. Since there is an odd number of sequences when $n=1$, it cannot be the case that half of the sequences have an even number of changes of direction.

When $n=2$, we can either have $p=2$ and $m=0$ (giving $d=2$) or $p=1$ and $m=1$ (giving $d=0$).
When $n=2$ and $d=2$, there is only 1 sequence, and so it cannot be the case that half of the sequences in this category have an even number of changes of direction.
When $n=2$ and $d=0$, there are 2 sequences (RL or LR), each of which has 1 change of direction, so it is not the case that half of the sequences in this category have an even number of changes of direction.

Therefore, we assume that $n\geq 3$. Since $n=p+m$ and $p\geq m$, then $p\geq 2$ as well.
We think about sequences of n letters that include p R's and m L's.
As in (b) Solution 1, the number of direction changes is even exactly when the number of blocks of letters is odd, which is exactly when the sequence begins and ends with the same letter.
For such a sequence to begin and end with R, the $n-2$ letters between include $(p-2)$ R's and m L's. (Note that $n-2\geq 0$ and $p-2\geq 0$ since $n\geq 2$ and $p\geq 0$.)

For such a sequence to begin and end with L, the $n-2$ letters between include p R's and $(m-2)$ L's. (Note that $n-2 \geq 0$. It is possible, though, that $m-2 < 0$, in which case we adopt the convention that there are 0 such sequences. In this case, we have $p = n-m = (n-2)-(m-2) > n-2$ and so $\binom{n-2}{p} = 0$, making the calculations that follow consistent with this convention.)

Thus, there are $\binom{n-2}{p-2} + \binom{n-2}{p}$ such sequences.

This is true for exactly half of all sequences when the following equivalent equations are true:

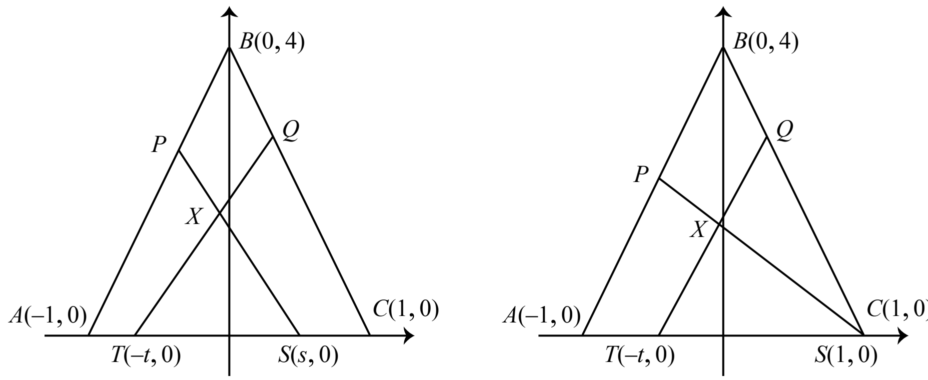
$$\begin{aligned}
\binom{n-2}{p-2} + \binom{n-2}{p} &= \frac{1}{2}\binom{n}{p} \\
\frac{(n-2)!}{(p-2)!m!} + \frac{(n-2)!}{p!(m-2)!} &= \frac{n!}{2 \cdot p!m!} \\
\frac{(n-2)!}{(p-2)!m!} + \frac{(n-2)!}{p!(m-2)!} &= \frac{n \cdot (n-1) \cdot (n-2)!}{2 \cdot p!m!} \\
\frac{1}{(p-2)!m!} + \frac{1}{p!(m-2)!} &= \frac{n(n-1)}{2 \cdot p!m!} \\
\frac{p!}{(p-2)!} + \frac{m!}{(m-2)!} &= \frac{n(n-1)}{2} \\
2p(p-1) + 2m(m-1) &= n(n-1) \\
2p^2 + 2m^2 &= n^2 - n + 2p + 2m \\
(p^2 + 2mp + m^2) + (p^2 - 2mp + m^2) &= n^2 - (p+m) + 2p + 2m \\
(p+m)^2 + (p-m)^2 &= n^2 + (p+m) \\
n^2 + d^2 &= n^2 + n \\
d^2 &= n
\end{aligned}$$

Therefore, exactly half of these sequences have an even number of changes of direction exactly when $n = d^2$.

To finish the problem, we now need to count the number of perfect squares (that is, possible values for n) in the correct range.

Since $2 \leq n \leq 2024$ and $d \geq 0$ and n is a perfect square, then the facts that $1^2 = 1$, $2^2 = 4$, $44^2 = 1936$, and $45^2 = 2025$ tell us that there are 43 perfect squares from 2 to 2024, inclusive, and so 43 such pairs (d, n). These are the pairs (d, d^2) for $d = 2, 3, \ldots, 43, 44$.

10. (a) Here is the general set-up for this problem along with the specific instance in (a) where $s = 1$:



Since SP and TQ divide $\triangle ABC$ into four regions of equal area, then $\triangle APS$, which is made up of two of these regions, has area equal to one-half of the area of $\triangle ABC$.
Since S and C coincide, then P is the midpoint of AB, which means that P has coordinates $(-\frac{1}{2}, 2)$.

Alternatively, we could note that the area of $\triangle ABC$ is $\frac{1}{2} \cdot 2 \cdot 4 = 4$ and so the area of $\triangle APS$ must be 2.
If P has y-coordinate p, then $\frac{1}{2} \cdot 2 \cdot p = 2$ which gives $p = 2$.
Since AB has slope 4 and y-intercept 4, its equation is $y = 4x + 4$.
Since P lies on AB and has y-coordinate 2, its x-coordinate satisfies $2 = 4x + 4$ and so $4x = -2$ or $x = -\frac{1}{2}$.
Thus, the coordinates of P are $(-\frac{1}{2}, 2)$.

(b) We note that the area of $\triangle ABC$ is $\frac{1}{2} \cdot 2 \cdot 4 = 4$. When the triangle is divided into four equal areas, each of these areas must be 1.
Suppose that (s, t) is a balancing pair.
This is true exactly when

- the area of $\triangle SXT$ is 1, and
- the area of $\triangle APS$ is 2, and
- the area of $\triangle CQT$ is 2.

(Bullets 1 and 2 tell us that the area of quadrilateral $APXT$ is 1.
Bullets 1 and 3 tell us that the area of quadrilateral $CQXS$ is 1.
Since three of the areas are 1, then the fourth must be 1.)

$\triangle SXT$ has base $TS = s + t$. If its height is h, then $\dfrac{1}{2}(s + t)h = 1$ and so $h = \dfrac{2}{s + t}$.
$\triangle APS$ has base $AS = 1 + s$. If its height is p, then $\dfrac{1}{2}(s + 1)p = 2$ and so $p = \dfrac{4}{s + 1}$.
$\triangle CQT$ has base $TC = 1 + t$. If its height is q, then $\dfrac{1}{2}(t + 1)q = 2$ and so $q = \dfrac{4}{t + 1}$.
The line through A and B has slope 4 and y-intercept 4, so its equation is $y = 4x + 4$.
The line through C and B has slope -4 and y-intercept 4, so its equation is $y = -4x + 4$.
Since P lies on the line with equation $y = 4x + 4$ and the y-coordinate of P is $\dfrac{4}{s + 1}$, then the x-coordinate of P satisfies $\dfrac{4}{s + 1} = 4x + 4$ which gives $\dfrac{1}{s + 1} = x + 1$, from which we obtain $x = \dfrac{1}{s + 1} - 1 = \dfrac{1 - s - 1}{s + 1} = -\dfrac{s}{s + 1}$.

Therefore, P has coordinates $\left(-\dfrac{s}{s+1}, \dfrac{4}{s+1}\right)$.

Since Q lies on the line with equation $y = -4x + 4$ and the y-coordinate of Q is $\dfrac{4}{t+1}$, then the x-coordinate of Q satisfies $\dfrac{4}{t+1} = -4x + 4$ which gives $\dfrac{1}{t+1} = -x + 1$ or $x = 1 - \dfrac{1}{t+1} = \dfrac{t+1-1}{t+1} = \dfrac{t}{t+1}$.

Therefore, Q has coordinates $\left(\dfrac{t}{t+1}, \dfrac{4}{t+1}\right)$.

Next, we find the coordinates of X by finding the equations of the lines through P and S, and through Q and T.

The slope of the line through S and P is

$$\frac{\dfrac{4}{s+1} - 0}{-\dfrac{s}{s+1} - s} = \frac{4}{-s - s(s+1)} = -\frac{4}{s^2 + 2s}$$

Since this line passes through $S(s, 0)$, its equation is $y = -\dfrac{4}{s^2+2s}(x - s)$.

The slope of the line through T and Q is

$$\frac{\dfrac{4}{t+1} - 0}{\dfrac{t}{t+1} - (-t)} = \frac{4}{t + t(t+1)} = \frac{4}{t^2 + 2t}$$

Since this line passes through $S(-t, 0)$, its equation is $y = \dfrac{4}{t^2+2t}(x + t)$.

To find the x-coordinate of X, we find the point of intersection of the line through S and P and the line through T and Q; thus, we solve

$$\begin{aligned}
-\frac{4}{s^2+2s}(x - s) &= \frac{4}{t^2+2t}(x + t) \\
\frac{4s}{s^2+2s} - \frac{4t}{t^2+2t} &= \left(\frac{4}{s^2+2s} + \frac{4}{t^2+2t}\right)x \\
4s(t^2+2t) - 4t(s^2+2s) &= (4(t^2+2t) + 4(s^2+2s))x \\
x &= \frac{st^2 - s^2t}{t^2+2t+s^2+2s}
\end{aligned}$$

Therefore, the y-coordinate of X is obtained from

$$\begin{aligned}
y &= \frac{4}{t^2+2t}\left(\frac{st^2 - s^2t}{t^2+2t+s^2+2s} + t\right) \\
&= \frac{4}{t^2+2t} \cdot \frac{st^2 - s^2t + t^3 + 2t^2 + ts^2 + 2st}{t^2+2t+s^2+2s} \\
&= \frac{4}{t^2+2t} \cdot \frac{t^3 + 2t^2 + st^2 + 2st}{t^2+2t+s^2+2s} \\
&= \frac{4}{t^2+2t} \cdot \frac{t(t^2+2t) + s(t^2+2t)}{t^2+2t+s^2+2s} \\
&= \frac{4(s+t)}{t^2+2t+s^2+2s}
\end{aligned}$$

Finally, the y-coordinate of X is equal to the height h from earlier, so

$$\begin{aligned}
\frac{2}{s+t} &= \frac{4(s+t)}{t^2+2t+s^2+2s} \\
\frac{1}{s+t} &= \frac{2(s+t)}{t^2+2t+s^2+2s} \\
t^2+2t+s^2+2s &= 2(s+t)^2 \\
t^2+2t+s^2+2s &= 2s^2+4st+2t^2 \\
-4st+2s+2t &= s^2+t^2
\end{aligned}$$

Therefore, the desired relationship is true, with $d=-4$, $e=f=2$ and $g=0$.

(c) We look for pairs of the form $(s,t)=(s,ks)$ where s and k are rational numbers.
From the relationship from (b),

$$\begin{aligned}
s^2+k^2s^2 &= -4ks^2+2s+2ks \\
s^2(k^2+4k+1) &= (2k+2)s \\
s(k^2+4k+1) &= 2k+2 \qquad \text{(since } s>0\text{)} \\
s &= \frac{2k+2}{k^2+4k+1}
\end{aligned}$$

Thus, $t=ks=\dfrac{2k^2+2k}{k^2+4k+1}$.
Since $k>0$, then $s>0$.
We need to have $s\le t$. This is equivalent to

$$\begin{aligned}
\frac{2k+2}{k^2+4k+1} &\le \frac{2k^2+2k}{k^2+4k+1} \\
2k+2 &\le 2k^2+2k \qquad \text{(since } k^2+4k+1>0\text{)} \\
2 &\le 2k^2 \\
1 &\le k \qquad \text{(since } k>0\text{)}
\end{aligned}$$

Thus, when $k\ge 1$, we have $0<s\le t$.
Finally, we need $t\le 1$. This is equivalent to

$$\begin{aligned}
\frac{2k^2+2k}{k^2+4k+1} &\le 1 \\
2k^2+2k &\le k^2+4k+1 \qquad \text{(since } k^2+4k+1>0\text{)} \\
k^2-2k-1 &\le 0
\end{aligned}$$

The roots of $k^2-2k-1=0$ are $k=\dfrac{2\pm\sqrt{(-2)^2-4(1)(-1)}}{2}=1\pm\sqrt{2}$.
Thus, $k^2-2k-1\le 0$ when $1-\sqrt{2}\le k\le 1+\sqrt{2}$.
Since $k>0$, then $t\le 1$ when $0<k\le 1+\sqrt{2}$.
Therefore, when $1\le k\le 1+\sqrt{2}$, we have $0<s\le t\le 1$.
We note that there are infinitely many rational numbers in any interval of non-zero length; in particular, there are infinitely many rational numbers k that satisfy $1\le k\le 1+\sqrt{2}$.
We do need to confirm that the values of $\dfrac{2k+2}{k^2+4k+1}$ are different for different values of k, which will confirm that there are infinitely many different values of s as k assumes these

infinitely many values itself.

To do this, we show that if $\dfrac{2k+2}{k^2+4k+1} = \dfrac{2j+2}{j^2+4j+1}$, then $k = j$; this fact will allow us to conclude that if $k \neq j$, then $\dfrac{2k+2}{k^2+4k+1} \neq \dfrac{2j+2}{j^2+4j+1}$.

When $k > 0$ and $j > 0$, then following equations are equivalent:

$$\begin{aligned}
\frac{2k+2}{k^2+4k+1} &= \frac{2j+2}{j^2+4j+1} \\
(2k+2)(j^2+4j+1) &= (2j+2)(k^2+4k+1) \\
(k+1)(j^2+4j+1) &= (j+1)(k^2+4k+1) \\
j^2k + 4jk + k + j^2 + 4j + 1 &= jk^2 + 4jk + j + k^2 + 4k + 1 \\
j^2k - jk^2 + 3j - 3k + j^2 - k^2 &= 0 \\
(j-k)(jk+j+k+3) &= 0
\end{aligned}$$

Since $jk + j + k + 3 > 0$, then it must be the case that $j - k = 0$ and so $j = k$.

This means that if $k \neq j$, then $\dfrac{2k+2}{k^2+4k+1} \neq \dfrac{2j+2}{j^2+4j+1}$ and so there are infinitely many different values of s.

Therefore, the pairs $(s,t) = \left(\dfrac{2k+2}{k^2+4k+1}, \dfrac{2k^2+2k}{k^2+4k+1}\right)$ where k is a rational number with $1 \leq k \leq 1+\sqrt{2}$ is an infinite family of solutions to the relationship from (b).