The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2024 Hypatia Contest

Thursday, April 4, 2024
(in North America and South America)

Friday, April 5, 2024
(outside of North America and South America)

Solutions

1. (a) Of the 4050 trucks sold, 32% were white or $\dfrac{32}{100} \cdot 4050 = 1296$ were white.

    (b) *Solution 1*

    Of the 4050 trucks sold, 24% were grey or $\dfrac{24}{100} \cdot 4050 = 972$ were grey.

    Since $\dfrac{1}{4}$ of the grey trucks sold were electric, then $\dfrac{1}{4} \cdot 972 = 243$ trucks sold were both grey and electric.

    Solution 2

    Since 24% of the trucks sold were grey, and $\dfrac{1}{4}$ of those were electric, then $\dfrac{24}{100} \cdot \dfrac{1}{4} = \dfrac{6}{100}$ (or 6%) were both grey and electric.

    Thus, of the 4050 trucks sold, $\dfrac{6}{100} \cdot 4050 = 243$ were both grey and electric.

    (c) *Solution 1*

    Of the 4050 trucks sold, 44% were black or $\dfrac{44}{100} \cdot 4050 = 1782$ were black.

    Thus, the total number of black trucks, sold and unsold, was $1782 + k$, and the total number of trucks, sold and unsold, was $4050 + k$.

    Since 46% of all trucks, sold and unsold, were black, then $\dfrac{1782 + k}{4050 + k} = \dfrac{46}{100}$.

    Solving, we get

    $$
    \begin{aligned}
    \frac{1782 + k}{4050 + k} &= \frac{46}{100} \\
    \frac{1782 + k}{4050 + k} &= \frac{23}{50} \\
    50(1782 + k) &= 23(4050 + k) \\
    89\,100 + 50k &= 93\,150 + 23k \\
    27k &= 4050 \\
    k &= 150
    \end{aligned}
    $$

    and so there were 150 unsold trucks, all of which were black.

    Solution 2

    Of the 4050 trucks sold, 44% were black and so $100\% - 44\% = 56\%$ were not black.

    Therefore, $\dfrac{56}{100} \cdot 4050 = 2268$ trucks sold were not black.

    Since all unsold trucks were black, then there were 2268 trucks, sold and unsold, that were not black.

    Since 46% of all trucks, sold and unsold, were black, then $100\% - 46\% = 54\%$ of all trucks, sold and unsold, were not black.

    The total number of trucks, sold and unsold, was $4050 + k$ and 54% of these trucks were not black, thus $\dfrac{2268}{4050 + k} = \dfrac{54}{100}$.

Solving, we get

$$
\begin{aligned}
\frac{2268}{4050 + k} &= \frac{54}{100} \\
\frac{2268}{4050 + k} &= \frac{27}{50} \\
50(2268) &= 27(4050 + k) \\
113\,400 &= 109\,350 + 27k \\
4050 &= 27k \\
k &= 150
\end{aligned}
$$

and so there were 150 unsold trucks, all of which were black.

2. (a) Evaluating, we get $f(132) = 132 + 1 + 3 + 2 = 138$.

(b) Suppose that n is equal to the 3-digit positive integer abc.
Then $f(n) = f(abc) = 100a + 10b + c + a + b + c = 101a + 11b + 2c$.
Since $f(n) = 175$, then $101a + 11b + 2c = 175$.
It cannot be the case that $a \geq 2$, since if we had $a \geq 2$, then $101a \geq 202$ which is too large, noting that $11b + 2c$ is always at least 0.
Therefore, $a < 2$ which means that $a = 1$.
When $a = 1$, we get $101 + 11b + 2c = 175$ or $11b + 2c = 74$.
It cannot be the case that $b \geq 7$, since if we had $b \geq 7$, then $11b \geq 77$ which is too large, noting that $2c$ is always at least 0.
Therefore, $b < 7$. If $b = 6$, then $66 + 2c = 74$ or $2c = 8$, and so $c = 4$.
If $b \leq 5$, then $11b \leq 55$, and so $2c \geq 74 - 55 = 19$, which is not possible since $c \leq 9$.
We can confirm that $f(164) = 164 + 1 + 6 + 4 = 175$, and so $n = 164$.

(c) Suppose that n is equal to the 3-digit positive integer pqr.
Then $f(pqr) = 100p + 10q + r + p + q + r$, and so $101p + 11q + 2r = 204$.
If $p \geq 3$, then $101p \geq 303$, and so $p = 1$ or $p = 2$.

If $p = 1$, then $101 + 11q + 2r = 204$ or $11q + 2r = 103$.
Since $r \leq 9$, then $2r \leq 18$ and so $11q \geq 103 - 18 = 85$.
Therefore, $q = 8$ or $q = 9$.
If $q = 8$, then $88 + 2r = 103$ or $2r = 15$, which is not possible since r is an integer.
If $q = 9$, then $99 + 2r = 103$ or $2r = 4$, and so $r = 2$.
In this case, $n = 192$ and we can confirm that $f(192) = 192 + 1 + 9 + 2 = 204$.

If $p = 2$, then $202 + 11q + 2r = 204$ or $11q + 2r = 2$.
The only possible solution to $11q + 2r = 2$ is $q = 0$ and $r = 1$.
In this case, $n = 201$ and we can confirm that $f(201) = 201 + 2 + 0 + 1 = 204$.
Therefore, if $f(n) = 204$, then the possible values of n are 192 and 201.

3. (a) To determine the coordinates of F, we find the point of intersection of the line through A and C and the line through B and E.
The line through $A(0,0)$ and $C(12,12)$ has slope $\frac{12-0}{12-0} = 1$.
Since it passes through $(0,0)$, this line has equation $y = x$.
The line through $B(12,0)$ and $E(0,6)$ has slope $\frac{6-0}{0-12} = -\frac{1}{2}$.
Since it passes through $(0,6)$, this line has equation $y = -\frac{1}{2}x + 6$.

To determine the x-coordinate of the point of intersection, F, we solve $x = -\frac{1}{2}x + 6$, which gives $\frac{3}{2}x = 6$ or $3x = 12$, and so $x = 4$.

Since F lies on the line with equation $y = x$, then the coordinates of F are $(4, 4)$.

(b) *Solution 1*

Consider $\triangle AEF$ as having base $AE = 6$.

Then $\triangle AEF$ has height equal to the perpendicular distance from F to AE, which is 4, the x-coordinate of F.

The area of $\triangle AEF$ is thus $\frac{1}{2} \cdot 6 \cdot 4 = 12$.

Solution 2

We can determine the area of $\triangle AEF$ by subtracting the area of $\triangle AFB$ from the area of $\triangle AEB$.

Consider $\triangle AFB$ as having base $AB = 12$.

Then $\triangle AFB$ has height equal to the perpendicular distance from F to AB, which is 4, the y-coordinate of F.

The area of $\triangle AFB$ is thus $\frac{1}{2} \cdot 12 \cdot 4 = 24$.

The area of $\triangle AEB$ is $\frac{1}{2} \cdot 12 \cdot 6 = 36$, and so the area of $\triangle AEF$ is $36 - 24 = 12$.

(c) To determine the area of quadrilateral $GDEF$, our strategy will be to subtract the area of $\triangle AEF$ and the area of $\triangle CDG$ from the area of $\triangle ACD$.

We need to find the area of $\triangle CDG$ still, which means finding the coordinates of G.

We can find the coordinates of G by determining the intersection of the line through A and C with the given circle.

Thus, we proceed by finding the equation of the circle.

Since the circle has diameter EB, then its centre is the midpoint of EB, which is $\left(\frac{0+12}{2}, \frac{6+0}{2}\right)$ or $(6, 3)$.

The diameter has length $EB = \sqrt{(12-0)^2 + (0-6)^2}$ or $EB = \sqrt{180}$, which simplifies to $EB = 6\sqrt{5}$.

Thus the radius of the circle is $r = \frac{1}{2} \cdot 6\sqrt{5} = 3\sqrt{5}$, and so the circle has equation $(x-6)^2 + (y-3)^2 = (3\sqrt{5})^2$ or $(x-6)^2 + (y-3)^2 = 45$.

Suppose the x-coordinate of G is g.

Since G lies on the line with equation $y = x$, then the coordinates of G are (g, g).

The point G also lies on the circle, and thus the coordinates of G satisfy the equation of the circle.

That is, $(g-6)^2 + (g-3)^2 = 45$, and solving for g, we get

$$
\begin{aligned}
g^2 - 12g + 36 + g^2 - 6g + 9 &= 45 \\
2g^2 - 18g + 45 &= 45 \\
2g^2 - 18g &= 0 \\
2g(g - 9) &= 0
\end{aligned}
$$

and so $g = 0$ or $g = 9$.

Since G is distinct from A, then $g = 9$ and G has coordinates $(9, 9)$.

We may now determine the area of $\triangle CDG$.

Consider $\triangle CDG$ as having base $CD = 12$.

Then $\triangle CDG$ has height equal to the perpendicular distance from G to CD, which is $12 - 9 = 3$, since CD lies along the line $y = 12$ and the y-coordinate of G is 9.

The area of $\triangle CDG$ is thus $\frac{1}{2} \cdot 12 \cdot 3 = 18$.

The area of $\triangle ACD$ is half the area of square $ABCD$ or $\frac{1}{2} \cdot 12^2 = 72$.

From part (b), the area of $\triangle AEF$ is 12, and so the area of $GDEF$ is $72 - 18 - 12 = 42$.

4. (a) Each Hewitt number, H, can be written as $H = (n-1)^3 + n^3 + (n+1)^3$ where n is an integer and $n \geq 2$.
(We chose $H = (n-1)^3 + n^3 + (n+1)^3$ instead of $H = n^3 + (n+1)^3 + (n+2)^3$, since the quadratic term and constant term subtract out when simplified, as shown below.)
Expanding and simplifying, we get

$$
\begin{aligned}
H &= (n-1)^3 + n^3 + (n+1)^3 \\
&= n^3 - 3n^2 + 3n - 1 + n^3 + n^3 + 3n^2 + 3n + 1 \\
&= 3n^3 + 6n \\
&= 3n(n^2 + 2)
\end{aligned}
$$

A Hewitt number is divisible by 10 exactly when its units digit is equal to 0.
If for example the units digit of n is 9, then the units digit of $3n$ is 7, the units digit of $n^2 + 2$ is 3, and so the units digit of $H = 3n(n^2 + 2)$ is 1 (since 7×3 has units digit 1).
For each possible units digit of n, we determine the units digit of H in the table below.

Units digit of n	0	1	2	3	4	5	6	7	8	9
Units digit of $3n$	0	3	6	9	2	5	8	1	4	7
Units digit of $n^2 + 2$	2	3	6	1	8	7	8	1	6	3
Units digit of $H = 3n(n^2 + 2)$	0	9	6	9	6	5	4	1	4	1

Thus, for a Hewitt number to be divisible by 10, the units digit of n must be 0, and so n must be divisible by 10.
When $n = 10$, $H = 3(10)(10^2 + 2) = 3060$ which is less than $10\,000$.
When $n = 20$, $H = 3(20)(20^2 + 2) = 24\,120$ which lies between $10\,000$ and $100\,000$.
When $n = 30$, $H = 3(30)(30^2 + 2) = 81\,180$ which lies between $10\,000$ and $100\,000$.
When $n \geq 40$, $H \geq 3(40)(40^2 + 2) = 192\,240$ which is greater than $100\,000$.
Thus, there are 2 Hewitt numbers between $10\,000$ and $100\,000$ that are divisible by 10.

(b) From part (a), each Hewitt number can be written as $H = 3n(n^2 + 2)$ where n is an integer and $n \geq 2$.
Since $216 = 2^3 \cdot 3^3$, then a Hewitt number is divisible by 216 exactly when $3n(n^2 + 2)$ is divisible by $2^3 \cdot 3^3$, or exactly when $n(n^2 + 2)$ is divisible by $2^3 \cdot 3^2$.
That is, we need $n(n^2 + 2)$ to be divisible by $2^3 = 8$ and by $3^2 = 9$.

We begin by considering what is required for $n(n^2 + 2)$ to be divisible by 8.
If $n(n^2 + 2)$ is divisible by 8, then $n(n^2 + 2)$ is divisible by 2 and thus is even.
If n is odd, then $n(n^2 + 2)$ is odd, and so n must be even.
Since n is even, then $n = 2a$ for some positive integer a, and so $n^2 + 2 = 4a^2 + 2$ which is 2 more than a multiple of 4 and so $n^2 + 2$ is not divisible by 4 (but it is divisible by 2).
Since $n(n^2 + 2)$ is divisible by 8 and $n^2 + 2$ contains exactly one factor of 2, then n must be divisible by 4.

Next, we consider what is required for $n(n^2 + 2)$ to be divisible by 9.
If $n(n^2 + 2)$ is divisible by 9, then at least one of the following must be true:

(i) n is divisible by 3 and $n^2 + 2$ is divisible by 3, or

(ii) n is divisible by 9, or

(iii) $n^2 + 2$ is divisible by 9.

Assume that n is divisible by 9, and thus divisible by 3.
Then $n = 3b$ for some positive integer b, and so $n^2 + 2 = 9b^2 + 2 = 3(3b^2) + 2$ which is 2 more than a multiple of 3 and so $n^2 + 2$ is not divisible by 3, and thus not divisible by 9.

This tells us that if n is divisible by 3, then $n^2 + 2$ is not divisible by 3, and so (i) cannot be true.

Further, if n is divisible by 9, then $n^2 + 2$ is not divisible by 9, and so exactly one of (ii) or (iii) is true.

Summarizing, we get $n(n^2 + 2)$ is divisible by 8 and by 9 (and thus a Hewitt number is divisible by 216) exactly when n is divisible by 4 and by 9, or when n is divisible by 4 and $n^2 + 2$ is divisible by 9.

Case 1: n is divisible by 4 and by 9
Since 4 and 9 share no common divisor larger than 1, then n is divisible by 4 and by 9 exactly when n is divisible by $4 \cdot 9 = 36$.
In this case, $n = 36k$ for positive integers k.
The first Hewitt number occurs when $n = 2$, and so the 2024th Hewitt number occurs when $n = 2025$.
That is, $2 \le n \le 2025$ or $2 \le 36k \le 2025$, and so $\frac{2}{36} \le k \le \frac{2025}{36}$.
Since k is an integer and $\frac{2025}{36} = 56.25$, then $1 \le k \le 56$.
Thus in this case, 56 of the smallest 2024 Hewitt numbers are divisible by 216.

Case 2: n is divisible by 4 and $n^2 + 2$ is divisible by 9
Since n is divisible by 4, then $n = 4m$ for some positive integer m, and so $n^2 + 2 = 16m^2 + 2$ is divisible by 9.
For some non-negative integers q and r, where $0 \le r \le 8$, every positive integer m can be written as $m = 9q + r$, depending on its remainder, r, when divided by 9.
Since $16m^2 + 2$ must be divisible by 9, then each of the following equivalent expressions must also be divisible by 9:

$$\begin{aligned} 16m^2 + 2 &= 16(9q + r)^2 + 2 \\ &= 16(9^2q^2 + 2 \cdot 9qr + r^2) + 2 \\ &= 16(9^2q^2 + 2 \cdot 9qr) + 16r^2 + 2 \\ &= 9 \cdot 16(9q^2 + 2qr) + 16r^2 + 2 \end{aligned}$$

which is divisible by 9 exactly when $16r^2 + 2$ is divisible by 9.
That is, n is divisible by 4 and $n^2 + 2 = 16m^2 + 2$ is divisible by 9 exactly when $16r^2 + 2$ is divisible by 9, where r is the remainder when m is divided by 9.
For each of the possible remainders $0 \le r \le 8$, we may determine the remainder when $16r^2 + 2$ is divided by 9.
For example, when $r = 2$, $16r^2 + 2 = 16(2)^2 + 2 = 66$ leaves remainder 3 when divided by 9.
Similarly, we determine the remainder when $16r^2 + 2$ is divided by 9 for each of the possible values of r:

Value of r	0	1	2	3	4	5	6	7	8
Remainder when $16r^2 + 2$ is divided by 9	2	0	3	2	6	6	2	3	0

Thus, $16r^2 + 2$ is divisible by 9 when $r = 1$ or when $r = 8$.
Summarizing, we get that n is divisible by 4 and $n^2 + 2$ is divisible by 9 exactly when $n = 4m = 4(9q + 1)$ or when $n = 4(9q + 8)$ for non-negative integers q.
For the smallest 2024 Hewitt numbers, $2 \le n \le 2025$ or $2 \le 4(9q + 1) \le 2025$, and so $0 \le q \le 56$ (since q is a non-negative integer).
In this case, 57 of the smallest 2024 Hewitt numbers are divisible by 216.
Similarly, $2 \le 4(9q + 8) \le 2025$, and so $0 \le q \le 55$.

In this case, 56 of the smallest 2024 Hewitt numbers are divisible by 216.

Of the smallest 2024 Hewitt numbers, $56 + 57 + 56 = 169$ are divisible by 216.

(c) From part (a), each Hewitt number is given by $H = 3n(n^2 + 2)$ where n is an integer and $n \geq 2$.

If S is the sum of two distinct Hewitt numbers, then $S = 3n(n^2 + 2) + 3m(m^2 + 2)$ for some integers m and n and for which we may assume that $2 \leq m < n$.

If there are two distinct Hewitt numbers whose sum is equal to $9 \cdot 2^k$ for some positive integer k, then we get the following equivalent equations

$$\begin{aligned}
S &= 3n(n^2 + 2) + 3m(m^2 + 2) \\
9 \cdot 2^k &= 3n(n^2 + 2) + 3m(m^2 + 2) \\
3 \cdot 2^k &= n(n^2 + 2) + m(m^2 + 2) \\
3 \cdot 2^k &= n^3 + m^3 + 2n + 2m \\
3 \cdot 2^k &= (n + m)(n^2 - nm + m^2) + 2(n + m) \\
3 \cdot 2^k &= (n + m)(n^2 - nm + m^2 + 2)
\end{aligned}$$

and so if there are two distinct Hewitt numbers whose sum is equal to $9 \cdot 2^k$, then $(n + m)(n^2 - nm + m^2 + 2) = 3 \cdot 2^k$ for some positive integers k, m, n where $2 \leq m < n$. If m and n have different parity (one is even and the other is odd), then $n + m$ is odd. Also, $n^2 - nm + m^2 + 2$ is odd since exactly one of n^2 or m^2 is odd and the remaining three terms in the sum are even.

In this case, $n + m$ and $n^2 - nm + m^2 + 2$ are both odd, and so their product is odd. However, $3 \cdot 2^k$ is even for all positive integers k. Therefore, m and n must have the same parity (they must both be odd or they must both be even).

Case 1: m and n are both odd
If m and n are both odd, then $n^2 - nm + m^2 + 2$ is odd.
Since the only odd factors of $3 \cdot 2^k$ are 1 and 3, then $n^2 - nm + m^2 + 2$ must equal 1 or 3. However, if m and n are both odd with $2 \leq m < n$, then $m \geq 3$ and $n \geq 5$ and $n - m \geq 2$. Therefore,

$$\begin{aligned}
n^2 - nm + m^2 + 2 &= n(n - m) + m^2 + 2 \\
&\geq 5(2) + 3^2 + 2 \\
&= 21
\end{aligned}$$

and so $n^2 - nm + m^2 + 2$ cannot equal 1 or 3.
Therefore, m and n cannot both be odd.

Case 2: m and n are both even
If m and n are both even, then $m = 2a$ and $n = 2b$ for some integers a and b with $1 \leq a < b$.
Substituting and simplifying, we get

$$\begin{aligned}
3 \cdot 2^k &= (n + m)(n^2 - nm + m^2 + 2) \\
3 \cdot 2^k &= (2b + 2a)(4b^2 - 4ab + 4a^2 + 2) \\
3 \cdot 2^k &= 4(b + a)(2b^2 - 2ab + 2a^2 + 1) \\
3 \cdot 2^{k-2} &= (b + a)(2b^2 - 2ab + 2a^2 + 1)
\end{aligned}$$

Since the right side is the product of two integers, then $k \geq 2$.

The factor $2b^2 - 2ab + 2a^2 + 1$ is one more than a multiple of 2, and thus is odd and so it must be equal to 1 or 3.

Since $1 \le a < b$, then $a \ge 1$, $b \ge 2$ and $b - a \ge 1$, and so

$$\begin{aligned}
2b^2 - 2ab + 2a^2 + 1 &= 2b(b - a) + 2a^2 + 1 \\
&\ge 4(1) + 2(1^2) + 1 \\
&= 7
\end{aligned}$$

and so $2b^2 - 2ab + 2a^2 + 1$ cannot equal 1 or 3.

Therefore, m and n cannot both be even.

We can conclude that there cannot be two distinct Hewitt numbers whose sum is equal to $9 \cdot 2^k$ for some positive integer k.