The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2023 Gauss Contests

(Grades 7 and 8)

Wednesday, May 17, 2023
(in North America and South America)

Thursday, May 18, 2023
(outside of North America and South America)

Solutions

Grade 7

1. Half of 24 is $24 \div 2 = 12$. Kiyana gives her friend 12 grapes.

   Answer: (D)

2. Reading from the graph, Friday had the highest temperature.

   Answer: (C)

3. At a cost of \$16.50 a basket, the cost to buy 4 baskets of strawberries is $4 \times \$16.50 = \$66.00$.

   Answer: (B)

4. The difference between 3 and -5 is $3 - (-5) = 3 + 5 = 8$. Therefore, it is now 8°C warmer.

   Answer: (A)

5. Since $5 \times 5 = 25$ and each of the given answers is greater than 25, then the integer that Sarah multiplied by itself must have been greater than 5.
   Further, $6 \times 6 = 36$ and each of the given answers is less than or equal to 36.
   Thus, of the given answers, only 36 could be the result of multiplying an integer by itself.
   Alternatively, we may have noted that the result of multiplying an integer by itself is a perfect square, and of the answers given, 36 is the only perfect square.

   Answer: (E)

6. Since the perimeter of $PQRS$ is 40 cm and $SR = 16$ cm, then the combined length of the remaining three sides is 40 cm $-$ 16 cm $=$ 24 cm.
   Each of the remaining three sides is equal in length, and so $PQ = \frac{24 \text{ cm}}{3} = 8$ cm.

   Answer: (C)

7. Dividing 52 by each of the given denominators, we get that $\frac{52}{4} = 13$ is the only whole number.

   Answer: (C)

8. The line segment with greatest length that joins two points on a circle is a diameter of the circle. Since the circle has a radius of 4 cm, then its diameter has length 2×4 cm $= 8$ cm, and so the greatest possible length of the line segment is 8 cm.

   Answer: (B)

9. The number of integers in the list is 10. Of these integers, 5 are even (they are 10, 12, 14, 16, and 18). Thus, the probability that the chosen integer is even is $\frac{5}{10}$.

   Answer: (C)

10. Before adding tax, the combined cost of the three items is $\$4.20 + \$7.60 + \$3.20 = \$15.00$.
    The 5% tax on \$15.00 is $0.05 \times \$15.00 = \$0.75$, and so the total cost of the three items, after tax is added, is $\$15.00 + \$0.75 = \$15.75$.
    Note that we could have calculated the 5% tax on each individual item, however doing so is less efficient than calculating tax on the \$15.00 total.

    Answer: (D)

11. Since BCD is a straight line segment, then $\angle BCD = 180^\circ$.
Therefore, $\angle ACB = \angle BCD - \angle ACD = 180^\circ - 75^\circ = 105^\circ$.
Since the sum of the three angles in $\triangle ABC$ is 180°, then $\angle ABC = 180^\circ - 105^\circ - 35^\circ = 40^\circ$.
Answer: (B)

12. Of the 100 small identical squares, 28 are presently unshaded, and so $100 - 28 = 72$ are shaded.
So that 75% of the area of $WXYZ$ is shaded, 75 of the 100 small squares must be shaded.
Therefore, $75 - 72 = 3$ more of the small squares must be shaded.
Answer: (A)

13. Suppose we call the unknown vertex V.
The side of the rectangle joining the points $(2, 1)$ and $(2, 5)$ is vertical, and so the opposite side of the rectangle (the side joining $(4, 1)$ to V) must also be vertical.
This means that V has the same x-coordinate as $(4, 1)$, which is 4.
Similarly, the side of the rectangle joining the points $(2, 1)$ and $(4, 1)$ is horizontal, and so the opposite side of the rectangle (the side joining $(2, 5)$ to V) must also be horizontal.
This means that V has the same y-coordinate as $(2, 5)$, which is 5.
Therefore, the coordinates of the fourth vertex of the rectangle are $(4, 5)$.
Answer: (D)

14. The prime numbers that are less than 10 are $2, 3, 5$, and 7.
Thus, the only two different prime numbers whose sum is 10 are 3 and 7.
The product of these two numbers is $3 \times 7 = 21$.
Answer: (D)

15. The given list, $2, 9, 4, n, 2n$ contains 5 numbers.
The average of these 5 numbers is 6, and so the sum of the 5 numbers is $5 \times 6 = 30$.
That is, $2 + 9 + 4 + n + 2n = 30$ or $15 + 3n = 30$, and so $3n = 15$ or $n = 5$.
Answer: (D)

16. The sum of P and Q is equal to 5, and so P and Q are (in some order) either equal to 1 and 4, or they are equal to 2 and 3.
The difference between R and S is equal to 5, and so R and S must be (in some order) equal to 1 and 6.
Since R and S are equal to 1 and 6, then neither P nor Q can be 1, which means that P and Q cannot be equal to 1 and 4, and so they must be equal to 2 and 3.
The only numbers from 1 to 6 not accounted for are 4 and 5.
Since T is greater than U, then the number that replaces T is 5.
Answer: (E)

17. *Solution 1*
The area of $\triangle AED$ is equal to one-half its base times its height.
Suppose the base of $\triangle AED$ is AE, then its height is BD (AE is perpendicular to BD).
Since $AB = BC = 24$ cm and E and D are the midpoints of their respective sides, then $AE = 12$ cm and $BD = 12$ cm.
Thus, the area of $\triangle AED$ is $\frac{1}{2} \times 12 \text{ cm} \times 12 \text{ cm} = 72 \text{ cm}^2$.

Solution 2
The area of $\triangle AED$ is equal to the area of $\triangle ABD$ minus the area of $\triangle EBD$.
Suppose the base of $\triangle EBD$ is BD, then its height is EB.
Since $AB = BC = 24$ cm and E and D are the midpoints of their respective sides, then $EB = 12$ cm and $BD = 12$ cm.

Thus, the area of $\triangle EBD$ is $\frac{1}{2} \times 12 \text{ cm} \times 12 \text{ cm} = 72 \text{ cm}^2$.
The area of $\triangle ABD$ is equal to $\frac{1}{2} \times BD \times AB = \frac{1}{2} \times 12 \text{ cm} \times 24 \text{ cm} = 144 \text{ cm}^2$.
Thus, the area of $\triangle AED$ is $144 \text{ cm}^2 - 72 \text{ cm}^2 = 72 \text{ cm}^2$.

Answer: (C)

18. The water is in the shape of a rectangular prism with a 2 cm by 5 cm base and depth 6 cm. Therefore, the volume of water is $2 \text{ cm} \times 5 \text{ cm} \times 6 \text{ cm} = 60 \text{ cm}^3$.
A face of the prism having the greatest area has dimensions 5 cm by 8 cm.
When the prism is tipped so that it stands on a 5 cm by 8 cm face, the water is once again in the shape of a rectangular prism with a 5 cm by 8 cm base and unknown depth.
Suppose that after the prism is tipped, the water's depth is d cm.
Since the volume of water is still 60 cm^3 when the prism is tipped, then $5 \text{ cm} \times 8 \text{ cm} \times d \text{ cm} = 60 \text{ cm}^3$ or $40d \text{ cm}^3 = 60 \text{ cm}^3$, and so $d = \dfrac{60}{40} = \dfrac{3}{2}$.
When the prism is tipped so that it stands on a face with the greatest area, the depth of the water is $\dfrac{3}{2}$ cm $= 1.5$ cm.

Answer: (D)

19. *Solution 1*
We begin by completing a table in which the ones digit of each possible product is listed.
For example, when the number on the first die is 3 and the number on the second die is 6, the entry in the table is 8 since $3 \times 6 = 18$ and the ones digit of 18 is 8.

Number on the Second Die

Number on the First Die

$\times$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	**0**	2
3	3	6	9	2	5	8
4	4	8	2	6	**0**	4
5	5	**0**	5	**0**	5	**0**
6	6	2	8	4	**0**	6

Of the 36 possible outcomes in the table above, 6 outcomes have a ones digit that is equal to 0. Thus, the probability that the ones digit of the product is 0 is $\dfrac{6}{36} = \dfrac{1}{6}$.

Solution 2
Since the ones digit of the product is 0, then the product is divisible by 5 and is even.
Since the possible numbers in the product are 1, 2, 3, 4, 5, 6, then one of the numbers rolled must be 5.
Since the product is even (and 5 is not), then the other number rolled must be one of the three even numbers, namely 2, 4, 6.
Thus, the possible pairs of numbers that can be rolled to give a product whose ones digit is 0,

are $(5,2)$, $(5,4)$, $(5,6)$ or $(2,5)$, $(4,5)$, $(6,5)$. (We note that the first number in the ordered pair represents the first number rolled, while the second number in the pair is the second number rolled.)
Since there are 6 possible rolls for each of the two dice, then there are $6 \times 6 = 36$ possible ordered pairs representing all possible outcomes.
Since 6 of these ordered pairs represent a product whose ones digit is 0, then the required probability is $\frac{6}{36} = \frac{1}{6}$.

ANSWER: (D)

20. Since a and b are positive integers, then each of $\frac{a}{7}$ and $\frac{2}{b}$ is greater than 0.
The sum of $\frac{a}{7}$ and $\frac{2}{b}$ is equal to 1, and so each is less than 1.
Since $\frac{a}{7}$ is greater than 0 and less than 1, then the possible values of a are $1, 2, 3, 4, 5, 6$.
By substituting each of these values for a into the equation one at a time, we can determine if there is a positive integer value of b for which the equation is true.
Substituting $a = 1$, we get $\frac{1}{7} + \frac{2}{b} = 1$ or $\frac{2}{b} = 1 - \frac{1}{7}$, and so $\frac{2}{b} = \frac{6}{7}$.
Since $\frac{2}{b} = \frac{6}{7}$, we can multiply the numerator and denominator of the first fraction by 3 (which is $6 \div 2$) to get $\frac{6}{3b} = \frac{6}{7}$.
This gives $3b = 7$ which does not have an integer solution ($b = \frac{7}{3}$).
Thus when $a = 1$, there is no positive integer value of b that satisfies the equation.

Substituting $a = 2$, we get $\frac{2}{7} + \frac{2}{b} = 1$ or $\frac{2}{b} = 1 - \frac{2}{7}$, and so $\frac{2}{b} = \frac{5}{7}$.
Since $\frac{2}{b} = \frac{5}{7}$, we can multiply the numerator and denominator of the first fraction by 5, and the numerator and denominator of the second fraction by 2 to get $\frac{10}{5b} = \frac{10}{14}$.
This gives $5b = 14$ which does not have an integer solution.
Thus when $a = 2$, there is no positive integer value of b that satisfies the equation.

Substituting $a = 3$, we get $\frac{3}{7} + \frac{2}{b} = 1$ or $\frac{2}{b} = 1 - \frac{3}{7}$, and so $\frac{2}{b} = \frac{4}{7}$.
Since $\frac{2}{b} = \frac{4}{7}$, we can multiply the numerator and denominator of the first fraction by 2 to get $\frac{4}{2b} = \frac{4}{7}$.
This gives $2b = 7$ which does not have an integer solution.
Thus when $a = 3$, there is no positive integer value of b that satisfies the equation.

Substituting $a = 4$ and simplifying, we get $\frac{2}{b} = \frac{3}{7}$.
Since $\frac{2}{b} = \frac{3}{7}$, we can multiply the numerator and denominator of the first fraction by 3, and the numerator and denominator of the second fraction by 2 to get $\frac{6}{3b} = \frac{6}{14}$.
This gives $3b = 14$ which does not have an integer solution.

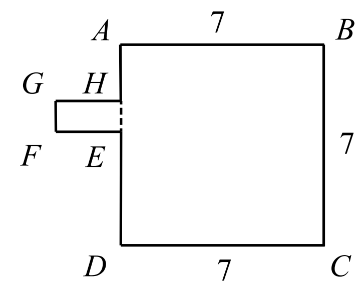
Thus when $a = 4$, there is no positive integer value of b that satisfies the equation.

Substituting $a = 5$ and simplifying, we get $\frac{2}{b} = \frac{2}{7}$.
Since the numerators are equal, then the denominators must be equal, and so $b = 7$ satisfies the equation.

Finally, substituting $a = 6$ and simplifying, we get $\frac{2}{b} = \frac{1}{7}$.
Since $\frac{2}{b} = \frac{1}{7}$, we can multiply the numerator and denominator of the second fraction by 2 to get $\frac{2}{b} = \frac{2}{14}$, and so $b = 14$.
Thus, there are two pairs of positive integers a and b that satisfy the given equation: $a = 5, b = 7$ and $a = 6, b = 14$.

ANSWER: (E)

21. Since $ABCD$ is a square and its side lengths are integers, then its area is equal to a perfect square.
Since the product of the areas of $ABCD$ and $EFGH$ (the rectangle) is equal to 98, then the area of $ABCD$ is a divisor of 98.
The positive divisors of 98 are 1, 2, 7, 14, 49, and 98.
There are exactly two divisors of 98 that are perfect squares, namely 1 and 49.
Since the area of $ABCD$ is greater than the area of $EFGH$, then the area of $ABCD$ is 49, and so the area of $EFGH$ is 2 (since $49 \times 2 = 98$).
Square $ABCD$ has area 49, and so $AB = BC = CD = DA = 7$.
The perimeter of $ABCDEFGH$ is equal to



$$\begin{aligned} & AB + BC + CD + DE + EF + FG + GH + HA \\ = \ & 7 + 7 + 7 + DE + EF + EH + GH + HA \quad \text{(since } EH = FG\text{)} \\ = \ & 21 + DE + EH + HA + EF + GH \quad \text{(reorganizing)} \\ = \ & 21 + DA + EF + GH \quad \text{(since } DE + EH + HA = DA\text{)} \\ = \ & 21 + 7 + EF + GH \quad \text{(since } DA = 7\text{)} \\ = \ & 28 + 2 \times GH \quad \text{(since } EF = GH\text{)} \end{aligned}$$

Since the side lengths are integers and the area of $EFGH$ is 2, then either $GH = 1$ (and $FG = 2$), or $GH = 2$ (and $FG = 1$).
If $GH = 1$, then the perimeter of $ABCDEFGH$ is $28 + 2 \times 1 = 30$.
Since 30 is not given as a possible answer, then $GH = 2$ and the perimeter is $28 + 2 \times 2 = 32$.

ANSWER: (B)

22. If a Gareth sequence begins 10, 8, then the 3rd number in the sequence is $10 - 8 = 2$, the 4th is $8 - 2 = 6$, the 5th is $6 - 2 = 4$, the 6th is $6 - 4 = 2$, the 7th is $4 - 2 = 2$, the 8th is $2 - 2 = 0$, the 9th is $2 - 0 = 2$, the 10th is $2 - 0 = 2$, and the 11th is $2 - 2 = 0$.
Thus, the resulting sequence is $10, 8, 2, 6, 4, 2, 2, 0, 2, 2, 0, \ldots$.
The first 5 numbers in the sequence are $10, 8, 2, 6, 4$, the next 3 numbers are $2, 2, 0$, and this block of 3 numbers appears to repeat.
Since each new number added to the end of this sequence is determined by the two previous numbers in the sequence, then this block of 3 numbers will indeed continue to repeat. (That

is, since the block repeats once, then it will continue repeating.)
The first 30 numbers of the sequence begins with the first 5 numbers, followed by 8 blocks of $2, 2, 0$, followed by one additional 2 (since $5 + 8 \times 3 + 1 = 30$).
The sum of the first 5 numbers is $10 + 8 + 2 + 6 + 4 = 30$.
The sum of each repeating block is $2 + 2 + 0 = 4$, and so the sum of 8 such blocks is $8 \times 4 = 32$.
Thus, the sum of the first 30 numbers in the sequence is $30 + 32 + 2 = 64$.

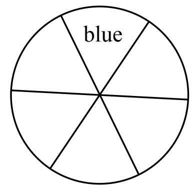
ANSWER: (E)

23. Suppose that the length, or the width, or the height of the rectangular prism is equal to 5.
The product of 5 with any of the remaining digits has a units (ones) digit that is equal to 5 or it is equal to 0.
This means that if the length, or the width, or the height of the rectangular prism is equal to 5, then at least one of the two-digit integers (the area of a face) has a units digit that is equal to 5 or 0.
However, 0 is not a digit that can be used, and each digit from 1 to 9 is used exactly once (that is, 5 cannot be used twice), and so it is not possible for one of the dimensions of the rectangular prism to equal 5.
Thus, the digit 5 occurs in one of the two-digit integers (the area of a face).
The digit 5 cannot be the units digit of the area of a face, since this would require that one of the dimensions be 5.
Therefore, one of the areas of a face has a tens digit that is equal to 5.
The two-digit integers with tens digit 5 that are equal to the product of two different one-digit integers (not equal to 5) are $54 = 6 \times 9$ and $56 = 7 \times 8$.
Suppose that two of the dimensions of the prism are 7 and 8, and so one of the areas is 56.
In this case, the digits $5, 6, 7$, and 8 have been used, and so the digits $1, 2, 3, 4$, and 9 remain.
Which of these digits is equal to the remaining dimension of the prism?
It cannot be 1 since the product of 1 and 7 does not give a two-digit area, nor does the product of 1 and 8.
It cannot be 2 since the product of 2 and 8 is 16 and the digit 6 has already been used.
It cannot be 3 since $3 \times 7 = 21$ and $3 \times 8 = 24$, and so the areas of two faces share the digit 2.
It cannot be 4 since $4 \times 7 = 28$ and the digit 8 has already been used.
Finally, it cannot be 9 since $9 \times 7 = 63$ and the digit 6 has already been used.
Therefore, it is not possible for 7 and 8 to be the dimensions of the prism, and thus 6 and 9 must be two of the three dimensions.
Using a similar systematic check of the remaining digits, we determine that 3 is the third dimension of the prism.
That is, when the dimensions of the prism are $3, 6$ and 9, the areas of the faces are $3 \times 6 = 18$, $3 \times 9 = 27$, and $6 \times 9 = 54$, and we may confirm that each of the digits from 1 to 9 has been used exactly once.
Since the areas of the faces are 18, 27 and 54, the surface area of the rectangular prism is $2 \times (18 + 27 + 54)$ or $2 \times 99 = 198$.

ANSWER: (D)

24. *Solution 1*

Begin by colouring the section at the top blue.
Since two circles have the same colouring if one can be rotated to match the other, it does not matter which section is coloured blue, so we arbitrarily choose the top section.

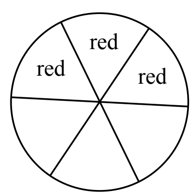


There are now 5 sections which can be coloured green.
After choosing the section to be coloured green, there are 4 sections remaining which can be coloured yellow.
Each of the remaining 3 sections must then be coloured red.
Thus, the total number of different colourings of the circle is $5 \times 4 = 20$.

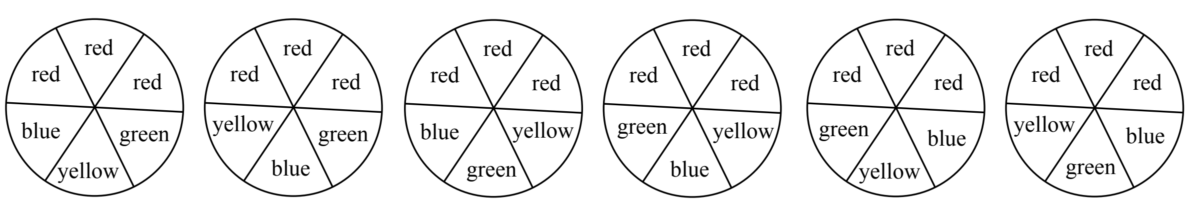
Solution 2

We begin by considering the locations of the three sections coloured red, relative to one another.
The three red sections could be adjacent to one another, or exactly two red sections could be adjacent to one another, or no red section could be adjacent to another red section.
We consider each of these 3 cases separately.

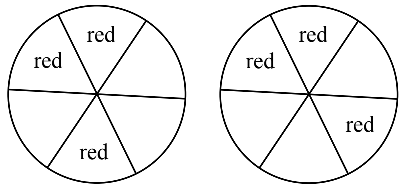
Case 1: All three red sections are adjacent to one another
Begin by colouring any three adjacent sections red.



Since two circles have the same colouring if one can be rotated to match the other, it does not matter which three adjacent sections are coloured red.
Consider the first section that follows the three red sections as we move clockwise around the circle.
There are 3 choices for the colour of this section: blue, green or yellow.
Continuing to move clockwise to the next section, there are now 2 choices for the colour of this section.
Finally, there is 1 choice for the colour of the final section, and thus there are $3 \times 2 \times 1 = 6$ different colourings of the circle in which all three red sections are adjacent to one another.
These 6 colourings are shown below.

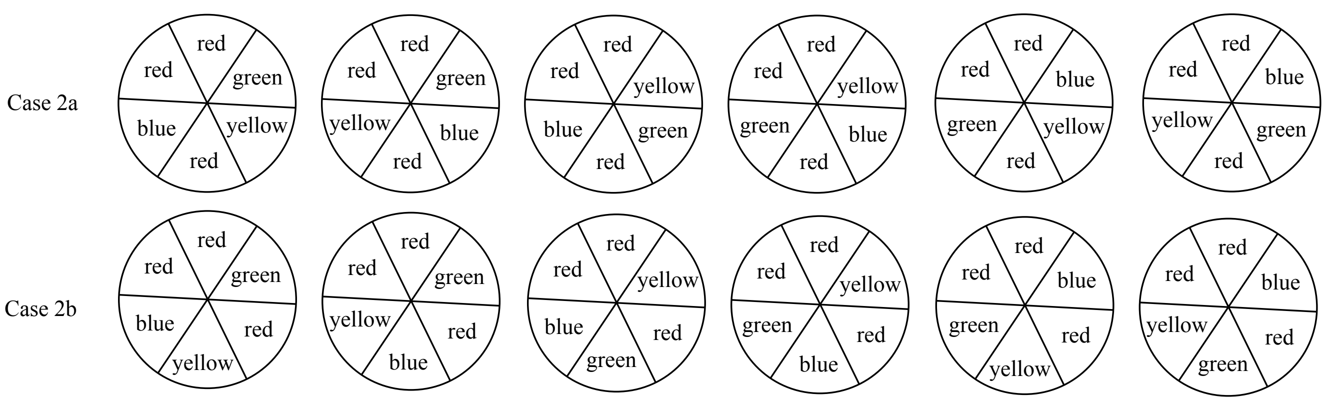


Case 2: Exactly two red sections are adjacent to one another
There are two different possible arrangements in which exactly two red sections are adjacent to one another.
In the first of these, the next two sections that follow the two adjacent red sections as we move clockwise around the circle, are both not red. We call this Case 2a.
In the second of these, the section that follows the two adjacent red sections as we move clockwise around the circle is not red, but the next section is. We call this Case 2b.
The arrangements for Cases 2a and 2b are shown below.

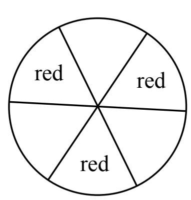


Case 2a Case 2b

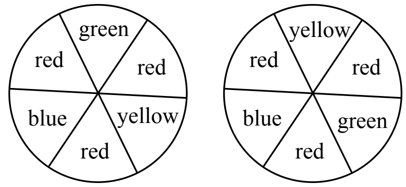
Notice that the first of these two circles cannot be rotated to match the second.
The number of colourings in Case 2a and in Case 2b are each equal to the number of colourings as in Case 1.
That is, there are 3 choices for the first uncoloured section that follows the two red sections as we move clockwise around the circle.
Continuing to move clockwise to the next uncoloured section, there are now 2 choices for the colour of this section.
Finally, there is 1 choice for the colour of the final section, and thus there are $3 \times 2 \times 1 = 6$ different colourings of the circle in Case 2a as well as in Case 2b.
These 12 colourings are shown below.



Case 3: No red section is adjacent to another red section
Begin by colouring any three non-adjacent sections red.



Since two circles have the same colouring if one can be rotated to match the other, it does not matter which three non-adjacent sections are coloured red.
In this case, there are 2 possible colourings as shown below.

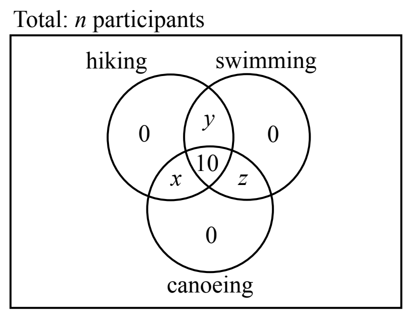


A circle with any other arrangement of the green, yellow and blue sections can be rotated to match one of the two circles above.

The total number of different colourings of the circle is $6 + 12 + 2 = 20$.

Answer: (E)

25. We can represent the given information in a Venn diagram by first introducing some variables.
Let x be the number of students that participated in hiking and canoeing, but not swimming.
Let y be the number of students that participated in hiking and swimming, but not canoeing.
Let z be the number of students that participated in canoeing and swimming, but not hiking.
Since 10 students participated in all three activities and no students participated in fewer than two activities, we complete the Venn diagram as shown.



Suppose that the total number of students participating in the school trip was n.
Since 50% of all students participated in at least hiking and canoeing, then $\frac{50}{100}n$ or $\frac{n}{2}$ participated in at least hiking and canoeing.
Since this number of students, $\frac{n}{2}$, is an integer, then n must be divisible by 2.
Similarly, $\frac{60}{100}n$ or $\frac{3n}{5}$ students participated in at least hiking and swimming.
Since this number of students, $\frac{3n}{5}$, is an integer, then n must be divisible by 5 (since 3 and 5 have no factors in common).
This means that n is divisible by both 2 and 5, and thus n is divisible by 10.
From the Venn diagram, we see that $x + 10 = \frac{n}{2}$, and $y + 10 = \frac{3n}{5}$.
Since the total number of participants is n, we also get that $x+y+z+10 = n$ or $z = n-10-x-y$.
We may now use these equations,

$$x = \frac{n}{2} - 10, \ y = \frac{3n}{5} - 10, \text{ and } z = n - 10 - x - y$$

and the fact that n is divisible by 10, to determine all possible values of z.
We can then use the values of z to determine all possible values of the positive integer k, where k% participated in at least canoeing and swimming.

Since n is a positive integer that is divisible by 10, its smallest possible value is 10.
However, substituting $n = 10$ into $x = \frac{n}{2} - 10$, we get $x = 5 - 10$ and so $x = -5$ which is not possible. (Recall that x is the number of students that participated in hiking and canoeing, but not swimming, and so $x \geq 0$.)
Next, we try $n = 20$.

When $n = 20$, $x = 10 - 10$ and so $x = 0$.
When $n = 20$, $y = \dfrac{3 \times 20}{5} - 10$ or $y = 12 - 10$, and so $y = 2$.
Finally, when $n = 20$, $x = 0$, and $y = 2$, we get $z = 20 - 10 - 0 - 2 = 8$.
When $z = 8$, the number of students who participated in at least canoeing and swimming is $8 + 10 = 18$ (since 10 students participated in all three), and so the percentage of students who participated in at least canoeing and swimming is $\dfrac{18}{20} \times 100\% = 90\%$, and so $k = 90$.
In the table below, we continue in this way by using successively greater multiples of 10 for the value of n.

n	$x = \dfrac{n}{2} - 10$	$y = \dfrac{3n}{5} - 10$	$z = n - 10 - x - y$	$k = \dfrac{z+10}{n} \times 100$
20	0	2	8	$k = \dfrac{8+10}{20} \times 100 = 90$
30	5	8	7	$k = \dfrac{7+10}{30} \times 100 \approx 56.7$
40	10	14	6	$k = \dfrac{6+10}{40} \times 100 = 40$
50	15	20	5	$k = \dfrac{5+10}{50} \times 100 = 30$
60	20	26	4	$k = \dfrac{4+10}{60} \times 100 \approx 23.3$
70	25	32	3	$k = \dfrac{3+10}{70} \times 100 \approx 18.6$
80	30	38	2	$k = \dfrac{2+10}{80} \times 100 = 15$
90	35	44	1	$k = \dfrac{1+10}{90} \times 100 \approx 12.2$
100	40	50	0	$k = \dfrac{0+10}{100} \times 100 = 10$

For values of n that are greater than 100, we get that $z < 0$, which is not possible.
Therefore, the sum of all such positive integers k is $90 + 40 + 30 + 15 + 10 = 185$.

Answer: (B)

Grade 8

1. The fraction $\frac{1}{4}$ is equivalent to $1 \div 4 = 0.25$.

   Answer: (B)

2. Reading from the graph, the forecast wind speed is less than 20 km/h on Monday, Tuesday, Wednesday, and Sunday.
   Thus, Jack will be able to sail alone on 4 days during this 7-day period.

   Answer: (A)

3. We note that $15 \times 10 = 150$, $15 \times 2 = 30$, $15 \times 3 = 45$, and $15 \times 4 = 60$.
   Since there is no integer n for which $15 \times n = 25$, then 25 is not a multiple of 15.

   Answer: (B)

4. Ordering the given list of integers from least to greatest, we get $-9, -7, 0, 9, 10$.
   The third integer in the ordered list is 0.

   Answer: (D)

5. *Solution 1*
   If $2n = 14$, then $n = \dfrac{14}{2} = 7$.
   When $n = 7$, the value of $10n$ is $10 \times 7 = 70$.

   Solution 2
   Multiplying both sides of the given equation by 5, we get $5 \times 2n = 5 \times 14$, and so $10n = 70$.

   Answer: (C)

6. There are 6 possible outcomes when Tallulah rolls a single standard die once.
   She loses if she rolls 2 of these 6 outcomes, and so the probability that she loses is $\dfrac{2}{6} = \dfrac{1}{3}$.

   Answer: (A)

7. *Solution 1*
   We may convert the given addition problem to a subtraction problem.
   That is, since $1013 + PQPQ = 2023$, then $PQPQ = 2023 - 1013$.
   The difference between 2023 and 1013 is $2023 - 1013 = 1010$, and so $P = 1$, $Q = 0$, and $P + Q = 1 + 0 = 1$.

   Solution 2
   The ones (units) digit of the sum 2023 is 3.
   Thus, the ones digit of the sum $3 + Q$ must equal 3.
   Since Q is a digit, the only possible value of Q is 0.
   The tens digit of the sum 2023 is 2.
   Since there is no "carry" from the ones column to the tens column, the ones digit of the sum $1 + P$ must equal 2.
   Since P is a digit, the only possible value of P is 1.
   We may confirm that when $P = 1$ and $Q = 0$, we get $1013 + 1010 = 2023$ as required.
   The value of $P + Q$ is $1 + 0 = 1$.

   Answer: (B)

8. Suppose the salad dressing initially contains 300 mL of oil.
Since the ratio of oil to vinegar is $3:1$, then the salad dressing initially contains one-third as much vinegar as oil, or $\frac{1}{3} \times 300$ mL $= 100$ mL of vinegar (note that $300:100 = 3:1$).
If the volume of vinegar is doubled, the new volume of vinegar is 2×100 mL $= 200$ mL, and so the new ratio of oil to vinegar is $300:200 = 3:2$.
Note: We chose to begin with 300 mL of oil, but performing the above calculations with any starting volume of oil will give the same $3:2$ ratio.

Answer: (A)

9. Before including tax, the combined cost of the three items is $\$4.20 + \$7.60 + \$3.20 = \$15.00$.
The 5% tax on \$15.00 is $0.05 \times \$15.00 = \$0.75$, and so the total cost of the three items, including tax, is $\$15.00 + \$0.75 = \$15.75$.
Note that we could have calculated the 5% tax on each individual item, however doing so is less efficient than calculating tax on the \$15.00 total.

Answer: (D)

10. When $(1, 3)$ is reflected in the y-axis, the reflected point is $(-1, 3)$.
In general, when a point is reflected in the y-axis, its x-coordinate changes sign, and its y-coordinate does not change.
Thus, the vertices of the reflected rectangle are $(-1, 3)$, $(-1, 1)$, $(-4, 1)$, and $(-4, 3)$.
Of the given possibilities, $(-3, 4)$ is not a vertex of the reflected rectangle.

Answer: (C)

11. In the leftmost rectangle, the length of the path along the rectangle's diagonal, d, and the sides with lengths 3 and 4, form a right-angled triangle.
Using the Pythagorean Theorem, we get $d^2 = 3^2 + 4^2$, and so $d = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ (these are "3-4-5" right-angled triangles).
The path from A to B includes one such diagonal, two vertical sides each of length 4, and three horizontal sides each of length 3, and thus has length $5 + (2 \times 4) + (3 \times 3) = 22$.

Answer: (A)

12. Since $\angle PQR$ is a straight angle, its measure is 180°, and so $\angle SQR = 180^\circ - 125^\circ = 55^\circ$.
Since $SQ = SR$, then $\angle SRQ = \angle SQR = 55^\circ$.
The sum of the angles inside $\triangle SQR$ is 180°, and so $x = 180 - 55 - 55 = 70$.

Answer: (B)

13. The number of peaches in the original pile is two more than a multiple of three.
Of the choices given, 29 is the only number which is two more than a multiple of three $(29 = 3 \times 9 + 2)$.

Answer: (D)

14. The sum of each block of 5 repeating integers is $4 - 3 + 2 - 1 + 0 = 2$.
In the first 23 integers, there are 4 such blocks of 5 integers, plus 3 additional integers (since $23 = 4 \times 5 + 3$).
The sum of the 4 blocks of 5 repeating integers is $4 \times 2 = 8$, and the next three integers in the list following these 4 blocks are $4, -3, 2$.
Thus, the sum of the first 23 integers is $8 + 4 - 3 + 2 = 11$.

Answer: (D)

15. The circumference of each of Bindu's bike tires is $2 \times \pi \times 30$ cm $= 60\pi$ cm.
If the bike tires rotate exactly five times, then the distance travelled by Bindu's bike is $5 \times 60\pi$ cm $= 300\pi$ cm.

Answer: (D)

16. *Solution 1*
The sum of all 8 numbers in the list is $41 + 35 + 19 + 9 + 26 + 45 + 13 + 28 = 216$.
When the 8 numbers are arranged in pairs, there are 4 pairs.
The sum of the numbers in each pair is the same, and so this sum is $\frac{216}{4} = 54$.
Therefore, the number paired with 13 is $54 - 13 = 41$.
Note: We may confirm that $45 + 9 = 54$, $41 + 13 = 54$, $35 + 19 = 54$, and $28 + 26 = 54$.

Solution 2
Since the sum of the numbers in each pair is the same, then the largest number in the list must be paired with the smallest number, the second largest with the second smallest, and so on.
(Can you reason why this must be true?)
That is, the largest and smallest numbers in the list, 45 and 9, must be paired.
The second largest and second smallest numbers, 41 and 13, must be paired, and so the number paired with 13 is 41.
Note: We may confirm that $45 + 9 = 54$, $41 + 13 = 54$, $35 + 19 = 54$, and $28 + 26 = 54$.

Answer: (E)

17. The mean (average) is determined by adding the 30 recorded temperatures, and dividing the sum by 30.
The sum of the temperatures for the first 25 days was $25 \times 21^\circ\text{C} = 525^\circ\text{C}$.
The sum of the temperatures for the last 5 days was $5 \times 15^\circ\text{C} = 75^\circ\text{C}$.
Thus, the mean of the recorded temperatures was $\frac{525^\circ\text{C} + 75^\circ\text{C}}{30} = \frac{600^\circ\text{C}}{30} = 20^\circ\text{C}$.

Answer: (C)

18. We begin by listing, in order, the smallest 2-digit positive divisors of 630.
We note that by first writing 630 as a product of its prime factors ($630 = 2 \times 3^2 \times 5 \times 7$), it may be easier to determine these divisors.
The smallest five 2-digit positive divisors of 630 are $10, 14, 15, 18$, and 21.
The next largest 2-digit positive divisor of 630 is 30, and we notice that $21 \times 30 = 630$.
That is, 21 and 30 are a pair of 2-digit positive integers whose product is 630, and they are consecutive in the ordered list of positive divisors.
Thus, each of $10, 14, 15$, and 18 must be paired with a divisor of 630 that is greater than 30.
We may check that the divisors that pair with each of $10, 14, 15$, and 18 is a 2-digit positive integer by dividing 630 by the smaller divisor.
That is, $\frac{630}{18} = 35$, $\frac{630}{15} = 42$, $\frac{630}{14} = 45$, and $\frac{630}{10} = 63$.
Thus, the pairs of 2-digit positive integers whose product is 630 are 21 and 30, 18 and 35, 15 and 42, 14 and 45, and 10 and 63, and so there are 5 such pairs.

Answer: (D)

19. Between 9 a.m. and 10 a.m., Ryan cut $\frac{7}{8} - \frac{1}{2} = \frac{7}{8} - \frac{4}{8} = \frac{3}{8}$ of his lawn.
Ryan cut $\frac{3}{8}$ of his lawn in 1 hour (60 minutes), and so he cut $\frac{1}{8}$ of his lawn in $\frac{60}{3}$ minutes or 20 minutes.
At 10 a.m., Ryan had cut $\frac{7}{8}$ of his lawn, and thus had $1 - \frac{7}{8} = \frac{1}{8}$ of his lawn left to cut.
Since Ryan cuts $\frac{1}{8}$ of his lawn in 20 minutes, then he finished at 10:20 a.m.

ANSWER: (C)

20. Begin by placing four tiles in the squares of the first row.
The only restriction is that the row must contain one tile of each colour.
Thus in the first row, there are 4 choices of tile colour for the first column, 3 choices for the second, 2 for the third, and 1 choice for the fourth.
That is, there are $4 \times 3 \times 2 \times 1 = 24$ different ways to cover the squares in the first row using one tile of each of the four colours.
For example, using R for red, B for black, G for green, and Y for yellow, the first row could contain tiles coloured $G\ \ Y\ \ R\ \ B$, in that order.
Suppose that the first row does contain tiles coloured $G\ \ Y\ \ R\ \ B$, in that order.
We will show that there is only one way to arrange the remaining 12 tiles in the grid.

Consider the first square in the second row, that is, the square directly below the tile coloured G. The tile placed in this square cannot be coloured G since it shares an edge with the tile coloured G in row 1.
Also, the tile placed in this square cannot be coloured Y since it touches the corner of the square containing the tile coloured Y in row 1.
Assume that the tile in the first square of row 2 is coloured B, as shown.

G	Y	R	B
B			

Next, consider the colour of the tiles that could be placed in the second square of row 2.
The tile in this square cannot be coloured Y since it shares an edge with the tile coloured Y in row 1.
Also, the tile in this square cannot be coloured G since it touches the corner of the square containing the tile coloured G in row 1.
Further, the tile in this square cannot be coloured R since it touches the corner of the square containing the tile coloured R in row 1.
Since the first square in this row contains a tile coloured B, then we have no possible tile that can be placed in the second square of row 2.
This means that the tile in the first square of row 2 cannot be coloured B, and thus it must be coloured R, as shown.

G	Y	R	B
R			

The tile in the second square of row 2 cannot be coloured Y or G (as noted earlier), and thus must be coloured B.
Continuing to move right along row 2, the next tile cannot be coloured Y since it touches the corner of the square containing the tile coloured Y in row 1, and so the tile in this square must be coloured G, with the final tile in the row being coloured Y.

That is, the positions of the 4 tiles in row 2 are completely determined by the tiles in row 1.
Thus for each of the 24 different ways to place the tiles in row 1, there is exactly one way to place the tiles in row 2.
Repeating the argument, the same is then true for the tiles in row 3 and row 4; that is, there

is exactly one choice for the location of each of the coloured tiles within each of these two rows as well.

The 4×4 in our example above is completed here, as shown.

You should justify for yourself that each of rows 3 and 4 must contain tiles exactly as shown.

G	Y	R	B
R	B	G	Y
G	Y	R	B
R	B	G	Y

For each of the 24 different ways to cover the squares in the first row using one tile of each of the four colours, there is exactly one way to cover all remaining squares in the grid.

Thus, there are 24 different ways that the tiles can be arranged.

ANSWER: (B)

21. Since OM is a radius of the circle, then $OM = 87$.
$\triangle MNO$ is a right-angled triangle, and so by the Pythagorean Theorem, we get $OM^2 = MN^2 + NO^2$ or $87^2 = 63^2 + NO^2$, and so $NO^2 = 87^2 - 63^2 = 3600$.
Since $NO > 0$, then $NO = \sqrt{3600} = 60$.

Since OP is also a radius, then $OP = 87$, and so $NP = NO + OP = 60 + 87 = 147$.
The area of $\triangle PMN$ is equal to $\frac{1}{2} \times NP \times MN = \frac{1}{2} \times 147 \times 63 = 4630.5$.

ANSWER: (D)

22. Nasrin's mean (average) speed is determined by dividing the total distance travelled, which is 9 km, by the total time.
It took Nasrin 2 hours and thirty minutes, or 150 minutes, to canoe into her camp.
On the return trip, it took her $\dfrac{1}{3} \times 150$ minutes or 50 minutes.
Thus, the total time for Nasrin to paddle to camp and back was 200 minutes.
Converting to hours, 200 minutes is 3 hours and 20 minutes, and since 20 minutes is $\dfrac{20}{60} = \dfrac{1}{3}$ hours, it took Nasrin $3\frac{1}{3}$ hours in total.
Thus, Nasrin's mean speed as she paddled to camp and back was $\dfrac{9 \text{ km}}{3\frac{1}{3} \text{ h}}$ or $\dfrac{9 \text{ km}}{\frac{10}{3} \text{ h}}$, which is equal to $9 \times \dfrac{3}{10}$ km/h $= \dfrac{27}{10}$ km/h $= 2.7$ km/h.

ANSWER: (E)

23. To begin, the volume of water in Cylinder B is $\pi \times (8 \text{ cm})^2 \times 50 \text{ cm} = 3200\pi \text{ cm}^3$.
After some water is poured from Cylinder B into Cylinder A, the total volume of water in the two cylinders will be $3200\pi \text{ cm}^3$ (since no water is lost).
Let h cm be the height of the water in each of the two cylinders when the height of the water in both cylinders is the same.
At this time, the volume of water in Cylinder B is $\pi \times (8 \text{ cm})^2 \times h \text{ cm} = 64\pi h \text{ cm}^3$.
At this time, the volume of water in Cylinder A is $\pi \times (6 \text{ cm})^2 \times h \text{ cm} = 36\pi h \text{ cm}^3$.
Thus, the total volume of water in the two cylinders is $64\pi h \text{ cm}^3 + 36\pi h \text{ cm}^3 = 100\pi h \text{ cm}^3$, and so $100\pi h = 3200\pi$ or $h = \dfrac{3200\pi}{100\pi} = 32$.
When the height of the water in both cylinders is the same, that height is 32 cm.

ANSWER: (C)

24. *Solution 1*

We begin by multiplying the given equation through by 20 to get $20 \times \frac{a}{4} + 20 \times \frac{b}{10} = 20 \times 7$, or $5a + 2b = 140$.

Since $5a = 140 - 2b$ and both 140 and $2b$ are even, then $5a$ is even, which means that a is even.

We start by trying $a = 20$ and $b = 20$ which is a solution, since $5 \times 20 + 2 \times 20 = 140$.

This pair for a and b satisfies all of the conditions except $a < b$.

We may find other solutions to the equation $5a + 2b = 140$ by adding two 5s and subtracting five 2s (this is the same as adding 10 and subtracting 10), or by subtracting two 5s and adding five 2s.

Adding two 5s is equivalent to increasing the value of a by 2.

Subtracting five 2s is equivalent to decreasing the value of b by 5.

Consider $a = 20 + 2 = 22$ and $b = 20 - 5 = 15$.

This is a solution since $5 \times 22 + 2 \times 15 = 140$.

However, in this case $a > b$ and every time we add two 5s and subtract five 2s, a becomes greater and b becomes smaller.

Thus, we need to go in the other direction.

Consider $a = 20 - 2 = 18$ and $b = 20 + 5 = 25$.

This is a solution since $5 \times 18 + 2 \times 25 = 140$.

Here, $a < b$ and $a + b = 43$, so this pair satisfies all of the conditions.

Next, consider $a = 18 - 2 = 16$ and $b = 25 + 5 = 30$.

This is a solution since $5 \times 16 + 2 \times 30 = 140$.

Here, $a < b$ and $a + b = 46$, so this pair satisfies all of the conditions.

Notice that the sum $a + b$ increases by 3 on each of these steps.

This means that doing this 17 more times gets us to $a = 16 - 17 \times 2 = -18$ and $b = 30 + 17 \times 5 = 115$.

This is a solution since $5 \times (-18) + 2 \times 115 = 140$.

Notice that it is still the case that $a < b$ and $a + b < 100$.

Repeating this process one more time, we get $a = -20$ and $b = 120$, which gives $a + b = 100$ and so there are no more pairs that work.

Since we know that a has to be even and we are considering all possible even values for a, there can be no other pairs that work.

In total, there are $1 + 1 + 17 = 19$ pairs of integers a and b that satisfy each of the given conditions and the given equation.

Solution 2

We begin by rearranging the given equation to isolate b.

Doing so, we get

$$\begin{aligned}
\frac{a}{4} + \frac{b}{10} &= 7 \\
\frac{b}{10} &= 7 - \frac{a}{4} \\
10 \times \frac{b}{10} &= 10 \times 7 - 10 \times \frac{a}{4} \qquad \text{(multiplying each term by 10)} \\
b &= 70 - \frac{10a}{4} \\
b &= 70 - \frac{5a}{2}
\end{aligned}$$

Since b is an integer, then $70 - \frac{5a}{2}$ is an integer, which means that $\frac{5a}{2}$ must be an integer.

Since 2 does not divide 5, then 2 must divide a and so a is even.

Since $a < b$ and $b = 70 - \frac{5a}{2}$, then

$$\begin{aligned} a &< 70 - \frac{5a}{2} \\ 2 \times a &< 2 \times 70 - 2 \times \frac{5a}{2} \qquad \text{(multiplying each term by 2)} \\ 2a &< 140 - 5a \\ 7a &< 140 \\ a &< 20 \end{aligned}$$

Further, since $a + b < 100$ and $b = 70 - \frac{5a}{2}$, then

$$\begin{aligned} a + 70 - \frac{5a}{2} &< 100 \\ 2 \times a + 2 \times 70 - 2 \times \frac{5a}{2} &< 2 \times 100 \qquad \text{(multiplying each term by 2)} \\ 2a + 140 - 5a &< 200 \\ -60 &< 3a \\ -20 &< a \end{aligned}$$

Thus a is an even integer that is greater than -20 and less than 20.
Since there are 19 even integers from -18 to 18 inclusive, we suspect that there are 19 pairs of integers a and b that satisfy the given equation.
It is a good idea (and good practice) to at least check that the largest and smallest of these values of a do indeed satisfy each of the given conditions.

When $a = -18$, we get $b = 70 - \frac{5(-18)}{2}$ or $b = 70 - 5(-9)$, and so $b = 115$.
This pair satisfies the given conditions that $a < b$ and $a + b < 100$.
Substituting $a = -18$ and $b = 115$ into the given equation, we get

$$\frac{a}{4} + \frac{b}{10} = \frac{-18}{4} + \frac{115}{10} = \frac{-9}{2} + \frac{23}{2} = \frac{14}{2} = 7,$$

and thus $a = -18$ and $b = 115$ is a solution.

When $a = 18$, we get $b = 70 - \frac{5(18)}{2}$ or $b = 70 - 5(9)$, and so $b = 25$.
This satisfies the given conditions that $a < b$ and $a + b < 100$.
Substituting $a = 18$ and $b = 25$ into the given equation, we get

$$\frac{a}{4} + \frac{b}{10} = \frac{18}{4} + \frac{25}{10} = \frac{9}{2} + \frac{5}{2} = \frac{14}{2} = 7,$$

and thus $a = 18$ and $b = 25$ is also a solution.

At this point we can be confident that for each of the 19 even integer values of a from -18 to 18 inclusive, there is an integer b for which the pair of integers a and b satisfy each of the given conditions and the given equation.

Answer: (B)

25. For any triangle, the sum of the lengths of two sides is always greater than the length of the third side. This property is known as the *triangle inequality*.

If for example the side lengths of a triangle are a, b and c, then the triangle inequality says that

$$a + b > c \text{ and } a + c > b \text{ and } b + c > a.$$

We begin by considering the number of different ways to choose three integers from $3, 4, 10, 13$ (without using n), and then forming a triangle whose side lengths are equal to those integers.
Consider choosing the integers $3, 4, 10$.
Since $3 + 4 < 10$, then it is not possible to form a triangle whose side lengths are $3, 4, 10$.
Consider choosing the integers $3, 4, 13$.
Since $3 + 4 < 13$, then it is not possible to form a triangle whose side lengths are $3, 4, 13$.
Consider choosing the integers $3, 10, 13$.
Since $3 + 10 = 13$, then it is not possible to form a triangle whose side lengths are $3, 10, 13$.
However, the remaining possible choice of three side lengths, $4, 10, 13$, does satisfy the triangle inequality since $4 + 10 > 13$ and $4 + 13 > 10$ and $10 + 13 > 4$.
Thus, without using the value of n, there is exactly one way to choose three integers and form a triangle whose side lengths are equal to those integers.
This means that we need to determine values of n for which there are exactly three different ways to choose two of the integers $3, 4, 10, 13$ and form a triangle whose side lengths are equal to those two integers and n.
There are six possible ways to choose two integers from the list $3, 4, 10, 13$.
Thus for each value of n, the triangles we need to consider have side lengths: $3, 4, n$ or $3, 10, n$ or $3, 13, n$ or $4, 10, n$ or $4, 13, n$ or $10, 13, n$.
For each value of n, we need exactly three of these six triangles to satisfy the triangle inequality.
Next, we use the triangle inequality to determine the restrictions on n for each of the six possible groups of triangles.

In a triangle with side lengths $3, 4, n$, we get $3 + n > 4$ or $n > 1$, and $3 + 4 > n$ or $n < 7$, and $4 + n > 3$ or $n > -1$.
To satisfy all three inequalities, n must be greater than 1 and less than 7.
Thus, the possible values of n for which a triangle has side lengths $3, 4, n$ are $n = 2, 3, 4, 5, 6$.
Since n must be different from all other numbers in the list, then $n = 2, 5, 6$.

In a triangle with side lengths $3, 10, n$, we get $3 + n > 10$ or $n > 7$, and $3 + 10 > n$ or $n < 13$, and $10 + n > 3$ or $n > -7$.
To satisfy all three inequalities, n must be greater than 7 and less than 13.
Thus, the possible values of n for which a triangle has side lengths $3, 10, n$ are $n = 8, 9, 11, 12$ ($n \neq 10$ since 10 is in the list).

In a triangle with side lengths $3, 13, n$, we get $3 + n > 13$ or $n > 10$, and $3 + 13 > n$ or $n < 16$, and $13 + n > 3$ or $n > -10$.
To satisfy all three inequalities, n must be greater than 10 and less than 16.
Thus, the possible values of n for which a triangle has side lengths $3, 13, n$ are $n = 11, 12, 14, 15$ ($n \neq 13$ since 13 is in the list).

In a triangle with side lengths $4, 10, n$, we get $4 + n > 10$ or $n > 6$, and $4 + 10 > n$ or $n < 14$, and $10 + n > 4$ or $n > -6$.
To satisfy all three inequalities, n must be greater than 6 and less than 14.
Thus, the possible values of n for which a triangle has side lengths $4, 10, n$ are $n = 7, 8, 9, 11, 12$.

In a triangle with side lengths $4, 13, n$, we get $4 + n > 13$ or $n > 9$, and $4 + 13 > n$ or $n < 17$, and $13 + n > 4$ or $n > -9$.
To satisfy all three inequalities, n must be greater than 9 and less than 17.

Thus, the possible values of n for which a triangle has side lengths $4, 13, n$ are $n = 11, 12, 14, 15, 16$.

In a triangle with side lengths $10, 13, n$, we get $10+n > 13$ or $n > 3$, and $10+13 > n$ or $n < 23$, and $13 + n > 10$ or $n > -3$.
To satisfy all three inequalities, n must be greater than 3 and less than 23.
Thus, the possible values of n for which a triangle has side lengths $10, 13, n$ are $n = 5, 6, 7, 8, 9, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22$.

Recall that for each value of n, we need exactly three of the six triangles to satisfy the triangle inequality (the triangle with side lengths $4, 10, 13$ is the fourth).
Clearly for values of n less than 7, there are too few triangles, and similarly for values of n greater than 16, there are also too few triangles. (There are at most two triangles in each of these two cases.)
In the table below, we summarize our work by placing a checkmark if the triangle satisfies the triangle inequality and then counting the number of such triangles.

n	$(3, 4, n)$	$(3, 10, n)$	$(3, 13, n)$	$(4, 10, n)$	$(4, 13, n)$	$(10, 13, n)$	$(4, 10, 13)$	number of triangles
7				✓		✓	✓	3
8		✓		✓		✓	✓	4
9		✓		✓		✓	✓	4
11		✓	✓	✓	✓	✓	✓	6
12		✓	✓	✓	✓	✓	✓	6
14			✓		✓	✓	✓	4
15			✓		✓	✓	✓	4
16					✓	✓	✓	3

Therefore, there are exactly four different values of n that satisfy the given conditions, and the sum of these values of n is $8 + 9 + 14 + 15 = 46$.

ANSWER: (A)

The CENTRE for EDUCATION
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2022 Gauss Contests

(Grades 7 and 8)

Wednesday, May 18, 2022
(in North America and South America)

Thursday, May 19, 2022
(outside of North America and South America)

Solutions

Centre for Education in Mathematics and Computing Faculty and Staff

Ed Anderson
Jeff Anderson
Terry Bae
Jacquelene Bailey
Shane Bauman
Ersal Cahit
Diana Castañeda Santos
Sarah Chan
Ashely Congi
Serge D'Alessio
Fiona Dunbar
Mike Eden
Sandy Emms
Barry Ferguson
Steve Furino
Lucie Galinon
Robert Garbary
Rob Gleeson
Sandy Graham
Conrad Hewitt
Lisa Kabesh
Jenn Kelebuda
Carrie Knoll
Wesley Korir
Judith Koeller
Laura Kreuzer
Bev Marshman
Josh McDonald
Paul McGrath
Comfort Mintah
Jen Nelson
Ian Payne
J.P. Pretti
Alexandra Rideout
Nick Rollick
Kim Schnarr
Tucker Seabrook
Ashley Sorensen
Ian VanderBurgh
Troy Vasiga
Christine Vender
Heather Vo
Bonnie Yi

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Robert Wong, Edmonton, AB
Chris Wu, Ledbury Park E. and M.S., Toronto, ON
Lori Yee, William Dunbar P.S., Pickering, ON

Grade 7

1. Arranging the five answers along with 10 from smallest to largest, we get $1, 5, 8, 10, 13, 19$. Since 10 is 2 more than 8 and 10 is 3 less than 13, then 8 is the closest number to 10.

   Answer: (C)

2. Reading from the graph, the greatest number of hours that Gabe spent riding his bike is 4, and this occurred on Tuesday.

   Answer: (B)

3. Of the given answers, 0 is the only value of x that is less than 5.

   Answer: (B)

4. Since $18 + 5 = 23$, and $23 + 5 = 28$, and $28 + 5 = 33$, the next three terms in the sequence are $23, 28, 33$.

   Answer: (C)

5. The faces shown are labelled with $1, 3$ and 5 dots. Therefore, the other three faces are labelled with $2, 4$ and 6 dots. The total number of dots on the other three faces is $2 + 4 + 6 = 12$.

   Answer: (D)

6. Since $\angle ABC$ measures 90°, and $\angle ABC = 44^\circ + x^\circ$, then $x = 90 - 44 = 46$.

   Answer: (A)

7. The largest height of the singers in Saura's choir is 183.5 cm.
   The smallest height of the singers in Saura's choir is 141 cm.
   Thus, the range of their heights is 183.5 cm $-$ 141 cm $=$ 42.5 cm.

   Answer: (A)

8. Beginning at the origin $(0, 0)$, the point $(3, -4)$ is located right 3 units and down 4 units.
   In the diagram, the point $(3, -4)$ is labelled T.

   Answer: (E)

9. When Emily jumps for 75 seconds, she jumps for $60 + 15$ seconds.
   Jumping at the rate of 52 times in 60 seconds, Emily jumps $52 \div 4 = 13$ times in $60 \div 4 = 15$ seconds.
   Since Emily jumps 52 times in 60 seconds and 13 times in 15 seconds, then Emily jumps $52 + 13 = 65$ times in $60 + 15 = 75$ seconds.

   Answer: (C)

10. In \$1.00 worth of dimes, there are $\dfrac{\$1.00}{\$0.10} = 10$ dimes.
    In \$1.00 worth of quarters, there are $\dfrac{\$1.00}{\$0.25} = 4$ quarters.
    The jar contains 10 dimes and a total of $10 + 4 = 14$ coins.
    If Terry randomly removes one coin from the jar, the probability that it is a dime is $\dfrac{10}{14} = \dfrac{5}{7}$.

    Answer: (E)

11. Since 42 is an even number, then 2 is a factor of 42.
Since $42 = 2 \times 21$ and $21 = 3 \times 7$, then $42 = 2 \times 3 \times 7$.
Each of $2, 3$ and 7 is a prime number and each is a factor of 42.
Thus, the sum of the prime factors of 42 is $2 + 3 + 7 = 12$.
(We note that $1, 6, 14, 21$, and 42 are also factors of 42, however they are not prime numbers.)
ANSWER: (C)

12. $\triangle PQR$ is isosceles with $PQ = PR$, and so $\angle PRQ = \angle PQR$.
The sum of the angles in $\triangle PQR$ is 180°.
Since $\angle QPR = 70^\circ$, then the other two angles in this triangle add to $180^\circ - 70^\circ = 110^\circ$.
Since these two angles are equal, they each measure $110^\circ \div 2 = 55^\circ$, and so $x = 55$.
Since $QRST$ is a rectangle, each of its interior angles is a right angle, and so $y = 90$.
The value of $x + y$ is $55 + 90 = 145$.
ANSWER: (D)

13. A two-digit number has at least one digit that is a 4 if its tens digit is a 4 or if its ones digit is a 4.
There are 10 two-digit numbers whose tens digit is a 4.
These are $40, 41, 42, 43, 44, 45, 46, 47, 48$, and 49.
There are 9 two-digit numbers whose ones digit is a 4.
These are $14, 24, 34, 44, 54, 64, 74, 84, 94$, including 44 which was counted in our previous list.
The number of two-digit numbers that have at least one digit that is a 4 is $10 + 9 - 1 = 18$.
ANSWER: (C)

14. The side lengths of each of the three identical squares are equal.
The perimeter of $WXYZ$ is 56 m and is comprised of 8 such side lengths.
Thus, the length of each side of the three identical squares is $56 \text{ m} \div 8 = 7 \text{ m}$.
The area of each of the three identical squares is $7 \text{ m} \times 7 \text{ m} = 49 \text{ m}^2$.
Therefore, the area of $WXYZ$ is $3 \times 49 \text{ m}^2 = 147 \text{ m}^2$.
ANSWER: (B)

15. The first Wednesday of a month must occur on one of the first 7 days of a month, that is, the 1^{st}, 2^{nd}, 3^{rd}, 4^{th}, 5^{th}, 6^{th}, or 7^{th} of the month.
The second Wednesday of a month occurs 7 days following the first Wednesday of that month, and the third Wednesday of a month occurs 14 days following the first Wednesday of that month.
Adding 14 days to each of the possible dates for the first Wednesday of the month, we get that the third Wednesday of a month must occur on the 15^{th}, 16^{th}, 17^{th}, 18^{th}, 19^{th}, 20^{th}, or 21^{st} of that month.
Of the given answers, the public holiday cannot occur on the 22^{nd} of that month.
ANSWER: (B)

16. *Solution 1*
When a standard fair coin is tossed three times, there are 8 possible outcomes.
If H represents head and T represents tail, these 8 outcomes are: HHH, HHT, HTH, THH, HTT, THT, TTH, and TTT.
Of these 8 outcomes, there are exactly 2 whose outcomes are all the same (HHH and TTT).
Therefore, the probability that the three outcomes are all the same is $\frac{2}{8} = \frac{1}{4}$.

Solution 2

The three outcomes are all the same exactly when each of the three tosses is a head or when each of the three tosses is a tail.

When a coin is tossed, the probability that the coin shows a head is $\frac{1}{2}$ and the probability that the coin shows a tail is also $\frac{1}{2}$ (there are two possible outcomes and each is equally probable).

Thus, the probability that each of the three tosses is a head is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

Similarly, the probability that each of the three tosses is a tail is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

Therefore, the probability that the three outcomes are all the same is $\frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$.

ANSWER: (C)

17. If the value of P is less than 9, then $QR + PPP + PPP$ is at most $99 + 888 + 888 = 1875$.
Since the given sum is 2022, then the value of P cannot be less than 9 and thus must equal 9.
When $P = 9$, we get $QR + 999 + 999 = 2022$, and so $QR = 2022 - 999 - 999 = 24$.
Therefore $Q = 2$, $R = 4$, and so $P + Q + R = 9 + 2 + 4 = 15$.

ANSWER: (C)

18. We begin by recognizing that moving a block from Box A to Box B, and then moving the same block from Box B back to Box A has does not change the total mass in each box.
Since such a pair of moves increases the number of blocks that Jasmine moves from Box A to Box B, then all such pairs of moves will not occur in counting the fewest number of blocks that Jasmine could have moved from Box A to Box B.
(Similarly, moving a block from Box B to Box A, and then moving that same block from Box A back to Box B is a pair of moves that will not occur.)
That is, once a block has been moved from one box to another, that block will not be moved in the opposite direction between the two boxes.

Next, we consider what total masses of blocks can be moved from Box B to Box A.
Box B contains one 50 g block and three 10 g blocks, and so Jasmine can move the following total masses of blocks from Box B to Box A: 10 g, 20 g, 30 g, 50 g, 60 g, 70 g, and 80 g. (Can you see how to get each of these masses using the blocks in Box B and why no other masses are possible?)
If Jasmine moves 10 g from Box B to Box A, then she must move blocks whose total mass is 65 g + 10 g = 75 g from Box A to Box B so that Box A contains 65 g less than it did originally and Box B contains 65 g more than it did originally.
For each of the other masses that may be moved from Box B to Box A, we summarize the total mass of blocks that Jasmine must move from Box A to Box B. Note that in each case, the mass moved from Box A to Box B must be 65 g greater than the mass moved from Box B to Box A.

Mass moved from Box B to Box A	Mass moved from Box A to Box B
10 g	75 g
20 g	85 g
30 g	95 g
50 g	115 g
60 g	125 g
70 g	135 g
80 g	145 g

Next, we determine if it is possible for Jasmine to choose blocks from Box A whose total mass is given in the second column in the table above.
Recall that Box A originally contains one 100 g block, one 20 g block, and three 5 g blocks.
Is it possible for Jasmine to choose blocks from Box A whose total mass is exactly 75 g?
Since the 100 g block is too heavy, and the remaining blocks have a total mass that is less than 75 g, then it is not possible for Jasmine to choose blocks from Box A whose total mass is exactly 75 g.
In the table below, we determine which of the exact total masses Jasmine is able to move from Box A to Box B.
For those masses which are possible, we list the number of blocks that Jasmine must move from Box A to Box B.

Total mass moved from Box B to Box A	Total mass moved from Box A to Box B	Is it possible to move this mass from Box A to Box B?	Number of blocks moved from Box A to Box B
10 g	75 g	No	
20 g	85 g	No	
30 g	95 g	No	
50 g	115 g	Yes; one 100 g, three 5 g	4
60 g	125 g	Yes; one 100 g, one 20 g, one 5 g	3
70 g	135 g	Yes; one 100 g, one 20 g, three 5 g	5
80 g	145 g	No	

From the table above, the fewest number of blocks that Jasmine could have moved from Box A to Box B is 3. In this case, Jasmine moves a total mass of $1(100\text{ g}) + 1(20\text{ g}) + 1(5\text{ g}) = 125$ g from Box A to Box B and she moves a total mass of $1(50\text{ g}) + 1(10\text{ g}) = 60$ g from Box B to Box A.
This confirms that Box B contains 125 g – 60 g = 65 g more than it did originally (and hence Box A contains 65 g less), as required.

ANSWER: (A)

19. The original ratio of red candies to blue candies is $3 : 5$, and so the number of red candies was a positive integer multiple of 3, and the number of blue candies was the same positive integer multiple of 5.
For example, there could have been $3 \times 1 = 3$ red candies and $5 \times 1 = 5$ blue candies, or $3 \times 2 = 6$ red candies and $5 \times 2 = 10$ blue candies, or $3 \times 3 = 9$ red candies and $5 \times 3 = 15$ blue candies, and so on.
We continue these possibilities in the table below, and consider the number of red and blue candies and the resulting ratio after three blue candies are removed from the dish.

Possible numbers of red and blue candies originally	3 red, 5 blue	6 red, 10 blue	9 red, 15 blue	12 red, 20 blue	15 red, 25 blue	18 red, 30 blue
Numbers of red and blue candies after 3 blue are removed	3 red, 2 blue	6 red, 7 blue	9 red, 12 blue	12 red, 17 blue	15 red, 22 blue	18 red, 27 blue
New ratio of the number of red candies to blue	$3 : 2$	$6 : 7$	$9 : 12 = 3 : 4$	$12 : 17$	$15 : 22$	$18 : 27 = 2 : 3$

If the dish originally contained 18 red candies and 30 blue (note that $18 : 30 = 3 : 5$), then when three blue candies are removed, the ratio of the number of red candies to blue candies

becomes 18 : 27 which is equal to 2 : 3, as required.
Therefore, there were $30 - 18 = 12$ more blue candies than red candies in the dish before any candies were removed.

Answer: (B)

20. Let Anyu, Brad, Chi, and Diego be represented by A, B, C, and D, respectively, and so their original order is $ABCD$.
When rearranged, A is not in the 1^{st} position, and so there are exactly 3 cases to consider: A is in the 2^{nd} position, or A is in the 3^{rd} position, or A is in the 4^{th} position.
For each of these 3 cases, we count the number of ways to arrange B, C, and D.

<u>Case 1</u>: A is in the 2^{nd} position
Since A is in the 2^{nd} position, B can be in any of the other 3 positions ($1^{\text{st}}, 3^{\text{rd}}$ or 4^{th}).
If B is in the 1^{st} position, then there is exactly one possible rearrangement: $BADC$ (since C and D cannot be in the 3^{rd} and 4^{th} positions respectively).
If B is in the 3^{rd} position, then there is exactly one possible rearrangement: $DABC$ (since D cannot be in the 4^{th} position).
If B is in the 4^{th} position, then there is exactly one possible rearrangement: $CADB$ (since C cannot be in the 3^{rd} position).
Thus there are exactly 3 possible rearrangements when A is in the 2^{nd} position.

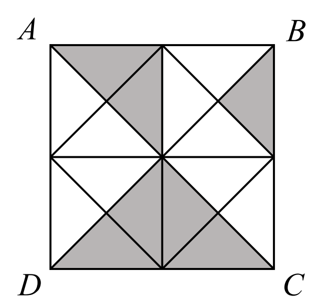
<u>Case 2</u>: A is in the 3^{rd} position
Since A is in the 3^{rd} position, C can be in any of the other 3 positions.
In a manner similar to Case 1, it can be shown that there are 3 possible rearrangements in this case: $CDAB$, $DCAB$, and $BDAC$.

<u>Case 3</u>: A is in the 4^{th} position
Since A is in the 4^{th} position, D can be in any of the other 3 positions.
Similarly, there are 3 possible rearrangements in this case: $DCBA$, $CDBA$, and $BCDA$.

So that each person is not in their original position, the four friends can rearrange themselves in $3 + 3 + 3 = 9$ different ways.
(Such a rearrangement of a list in which no element appears in its original position is called a *derangement*.)

Answer: (B)

21. We begin by constructing the three diagonals inside the smaller squares that were missing from the diagram given in the question, as shown.
The two diagonals inside each of these 4 smaller squares divide each smaller square into 4 identical triangles having equal area.
Thus, square $ABCD$ is divided into $4 \times 4 = 16$ such triangles.
Since 7 of these triangles are shaded, then the fraction of $ABCD$ that is shaded is $\frac{7}{16}$.



Answer: (C)

22. *Solution 1*
Since the sum of p, q, r, s and the sum of q, r, s, t are both equal to 35, and q, r, s are added in each sum, then $p = t$.
Similarly, since the sum of q, r, s, t and the sum of r, s, t, u are both equal to 35, and r, s, t are added in each sum, then $q = u$.

It can similarly be shown that $r = v$ and $s = w$.
Using the above observations, the sequence p, q, r, s, t, u, v, w can be written as p, q, r, s, p, q, r, s.
Since the sum of q and v is 14 and $r = v$, then the sum of q and r is 14.
Since the sum of p, q, r, s is 35 and the sum of q and r is 14, then the sum of p and s is $35 - 14 = 21$.
The value of p is as large as possible exactly when the value of s is as small as possible.
Since s is a positive integer, its smallest possible value is 1.
Therefore, the largest possible value of p is $21 - 1 = 20$.
(We note that $20, 4, 10, 1, 20, 4, 10, 1$ is an example of such a list.)

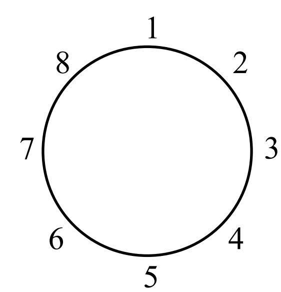
Solution 2
The sum of the values of each group of four consecutive letters is 35.
Thus, $p+q+r+s = 35$ and $t+u+v+w = 35$, and so $(p+q+r+s)+(t+u+v+w) = 35+35 = 70$.
Rearranging the sum of these eight letters, we get

$$p + q + r + s + t + u + v + w = (p + w) + (q + v) + (r + s + t + u) = 70$$

However, $r + s + t + u = 35$ (the sum of the values of four consecutive letters), and $q + v = 14$.
Substituting, we get $(p + w) + 14 + 35 = 70$, and so $p + w = 21$.
The value of p is as large as possible exactly when the value of w is as small as possible.
Since w is a positive integer, its smallest possible value is 1.
Substituting, we get $p + 1 = 21$, and so the largest possible value of p is 20.
(We note that $20, 12, 2, 1, 20, 12, 2, 1$ is an example of such a list.)

ANSWER: (C)

23. Katharina placed the 8 letters in a mixed-up order around the circle.
We number the position of L at the top the circle as 1, the next number moving clockwise from L position 2, and so on through to position 8, as shown.
Jaxon begins at position 1, writes down the letter L, and moves clockwise writing down every third letter that he has not yet written down.
Thus, the first 3 letters that Jaxon writes down are the letters in positions 1, 4 and 7.



Continuing to move clockwise around the circle, the next three positions at which letters have not yet been written down are positions 8, 2 and 3 (the letter in position 1 was already written down), and so Jaxon writes down the letter in position 3.
The next three positions at which letters have not yet been written down are 5, 6 and 8 (the letters in positions 4 and 7 were already written down), and so Jaxon writes down the letter in position 8.
At this point, Jaxon has written the letters in positions 1, 4, 7, 3, and 8.
The next three positions at which letters have not yet been written down are 2, 5 and 6 (the letters in positions 1, 3 and 4 were already written down), and so Jaxon writes down the letter in position 6.
At this point, only the letters in positions 2 and 5 have not been written.
Since Jaxon left off at position 6, he skips the letter at position 2, skips the letter at position 5, and then writes down the letter at position 2.
Finally, Jaxon writes the final letter at position 5.
Thus in order, Jaxon writes the letters in positions 1, 4, 7, 3, 8, 6, 2, and 5.

Since Jaxon's list is L, M, N, O, P, Q, R, S, then L is the letter in position 1 of Katharina's ordering, M is the letter in position 4, N is the letter in position 7, and so on.
Therefore, Katharina's clockwise order is L, R, O, M, S, Q, N, P.

ANSWER: (C)

24. A palindrome greater than 10 000 and less than 100 000 is a 5-digit positive integer of the form $abcba$, where a, b and c are digits and $a \neq 0$.
A positive integer is a multiple of 18 if it is a multiple of both 2 and 9 (and a positive integer that is a multiple of both 2 and 9 is a multiple of 18).
A positive integer is a multiple of 2 if it is even, and thus the digit a is equal to $2, 4, 6$ or 8 (recall $a \neq 0$).
A positive integer is a multiple of 9 exactly when the sum of its digits is a multiple of 9, and thus $a + b + c + b + a$ or $2a + 2b + c$ is a multiple of 9.
Next we consider four possible cases, one case for each of the possible values of a.

Case 1: $a = 2$
When $a = 2$, we require that $2a + 2b + c = 4 + 2b + c$ be a multiple of 9.
Since $4 + 2b + c \geq 4$, then the smallest possible multiple of 9 that $4 + 2b + c$ can equal is 9.
Since $b \leq 9$ and $c \leq 9$, then $4 + 2b + c$ is at most $4 + 2(9) + 9 = 31$.
Thus, $4 + 2b + c$ can equal $9, 18$ or 27, which gives $2b + c$ equal to $5, 14$ or 23 respectively.
Next, we determine the possible values of b and c so that $2b + c$ is equal to $5, 14$ or 23.

$2b + c = 5$	$2b + c = 14$	$2b + c = 23$
$b = 2, c = 1$	$b = 7, c = 0$	$b = 9, c = 5$
$b = 1, c = 3$	$b = 6, c = 2$	$b = 8, c = 7$
$b = 0, c = 5$	$b = 5, c = 4$	$b = 7, c = 9$
	$b = 4, c = 6$	
	$b = 3, c = 8$	

Thus when $a = 2$, there are $3 + 5 + 3 = 11$ such palindromes.

Case 2: $a = 4$
When $a = 4$, we require that $2a + 2b + c = 8 + 2b + c$ be a multiple of 9.
Since $8 + 2b + c \geq 8$, then the smallest possible multiple of 9 that $8 + 2b + c$ can equal is 9.
Since $b \leq 9$ and $c \leq 9$, then $8 + 2b + c$ is at most $8 + 2(9) + 9 = 35$.
Thus, $8 + 2b + c$ can equal $9, 18$ or 27, which gives $2b + c$ equal to $1, 10$ or 19 respectively.
Next, we determine the possible values of b and c so that $2b + c$ is equal to $1, 10$ or 19.

$2b + c = 1$	$2b + c = 10$	$2b + c = 19$
$b = 0, c = 1$	$b = 5, c = 0$	$b = 9, c = 1$
	$b = 4, c = 2$	$b = 8, c = 3$
	$b = 3, c = 4$	$b = 7, c = 5$
	$b = 2, c = 6$	$b = 6, c = 7$
	$b = 1, c = 8$	$b = 5, c = 9$

Thus when $a = 4$, there are $1 + 5 + 5 = 11$ such palindromes.

Case 3: $a = 6$
When $a = 6$, we require that $2a + 2b + c = 12 + 2b + c$ be a multiple of 9.
Since $12 + 2b + c \geq 12$ and $12 + 2b + c \leq 12 + 2(9) + 9 = 39$, then $12 + 2b + c$ can equal 18, 27 or 36, which gives $2b + c$ equal to 6, 15 or 24, respectively.

$2b+c=6$	$2b+c=15$	$2b+c=24$
$b=3,c=0$	$b=7,c=1$	$b=9,c=6$
$b=2,c=2$	$b=6,c=3$	$b=8,c=8$
$b=1,c=4$	$b=5,c=5$	
$b=0,c=6$	$b=4,c=7$	
	$b=3,c=9$	

Thus when $a=6$, there are $4+5+2=11$ such palindromes.

Case 4: $a=8$
When $a=8$, we require that $2a+2b+c=16+2b+c$ be a multiple of 9.
Since $16+2b+c \geq 16$ and $16+2b+c \leq 16+2(9)+9=43$, then $16+2b+c$ can equal 18, 27 or 36, which gives $2b+c$ equal to 2, 11 or 20, respectively.

$2b+c=2$	$2b+c=11$	$2b+c=20$
$b=1,c=0$	$b=5,c=1$	$b=9,c=2$
$b=0,c=2$	$b=4,c=3$	$b=8,c=4$
	$b=3,c=5$	$b=7,c=6$
	$b=2,c=7$	$b=6,c=8$
	$b=1,c=9$	

Thus when $a=8$, there are $2+5+4=11$ such palindromes.
Therefore, the number of palindromes that are greater than 10 000 and less than 100 000 and that are multiples of 18 is $11+11+11+11=44$.

ANSWER: (D)

25. After all exchanges there are 4 balls in each bag, and so if a bag contains exactly 3 different colours of balls, then it must contain exactly 2 balls of the same colour and 2 balls each having a colour that is different than all other balls in the bag.
Among the 8 balls in the two bags, there are 2 red balls and 2 black balls and each of the remaining balls has a colour that is different than all the other balls.
Thus after all exchanges, one bag must contain the 2 red balls and the other bag must contain the 2 black balls.
Since Becca's bag initially contains both black balls, and Becca moves only 1 ball from her bag to Arjun's bag, it is not possible for Arjun's bag to contain the 2 black balls after all exchanges.
This tells us that if each bag contains exactly 3 different colours of balls after all exchanges, then Arjun's bag contains the 2 red balls and Becca's bag contains the 2 black balls.
We let the first letter of each colour represent a ball of that colour.
Then initially, Arjun's bag contains $RRGYV$ and Becca's bag contains BBO.
For the first ball chosen to be moved, there are exactly two cases to consider:
Case 1: The first ball moved from Arjun's bag to Becca's bag is R, or
Case 2: The first ball moved from Arjun's bag to Becca's bag is not R (thus it is G, Y or V).
We begin with Case 1 and determine the probability that after all exchanges, each bag contains exactly 3 different colours of balls.

Case 1: The first ball moved from Arjun's bag to Becca's bag is R

Since Arjun's bag initially contains 5 balls, 2 of which are R, the probability that R is chosen as the first ball to move is $\frac{2}{5}$.
After R is moved from Arjun's bag to Becca's, Arjun's bag contains $RGYV$ and Becca's bag

contains $BBOR$.
Since Arjun's bag must contain both R's after all exchanges, and Becca moves only 1 ball from her bag to Arjun's, then the next ball chosen to move must be R.
Becca's bag contains 4 balls, 1 of which is R, and so in this case the probability that R is chosen as the second ball to move is $\frac{1}{4}$.
After the first two balls are moved, Arjun's bag contains $RRGYV$ and Becca's bag contains BBO.
Finally, a ball is chosen from Arjun's bag and moved to Becca's.
Since Arjun's bag must contain both R's after all exchanges, then the ball that is chosen must be G, Y or V which occurs with probability $\frac{3}{5}$.
If the first ball moved from Arjun's bag to Becca's bag is R, then the probability that each bag contains exactly 3 different colours of balls after all exchanges is given by the product of the individual probabilities of choosing each of the 3 balls or $\frac{2}{5} \times \frac{1}{4} \times \frac{3}{5} = \frac{6}{100} = \frac{3}{50}$.

Case 2: The first ball moved from Arjun's bag to Becca's bag is G, Y or V

Since Arjun's bag initially contains 5 balls, the probability that G, Y or V is chosen as the first ball to move is $\frac{3}{5}$.
After one of G, Y or V is moved from Arjun's bag to Becca's, Arjun's bag contains RR and two of G, Y and V, and Becca's bag contains BBO and one of G, Y or V.
For the second ball chosen to be moved, there are exactly two cases to consider and so we split Case 2 into these two separate cases.
Case 2a: The first ball moved from Arjun's bag to Becca's bag is G, Y or V, and the ball moved from Becca's bag to Arjun's bag is B, or
Case 2b: The first ball moved from Arjun's bag to Becca's bag is G, Y or V, and the ball moved from Becca's bag to Arjun's bag is not B (that is, it is O or the first ball moved).
We begin with Case 2a and determine the probability that after all exchanges, each bag contains exactly 3 different colours of balls.

Case 2a: The first ball moved from Arjun's bag to Becca's bag is G, Y or V, and the ball moved from Becca's bag to Arjun's bag is B
We previously determined that the probability that G, Y or V is chosen as the first ball to move is $\frac{3}{5}$.
At this point, Becca's bag contains 4 balls, 2 of which are B's, and so the probability that B is chosen is $\frac{2}{4} = \frac{1}{2}$.
After B is moved from Becca's bag to Arjun's, Arjun's bag contains RRB and two of G, Y and V and Becca's bag contains BO and one of G, Y or V.
Since Becca's bag must contain both B's after all exchanges, then the final ball chosen to move must be B.
Arjun's bag contains 5 balls, 1 of which is B, and so in this case the probability that B is chosen as the final ball to move is $\frac{1}{5}$.
If the first ball moved from Arjun's bag to Becca's bag is G, Y or V, and the ball moved from Becca's bag to Arjun's bag is B, then the probability that each bag contains exactly 3 different colours of balls after all exchanges is given by the product of the individual probabilities of choosing each of the 3 balls or $\frac{3}{5} \times \frac{1}{2} \times \frac{1}{5} = \frac{3}{50}$.

Case 2b: The first ball moved from Arjun's bag to Becca's bag is G, Y or V, and the ball moved from Becca's bag to Arjun's bag is O or the first ball moved (G, Y or V)
The probability that G, Y or V is chosen as the first ball to move is $\frac{3}{5}$.
At this point, Becca's bag contains 4 balls, 2 of which are B, and so the probability that B is

not chosen is $\frac{2}{4} = \frac{1}{2}$.
After this ball (O, G, Y or V) is moved from Becca's bag to Arjun's, Arjun's bag contains RR and three of O, G, Y and V and Becca's bag contains two B's and one of O, G, Y or V.
Since Arjun's bag must contain both R's after all exchanges, then the final ball chosen to move must be one of the 3 balls in Arjun's bag that is not R.
Arjun's bag contains 5 balls, 2 of which are R, and so in this case the probability that R is not chosen as the final ball to move is $\frac{3}{5}$.
If the first ball moved from Arjun's bag to Becca's bag is G, Y or V, and the ball moved from Becca's bag to Arjun's bag is not B, then the probability that each bag contains exactly 3 different colours of balls after all exchanges is given by the product of the individual probabilities of choosing each of the 3 balls or $\frac{3}{5} \times \frac{1}{2} \times \frac{3}{5} = \frac{9}{50}$.

Finally, the probability that each bag contains exactly 3 different colours of balls after all exchanges is $\frac{3}{50} + \frac{3}{50} + \frac{9}{50} = \frac{15}{50} = \frac{3}{10}$.

Answer: (A)

Grade 8

1. The regular pentagon shown has 5 sides, each with length 2 cm.
   The perimeter of the pentagon is 5×2 cm $= 10$ cm.

   ANSWER: (E)

2. The faces shown are labelled with $1, 3$ and 5 dots. Therefore, the other three faces are labelled with $2, 4$ and 6 dots. The total number of dots on the other three faces is $2 + 4 + 6 = 12$.

   ANSWER: (D)

3. If the number is n, then "a number increased by five" is best represented by the expression $n + 5$. The equation that best represents "a number increased by five equals 15" is $n + 5 = 15$.

   ANSWER: (C)

4. Reading from the graph, the approximate number of bobbleheads sold in the years 2016, 2017, 2018, 2019, 2020, and 2021 were 20, 35, 40, 38, 60, and 75, respectively.
   Beginning with 2016 and 2017, the increases (or decreases) in the sale of bobbleheads between consecutive years are approximately $35 - 20 = 15$, $40 - 35 = 5$, $38 - 40 = -2$ (a decrease of 2), $60 - 38 = 22$, and $75 - 60 = 15$.
   Thus the greatest increase in the sale of bobbleheads was approximately 22 and this occurred between 2019 and 2020.

   ANSWER: (D)

5. Continuing to count down by 11, we get

   $$72, 61, 50, 39, 28, 17, 6, -5, \ldots$$

   The last number that Aryana will count that is greater than 0 is 6.

   ANSWER: (C)

6. Since $\angle ABC = 90^\circ$, then $44 + x + x = 90$ or $2x = 90 - 44$, and so $2x = 46$ or $x = 23$.

   ANSWER: (B)

7. Since $\frac{5}{4} = 1\frac{1}{4}$ and $1^2 = 1$, then each of the choices (A) -1, (B) $\frac{5}{4}$ and (C) 1^2 is at least 1 away from 0.
   Since $-\frac{4}{5} = -0.8$ and -0.8 is closer to 0 than 0.9 is to 0, then $-\frac{4}{5}$ is the closest value to zero.

   ANSWER: (D)

8. *Solution 1*
   The value of 4 quarters is \$1.00.
   Thus, the value of $4 \times 100 = 400$ quarters is $\$1.00 \times 100 = \$100.00$.
   Since the jar initially contains 267 quarters, then $400 - 267 = 133$ quarters must be added to the jar for the total value of the quarters to equal \$100.00.

   Solution 2
   The value of 267 quarters is $\$0.25 \times 267 = \$66.75$ and so $\$100.00 - \$66.75 = \$33.25$ must be added to the jar so that it contains \$100.00.
   Thus, the number of quarters that must be added to the jar is $\dfrac{\$33.25}{\$0.25} = 133$.

   ANSWER: (D)

9. Each package of greeting cards comes with $10 - 8 = 2$ more envelopes than cards.
Thus, 3 packages of greeting cards comes with $3 \times 2 = 6$ more envelopes than cards, and 4 packages of greeting cards comes with $4 \times 2 = 8$ more envelopes than cards.
Kirra began with 7 cards and no envelopes.
To have more envelopes than cards, Kirra must buy enough packages to make up the difference between the number of cards and the number of envelopes (which is 7).
Thus, the smallest number of packages that Kirra must buy is 4.
Note: We can check that if Kirra buys 3 packages she has $3 \times 8 + 7 = 31$ cards and $3 \times 10 = 30$ envelopes, and thus fewer envelopes than cards. However, if she buys 4 packages, she has $4 \times 8 + 7 = 39$ cards and $4 \times 10 = 40$ envelopes, and thus more envelopes than cards, as required.

ANSWER: (B)

10. The horizontal distance between the point (a, b) and the y-axis is a units, and the horizontal distance between the point (c, d) and the y-axis is c units.
Since (a, b) is horizontally farther from the y-axis than (c, d) is from the y-axis, then $a > c$ and so statement (E) is true.

Let us consider why each of the other statements is false.
The points (a, b) and (c, d) each lie above the x-axis and thus $b > 0$ and $d > 0$.
The point (e, f) lies below the x-axis and thus $f < 0$.
Since $b > 0$ and $f < 0$, then $b > f$ and so statement (C) is false.
Further, the vertical distance between the point (a, b) and the x-axis is b units, and the vertical distance between the point (c, d) and the x-axis is d units.
Since (a, b) is vertically farther from the x-axis than (c, d) is from the x-axis, then $b > d$ and so statement (B) is false.

Similarly, the points (a, b) and (c, d) each lie to the right of the y-axis and thus $a > 0$ and $c > 0$.
The point (e, f) lies to the left of the y-axis and thus $e < 0$.
Since $a > 0$ and $e < 0$, then $a > e$ and so statement (D) is false.
Since $c > 0$ and $e < 0$, then $c > e$ and so statement (A) is also false.

ANSWER: (E)

11. In the sequence, the letters of the alphabet repeat in blocks of 26 letters.
Thus, 10 of these blocks gives a sequence that contains $10 \times 26 = 260$ letters.
Each block of 26 letters ends with the letter Z, and so the 260^{th} letter is a Z.
Moving backward in the sequence from this Z, we get that the 259^{th} letter is a Y, and the 258^{th} letter is an X.

ANSWER: (C)

12. The first Wednesday of a month must occur on one of the first 7 days of a month, that is, the 1^{st}, 2^{nd}, 3^{rd}, 4^{th}, 5^{th}, 6^{th}, or 7^{th} of the month.
The second Wednesday of a month occurs 7 days following the first Wednesday of that month, and the third Wednesday of a month occurs 14 days following the first Wednesday of that month.
Adding 14 days to each of the possible dates for the first Wednesday of the month, we get that the third Wednesday of a month must occur on the 15^{th}, 16^{th}, 17^{th}, 18^{th}, 19^{th}, 20^{th}, or 21^{st} of that month.
Of the given answers, the public holiday cannot occur on the 22^{nd} of that month.

ANSWER: (B)

13. The probability that the arrow stops on the largest section is 50% or $\frac{1}{2}$.
The probability that it stops on the next largest section is 1 in 3 or $\frac{1}{3}$.
Thus, the probability that the arrow stops on the smallest section is $1 - \frac{1}{2} - \frac{1}{3} = \frac{6}{6} - \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$.

ANSWER: (C)

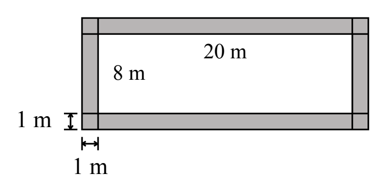
14. A positive number is divisible by both 3 and 4 if it is divisible by 12 (and a positive number that is divisible by 12 is divisible by both 3 and 4).
The positive two-digit numbers that are divisible by 12 (and thus 3 and 4) are $12, 24, 36, 48, 60$, $72, 84$, and 96.
Of these, $60, 72, 84$, and 96 have a tens digit that is greater than its ones digit.
Thus, there are 4 positive two-digit numbers that satisfy the given property.

ANSWER: (A)

15. *Solution 1*
The area of the walkway is equal to the area of the pool subtracted from the combined area of the pool and walkway.
That is, if the area of the walkway is A_w and the area of the pool is A_p, then $A_w = (A_w + A_p) - A_p$.
The combined area of the pool and walkway is equal to the area of the rectangle with length 22 m and width 10 m.
The length, 22 m, is given by the 20 m pool length plus the 1 m wide walkway on each end of the pool.
Similarly, the 10 m width is given by the 8 m pool width plus the 1 m wide walkway on each side of the pool.
Thus the area of the walkway is

$$A_w = (A_w + A_p) - A_p = (22 \text{ m} \times 10 \text{ m}) - (20 \text{ m} \times 8 \text{ m}) = 220 \text{ m}^2 - 160 \text{ m}^2 = 60 \text{ m}^2$$

Solution 2
We begin by extending each side of the pool 1 m in each direction.
This divides the area of the walkway into four 1 m by 1 m squares (in the corners), and two 20 m by 1 m rectangles (along the 20 m sides of the pool), and two 8 m by 1 m rectangles (along the 8 m sides of the pool), as shown.



Thus the area of the walkway is

$$4 \times (1 \text{ m} \times 1 \text{ m}) + 2 \times (20 \text{ m} \times 1 \text{ m}) + 2 \times (8 \text{ m} \times 1 \text{ m}) = 4 \text{ m}^2 + 40 \text{ m}^2 + 16 \text{ m}^2 = 60 \text{ m}^2$$

ANSWER: (B)

16. Reading from the Venn diagram, 5 students participate in both music and sports, 15 students participate in music (and not sports), and 20 students participate in sports (and not music).
Thus, there are $5 + 15 + 20 = 40$ students that participate in music or sports or both, and so there are $50 - 40 = 10$ students that do not participate in music and do not participate in sports.
Of the 50 students, $\frac{10}{50} \times 100\% = 20\%$ do not participate in music and do not participate in sports.

ANSWER: (C)

17. If the number of golf balls in Bin G is x, then the number of golf balls in Bin F is $\frac{2}{3}x$.
In this case, the total number of golf balls is $x + \frac{2}{3}x = \frac{3}{3}x + \frac{2}{3}x = \frac{5}{3}x$, and so $\frac{5}{3}x = 150$.
Multiplying both sides of this equation by 3, we get $5x = 150 \times 3 = 450$.
Dividing both sides by 5, we get $x = \frac{450}{5} = 90$ and so there are 90 golf balls in Bin G.
The number of golf balls in Bin F is $150 - 90 = 60$, and so there are $90 - 60 = 30$ fewer golf balls in Bin F than in Bin G.
(Alternately, we may have begun by letting there be $3x$ golf balls in Bin G and $2x$ golf balls in Bin F.)

Answer: (B)

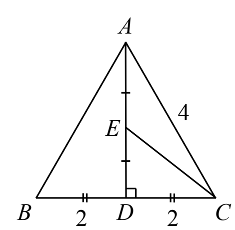
18. Figure 1 is formed with 7 squares.
Figure 2 is formed with $5 + 7$ squares.
Figure 3 is formed with $5 + 5 + 7 = 2 \times 5 + 7$ squares.
Figure 4 will be formed with $5 + 5 + 5 + 7 = 3 \times 5 + 7$ squares.
Figure 5 will be formed with $5 + 5 + 5 + 5 + 7 = 4 \times 5 + 7$ squares.
Thus, the number of groups of 5 squares needed to help form each figure is increasing by 1.
Also, in each case the number of groups of 5 squares needed is one less than the Figure number.
For example, Figure 6 will be formed with 5 groups of 5 squares plus 7 additional squares.
In general, we can say that Figure N will be formed with $N - 1$ groups of 5 squares, plus 7 additional squares.
Since $2022 = 403 \times 5 + 7$, the figure with 2022 squares has 403 groups of 5 squares, plus 7 additional squares.
Thus, $N - 1 = 403$ and so $N = 404$. The number of the figure that has 2022 squares is 404.

Answer: (C)

19. There are 60 minutes in an hour, and so the number of minutes between 7 a.m. and 11 a.m. is $4 \times 60 = 240$.
Since Mateo stopped for a 40 minute break, he drove for $240 - 40 = 200$ minutes.
Thus, the average speed for Mateo's 300 km trip was $\dfrac{300 \text{ km}}{200 \text{ minutes}} = 1.5$ km/min.
Since there are 60 minutes in an hour, if Mateo averaged 1.5 km per minute, then he averaged 1.5×60 km per hour or 90 km/h.

Answer: (C)

20. Since $\triangle ABC$ is equilateral and has sides of length 4, then $AB = BC = AC = 4$.
The midpoint of BC is D and so $BD = CD = 2$.
The midpoint of AD is E and so $AE = ED$.
Since $AB = AC$ and D is the midpoint of BC, then AD is perpendicular to BC, as shown.



Triangle ADC is a right-angled triangle, and so by the Pythagorean Theorem, we get $(AC)^2 = (AD)^2 + (DC)^2$ or $4^2 = (AD)^2 + 2^2$, and so $(AD)^2 = 16 - 4 = 12$.
Similarly, $\triangle EDC$ is right-angled, and so by the Pythagorean Theorem, we get $(EC)^2 = (ED)^2 + (DC)^2$ or $(EC)^2 = (ED)^2 + 2^2$.
Since $ED = \frac{1}{2}AD$, then $(ED)^2 = \frac{1}{2}AD \times \frac{1}{2}AD$ or $(ED)^2 = \frac{1}{4}(AD)^2$.
Since $AD^2 = 12$, then $(ED)^2 = \frac{1}{4} \times 12 = 3$.
Substituting, we get $(EC)^2 = 3 + 2^2$, and so $(EC)^2 = 7$.

Answer: (A)

21. A perfect square is a number that can be expressed as the product of two equal integers.
By this definition, 0 is a perfect square since $0 \times 0 = 0$.
Since the product of 0 and every positive integer is 0, then every positive integer is a factor of 0, and so 0 has an infinite number of positive factors.
The next three smallest perfect squares are $1^2 = 1$, $2^2 = 4$ and $3^2 = 9$.
Each of these has at most three positive factors.
The next largest perfect square is $4^2 = 16$.
The positive factors of 16 are 1, 2, 4, 8, and 16, and so 16 is a perfect square that has exactly five positive factors.
The remaining perfect squares that are less than 100 are 25, 36, 49, 64, and 81.
Both 25 and 49 each have exactly three positive factors.
The positive factors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, and 36.
The positive factors of 64 are 1, 2, 4, 8, 16, 32, and 64.
Thus, both 36 and 64 each have more than five positive factors.
Finally, the positive factors of 81 are 1, 3, 9, 27, and 81.
The two perfect squares that are less than 100 and that have exactly five positive factors are 16 and 81, and their sum is $16 + 81 = 97$.

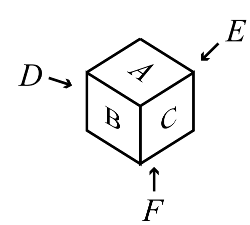
Answer: (E)

22. The sum of the values of each group of three consecutive letters is 35.
Thus, $r + s + t = 35$ and $s + t + u = 35$.
Since each of these equations is equal to 35, then the left sides of the two equations are equal to each other.
That is, $r + s + t = s + t + u$ and since $s + t$ is common to both sides of this equation, then $r = u$.
Since $q + u = 15$ and $r = u$, then $q + r = 15$.
Since $p + q + r = 35$ and $q + r = 15$, then $p = 35 - 15 = 20$.
Finally, we get

$$p + q + r + s + t + u + v = p + (q + r + s) + (t + u + v) = 20 + 35 + 35 = 90$$

Answer: (D)

23. Consider folding the net into the cube and positioning the cube with the face labelled F down (on the bottom), and so the face labelled A is on top and the four remaining vertical faces are as shown.



Beginning at A, the ant has 4 choices for which face to visit next.
That is, the ant can walk from A to any of the vertical faces, B, C, D, or E.
From each of these vertical faces, the ant can walk to F (the bottom face), or the ant can walk to an adjacent vertical face.
We call these two possibilities Case 1 and Case 2, and for each case we consider the number of possible orders in which the ant can visit the faces.

Case 1: The ant's 2nd move is to F, the bottom face

We begin by recognizing that there is only 1 choice for the ant's 2nd move in this case.
From F, the ant's 3rd move must be back to a vertical face.
The ant cannot return to the vertical face that it has already visited.
Also, the ant cannot move to the vertical face that is opposite the vertical face it has already

visited. Why?
Consider for example that the ant visits, in order, A, B, F.
If the ant walks to E (the face opposite B) on its 3rd move, then its 4th move must be to C or to D (since it has visited the other four faces).
However, once at C or D, the ant is "trapped" since it has already visited all four adjacent faces, and thus cannot get to the sixth face.
Thus from F, the ant's 3rd move must be to a vertical face that is not the face already visited, and is not the face opposite the face already visited, and so there are 2 choices for the next move.
From this vertical face, the ant must walk to another vertical face (since it has already visited A and F).
One such adjacent face has already been visited and the other has not, and so the ant's 1 choice is for its 4th move to be to the adjacent vertical face it has not visited.
For example, if the order is A, B, F, C, then the ant must move to E next.
Finally, the ant's final move must be to the adjacent vertical face that it has not visited.
Summarizing Case 1, there are 4 choices for the first move from A to one of the vertical faces, 1 choice for the 2nd move (to F), 2 choices for the 3rd move, and 1 choice for each of the last two moves, and so there are $4 \times 1 \times 2 \times 1 \times 1 = 8$ possible orders.

Case 2: The ant's 2nd move is to an adjacent vertical face

There are two vertical faces adjacent to each vertical face, and so the ant has 2 choices for its 2nd move.
For its 3rd move, the ant can walk to the bottom face F, or the ant can walk to the adjacent face that has not been visited.
We call the first of these Case 2a and the second Case 2b.

Case 2a: The ant's 3rd move is to F

We begin by recognizing that there is only 1 choice for the ant's 3rd move in this case.
From F, the ant's 4th move must be back to a vertical face.
The ant cannot return to either of the two vertical faces that it has already visited, and so there are 2 choices for the ant's 4th move.
For example, if the order is A, B, C, F, the ant's 4th move can be to D or to E.
The ant's final move is to the final vertical face and thus there is only 1 choice.
Summarizing Case 2a, there are 4 choices for the first move from A to one of the vertical faces, 2 choices for the 2nd move (to an adjacent vertical face), 1 choice for the 3rd move (to F), 2 choices for the 4th move, and 1 choice for the last move, and so there are $4 \times 2 \times 1 \times 2 \times 1 = 16$ possible orders.

Case 2b: The ant's 3rd move is to the adjacent face that has not been visited

We begin by recognizing that there is only 1 choice for the ant's 3rd move in this case.
At this point the ant has visited three vertical faces.
The ant's 4th move can be to F or to the final vertical face.
That is, the ant has 2 choices for its 4th move.
If the ant's 4th move is to F, then its last move is to the remaining vertical face.
If the ant's 4th move is to the final vertical face, then its last move is to F.
That is, once the ant chooses its 4th move, it has only 1 choice for its final move.
Summarizing Case 2b, there are 4 choices for the first move from A to one of the

vertical faces, 2 choices for the 2nd move (to an adjacent vertical face), 1 choice for the 3rd move (to the adjacent vertical face), 2 choices for the 4th move (to F or to the final vertical face), and 1 choice for the last move, and so there are $4 \times 2 \times 1 \times 2 \times 1 = 16$ possible orders.

Thus, if the ant starts at A and visits each face exactly once, there are $8 + 16 + 16 = 40$ possible orders.

Answer: (E)

24. *Solution 1*
We begin by recognizing that numbers with the given property cannot have two digits that are zero. Can you see why?
Thus, numbers with this property have exactly one zero or they have no zeros.
We consider each of these two cases separately.

Case 1: Suppose the number has exactly one digit that is a zero.
Each of the numbers greater than 100 and less than 999 is a three-digit number and so in this case, the number also has two non-zero digits.
Since one of the digits is equal to the sum of the other two digits and one of the digits is zero, then the two non-zero digits must be equal to one another.
The two non-zero digits can equal any integer from 1 to 9, and thus there are 9 possible values for the non-zero digits.
For each of these 9 possibilities, the zero digit can be the second digit in the number or it can be the third digit (the first digit cannot be zero).
That is, there are 9 possible values for the non-zero digits and 2 ways to arrange the three digits, and thus $9 \times 2 = 18$ numbers of this form satisfy the given property.
For example, numbers of this form are 101 and 110, 202 and 220, and so on.

Case 2: Suppose the number has no digits that are equal to zero.
Let the three digits of the number be a, b and c, arranged in some order.
Assume that a is the largest digit, and so $a = b + c$.
If $a = b$, then $c = 0$ which contradicts our assumption that no digit is equal to 0.
Similarly, if $a = c$, then $b = 0$ and the same contradiction arises.
Thus $a > b$ and $a > c$.
If $a = 1$, then $b = c = 0$ (since $a > b$ and $a > c$), but then $a \neq b + c$.
Therefore, a is at least 2.
If $a = 2$, then $b = c = 1$ and these are the only possible values for b and c when $a = 2$.
In this case, the 3 ways to arrange these digits give the numbers 112, 121 and 211, each with the desired property.
If $a = 3$, then b and c equal 1 and 2, in some order.
In this case, the 6 ways to arrange these digits give the numbers 123, 132, 213, 231, 312, and 321, each with the desired property.
From these cases, we recognize that if $b = c$, then there are 3 possible arrangements of the digits.
However, if the three digits are different from one another, then there are 3 choices for the first digit, 2 choices for second digit and 1 choice for the third digit, and thus $3 \times 2 \times 1 = 6$ ways to arrange the digits.
We consider all possible values of a, b, c and count the arrangements of these digits in the table below.

Values for a	Values for b and c with the number of arrangements in brackets [ ]				Total number of arrangements
2	1,1 [3]				3
3	$1,2$ [6]				6
4	$1,3$ [6]	$2,2$ [3]			9
5	$1,4$ [6]	$2,3$ [6]			12
6	$1,5$ [6]	$2,4$ [6]	$3,3$ [3]		15
7	$1,6$ [6]	$2,5$ [6]	$3,4$ [6]		18
8	$1,7$ [6]	$2,6$ [6]	$3,5$ [6]	$4,4$ [3]	21
9	$1,8$ [6]	$2,7$ [6]	$3,6$ [6]	$4,5$ [6]	24

Thus, there are $18 + 3 + 6 + 9 + 12 + 15 + 18 + 21 + 24 = 126$ numbers that satisfy the given property.

Solution 2
We begin by recognizing that numbers with the given property cannot have three equal digits. Can you see why?
Thus, numbers with this property have exactly two digits that are equal or all three digits are different.
We consider each of these two cases separately.

Case 1: Suppose the number has exactly two digits that are equal.
The two equal digits cannot be 0 since then the third digit would also be 0.
If for example the two equal digits are 1s, there are two possibilities for the third digit.
The third digit can be 2 (since $1 + 1 = 2$) or the third digit can be 0 (since $1 + 0 = 1$).
In the cases for which the third digit is equal to the sum of the two equal digits, the equal digits can be $1, 2, 3$, or 4 and the third digit is $2, 4, 6, 8$, respectively.
(We note that the equal digits cannot be greater than 4 since their sum is greater than 9.)
For each of these 4 possibilities, there are 3 ways to arrange the digits.
For example, when the equal digits are 1s and the third digit is 2, the numbers 112, 121 and 211 have the desired property.
Thus, there are $4 \times 3 = 12$ such numbers for which the third digit is equal to the sum of the two equal digits.
In the cases for which two of the digits are equal and the third digit is 0, the equal digits can be any integer from 1 to 9 inclusive.
For each of these 9 possibilities, there are 2 ways to arrange the digits.
For example, when the equal digits are 1s and the third digit is 0, the numbers 101 and 110 have the desired property.
Thus, there are $9 \times 2 = 18$ numbers which have two equal digits and the third digit is 0.
In total, there are $12 + 18 = 30$ numbers which have two equal digits and satisfy the given property.

Case 2: Suppose all three digits are different from one another.
Let the three digits of the number be a, b and c, arranged in some order with $a > b > c$.
Since a is the largest digit, then $a = b + c$.
If $a = 1$, then $b = c = 0$ (since $a > b$ and $a > c$), but this is not possible since $b > c$.
Similarly, if $a = 2$, then $b = 1$ and $c = 0$, however these digits do not satisfy the given property.
Therefore, a is at least 3.
If $a = 3$, then $b = 2$ and $c = 1$.

In this case, the 6 ways to arrange these digits give the numbers 123, 132, 213, 231, 312, 321 and each has the desired property.
We consider all possible values of a, b, c in the table below.

Values for a	Values for b, c			
3	$2, 1$			
4	$3, 1$			
5	$4, 1$	$3, 2$		
6	$5, 1$	$4, 2$		
7	$6, 1$	$5, 2$	$4, 3$	
8	$7, 1$	$6, 2$	$5, 3$	
9	$8, 1$	$7, 2$	$6, 3$	$5, 4$

For each of these 16 possibilities in the table above, there are 6 ways to arrange the three digits, and so there are $16 \times 6 = 96$ such numbers.
Thus, there are a total of $30 + 96 = 126$ numbers that satisfy the given property.

ANSWER: (B)

25. Of the 4200 samples to test, let the number of samples that contain blueberry be x.
Since each sample either contains blueberry or it does not, then there are $4200 - x$ samples that do not contain blueberry.
Student A reports correctly on 90% of the x samples that contain blueberry.
Thus, Student A reports $\frac{90}{100}x$ of these samples contain blueberry.
Student A reports correctly on 88% of the $4200 - x$ samples not containing blueberry and thus incorrectly reports that $100\% - 88\% = 12\%$ of these samples contain blueberry.
(That is, when a student is wrong when reporting "no blueberry", it means that there is blueberry.)
Therefore, Student A reports $\frac{12}{100}(4200 - x)$ of these samples contain blueberry.
In total, Student A reports that $\frac{90}{100}x + \frac{12}{100}(4200 - x)$ samples contain blueberry.
Similarly, Student B reports that $\frac{98}{100}x + \frac{14}{100}(4200 - x)$ samples contain blueberry, and Student C reports that $\frac{2m}{100}x + \left(\frac{100-4m}{100}\right)(4200 - x)$ samples contain blueberry.
Student B reports 315 more samples as containing blueberry than Student A, and so

$$\left(\tfrac{98}{100}x + \tfrac{14}{100}(4200 - x)\right) - \left(\tfrac{90}{100}x + \tfrac{12}{100}(4200 - x)\right) = 315$$

Clearing fractions by multiplying by 100, the equation becomes

$$98x + 14(4200 - x) - (90x + 12(4200 - x)) = 31\,500$$

Solving this equation, we get

$$\begin{aligned} 98x + 14(4200 - x) - (90x + 12(4200 - x)) &= 31\,500 \\ 98x + 58\,800 - 14x - (90x + 50\,400 - 12x) &= 31\,500 \\ 98x + 58\,800 - 14x - 90x - 50\,400 + 12x &= 31\,500 \\ 6x + 8400 &= 31\,500 \\ 6x &= 23\,100 \\ x &= 3850 \end{aligned}$$

Thus, there are 3850 samples that contain blueberry and $4200 - 3850 = 350$ samples that do not contain blueberry.
Together, the three students report

$$\left(\tfrac{98}{100}(3850) + \tfrac{14}{100}(350)\right) + \left(\tfrac{90}{100}(3850) + \tfrac{12}{100}(350)\right) + \left(\tfrac{2m}{100}(3850) + \left(\tfrac{100-4m}{100}\right)(350)\right)$$

samples as containing blueberry.
Simplifying this expression, we get

$$\begin{aligned} &\left(\tfrac{98}{100}(3850) + \tfrac{14}{100}(350)\right) + \left(\tfrac{90}{100}(3850) + \tfrac{12}{100}(350)\right) + \left(\tfrac{2m}{100}(3850) + \left(\tfrac{100-4m}{100}\right)(350)\right) \\ = \;& 3773 + 49 + 3465 + 42 + 77m + 350 - 14m \\ = \;& 7679 + 63m \end{aligned}$$

For some positive integers m, together the three students report $7679 + 63m$ samples as containing blueberry.
If $7679 + 63m$ is greater than 8000, then $63m$ is greater than $8000 - 7679 = 321$ and so m is greater than $\frac{321}{63} \approx 5.09$.
Since m is a positive integer, then m is greater than or equal to 6.
Similarly, if $7679 + 63m$ is less than 9000, then $63m$ is less than $9000 - 7679 = 1321$ and so m is less than $\frac{1321}{63} \approx 20.97$.
Since m is a positive integer, then m is less than or equal to 20.
Thus, we want all integers m from 6 to 20 inclusive for which $7679+63m$ is equal to a multiple of 5.
An integer is a multiple of 5 exactly when its units (ones) digit is 0 or 5.
The units digit of the total number of samples, $7679 + 63m$, is determined by the sum of the units digits of 7679, which is 9, and the units digit of the value of $63m$.
Thus, $7679 + 63m$ is a multiple of 5 exactly when the value of $63m$ has a units digit of 1 or 6 (since $9 + 1$ has a units digit of 0 and $9 + 6$ has a units digit of 5).
The value of $63m$ has a units digit of 1 exactly when m has a units digit of 7 (since 3×7 has a units digit of 1).
The value of $63m$ has a units digit of 6 exactly when m has a units digit of 2 (since 3×2 has a units digit of 6).
The values of m from 6 to 20 inclusive having a units digit of 7 or 2 are $7, 12$ and 17.
We can confirm that when m is equal to $7, 12$ and 17, the values of $7679 + 63m$ are 8120, 8435 and 8750 respectively, as required.
The sum of all values of m for which the total number of samples that the three students report as containing blueberry is a multiple of 5 between 8000 and 9000 is $7 + 12 + 17 = 36$.

Answer: (B)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2021 Gauss Contests

(Grades 7 and 8)

Wednesday, May 12, 2021
(in North America and South America)

Thursday, May 13, 2021
(outside of North America and South America)

Solutions

Centre for Education in Mathematics and Computing Faculty and Staff

Ed Anderson
Jeff Anderson
Terry Bae
Jacquelene Bailey
Shane Bauman
Jenn Brewster
Ersal Cahit
Diana Castañeda Santos
Sarah Chan
Ashely Congi
Serge D'Alessio
Fiona Dunbar
Mike Eden
Sandy Emms
Barry Ferguson
Steve Furino
Lucie Galinon
Robert Garbary
Rob Gleeson
Sandy Graham
Conrad Hewitt
Angie Hildebrand
Carrie Knoll
Wesley Korir
Judith Koeller
Laura Kreuzer
Bev Marshman
Paul McGrath
Jen Nelson
Ian Payne
J.P. Pretti
Alexandra Rideout
Nick Rollick
Kim Schnarr
Ashley Sorensen
Ian VanderBurgh
Troy Vasiga
Christine Vender
Heather Vo
Bonnie Yi

Gauss Contest Committee

Ashley Sorensen (Chair), University of Waterloo, Waterloo, ON
Kevin Grady (Assoc. Chair), Cobden District P.S., Cobden, ON
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David Matthews, Waterloo, ON
John McLoughlin, University of New Brunswick, Fredericton, NB
Nick Rollick, University of Waterloo, Waterloo, ON
David Switzer, Scott Central P.S., Sandford, ON
Rachael Verbruggen, University of Waterloo, Waterloo, ON
Robert Wong, Edmonton, AB
Chris Wu, Ledbury Park E. and M.S., Toronto, ON
Lori Yee, William Dunbar P.S., Pickering, ON

Grade 7

1. Arranging the five numbers from largest to smallest, we get $10\,000, 1000, 100, 10, 1$.
   The middle number is 100.

   Answer: (D)

2. Each side of the square has length 5 cm.
   The perimeter of the square is 4×5 cm $= 20$ cm.

   Answer: (A)

3. The right side of the equation is $10 + 20 = 30$.
   The equation is true when the left side is also equal to 30.
   Since $5 + 25 = 30$, the value that goes in the box to make the equation true is 25.

   Answer: (E)

4. Reading from the graph, Dan spent 6 hours on homework, Joe spent 3 hours, Bob spent 5 hours, Susie spent 4 hours, and Grace spent 1 hour.
   Adding their times together, Bob and Grace spent the same amount of time on homework as Dan.

   Answer: (C)

5. Each of the five fractions is positive and so the smallest of these fractions is the fraction that is closest to 0.
   Since each fraction has a numerator equal to 1, the smallest of these fractions is the one with the largest denominator.
   Of those given, the fraction that is closest to 0 is thus $\frac{1}{9}$.

   Answer: (E)

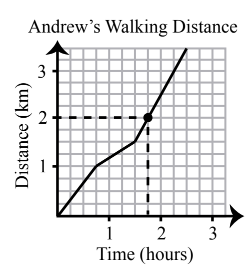
6. If the bag contained a total of 6 candies and exactly 5 of these candies were red, then the probability of Judith choosing a red candy from the bag would be $\frac{5}{6}$.
   Therefore, the total number of candies in the bag could be 6.
   Can you explain why each of the other four answers is not possible?

   Answer: (D)

7. Each point that lies to the right of the y-axis has an x-coordinate that is positive.
   Each point that lies below the x-axis has a y-coordinate that is negative.
   Since $P(x, y)$ lies to the right of the y-axis and below the x-axis, then the value of x is positive and the value of y is negative.

   Answer: (B)

8. Begin by locating 2 km on the vertical (Distance) axis.
   Next, locate the point on the line graph for which Andrew's distance walked is 2 km, as shown.
   The time in hours corresponding to this point is $\frac{3}{4}$ of an hour greater than 1 hour.
   Since $\frac{1}{4}$ of an hour is equal to 15 minutes, then $\frac{3}{4}$ of an hour is 45 minutes, and so it takes Andrew 1 hour, 45 minutes to walk the first 2 km.

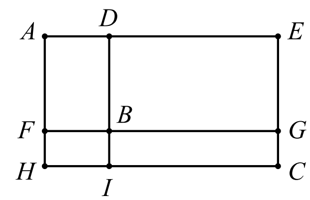


   Answer: (C)

9. The five numbers $5, 6, 7, 8, 9$ repeat to form the pattern shown.
Thus, the 5^{th} number in the pattern is 9, the 10^{th} number in the pattern is 9, the 15^{th} number in the pattern is 9, and so on.
Since 220 is a multiple of 5, the 220^{th} number in the pattern is also a 9 and so the 221^{st} number in the pattern is 5.

Answer: (A)

10. We begin by labelling additional points, as shown.
Beginning at A, the ant may travel right (to D) or down (to F).



Assume the ant begins by travelling right, to D.
From D, if the ant continues travelling right (to E) the path cannot pass through B (since the ant can travel only right or down).
Thus, from D, the ant must travel down to B.

From B, there are two paths that end at C, one travelling right to G and then down to C, and another travelling down to I and then right to C.
Therefore, there are two paths in which the ant begins by travelling right: $A-D-B-G-C$ and $A-D-B-I-C$.

Assume the ant begins by travelling down, to F.
From F, if the ant continues travelling down (to H) the path cannot pass through B (since the ant can travel only right or down). Thus, from F, the ant must travel right to B.
From B, there are two paths that end at C, as discussed above.
Therefore, there are two paths in which the ant begins by travelling down: $A-F-B-G-C$ and $A-F-B-I-C$.

There are 4 different paths from A to C that pass through B.

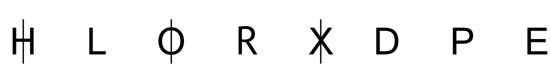
Answer: (C)

11. *Solution 1*
Writing the numbers that appear in the list, we get
$$4, 11, 18, 25, 32, 39, 46, 53, \ldots$$
Of the given answers, the number that appears in Laila's list is 46.

Solution 2
Laila begins her list at 4 and each new number is 7 more than the previous number.
Therefore, each of the numbers in her list will be 4 more than a multiple of 7.
Since 42 is a multiple of 7 ($6 \times 7 = 42$), then $42 + 4 = 46$ will appear in Laila's list.

Answer: (B)

12. If a letter is folded along its vertical line of symmetry, both halves of the letter would match exactly.
Three of the given letters, H, O and X, have a vertical line of symmetry, as shown.



Answer: (C)

13. Since $\triangle BCE$ is equilateral, then each of its three interior angles measures 60°.
Vertically opposite angles are equal in measure, and so $\angle DEA = \angle BEC = 60^\circ$.
In $\triangle ADE$, the sum of the three interior angles is 180°.
Thus, $x^\circ + 90^\circ + 60^\circ = 180^\circ$ or $x + 150 = 180$ and so $x = 30$.

Answer: (E)

14. Given three consecutive integers, the smallest integer is one less than the middle integer and the largest integer is one more than the middle integer.
For example, 10, 11 and 12 are three consecutive integers, and 10 is one less than the middle integer 11, and 12 is one more than 11.
So, the sum of the smallest and largest integers is twice the middle integer.
(In the example, $10 + 12 = 2 \times 11$.)
Then, the sum of three consecutive integers is equal to 3 times the middle integer and so the sum of three consecutive integers is a multiple of 3.
Of the answers given, the only number that is a multiple of 3 is 21.
Alternately, we could use trial and error to solve this problem to find that $6 + 7 + 8 = 21$.

ANSWER: (D)

15. There is no integer greater than 13931 and less than 14000 that is a palindrome. (You should consider why this is true before reading on.)
Let the next palindrome greater than 13931 be N.
We proceed under the assumption that N is between 14000 and 15000 and will show that this assumption is correct.
A 5-digit palindrome is a number of the form $abcba$. That is, the ten thousands digit, a, must equal the ones digit and the thousands digit, b, must equal the tens digit.
Since N is at least 14000, the smallest possible value of a (the ten thousands digit) is 1.
Since the smallest possible value of a is 1 and N is at least 14000, the smallest possible value of b (the thousands digit) is 4.
Thus N is a number of the form $14c41$.
Letting the hundreds digit, c, be as small as possible we get that N is 14041 and has digit sum $1 + 4 + 0 + 4 + 1 = 10$.

ANSWER: (D)

16. The positive factors of 14 are $1, 2, 7$, and 14.
The positive factors of 21 are $1, 3, 7$, and 21.
The positive factors of 28 are $1, 2, 4, 7, 14$, and 28.
The positive factors of 35 are $1, 5, 7$, and 35.
The positive factors of 42 are $1, 2, 3, 6, 7, 14, 21$, and 42.
There are 3 numbers in the list (14, 21 and 35) that have exactly 4 positive factors.

ANSWER: (C)

17. The percentage discount of the third price off the original price does not depend on the original price of the shirt.
That is, we may choose an original price for the shirt and calculate the combined percentage discount.
Since the discounts are given as percentages, letting the original price of the shirt be \$100 might make the calculations simpler.
If the original price of the shirt is \$100 and this price is reduced by 50%, the discounted price is half of \$100 or \$50.
A further 40% reduction on \$50 is equal to a $\frac{40}{100} \times \$50 = 0.40 \times \$50 = \$20$ discount.
After both price reductions, the \$100 shirt is priced at $\$50 - \$20 = \$30$ and thus the total discount is $\$100 - \$30 = \$70$.
The original price of the shirt was \$100, the final discounted price is \$70 less, and so the discount of the third price off the original price is $\frac{\$70}{\$100} \times 100\% = 70\%$.

ANSWER: (C)

18. The perimeter of $\triangle ABC$ is equal to $AB + BC + CA$ or $AB + BM + MC + CA$.
Since $AB = CA$ and $BM = MC$, then half of the perimeter of $\triangle ABC$ is equal to $AB + BM$.
The perimeter of $\triangle ABC$ is 64 and so $AB + BM = \frac{1}{2} \times 64 = 32$.
The perimeter of $\triangle ABM$ is 40, and so $AM + AB + BM = 40$.
Since $AB + BM = 32$, then $AM = 40 - 32 = 8$.

ANSWER: (B)

19. We begin by calling the missing digits A and B, as shown. Each of the digits A and B is chosen from the digits 1 to 9 and $A \neq B$.

$$\begin{array}{r} 5\,\boxed{A} \\ -\ \boxed{B}\,5 \\ \hline \end{array}$$

The top 2-digit number $5A$ is at most 59.
Thus, B cannot equal 6, 7, 8, or 9 since the result of subtracting the bottom number from the top would be negative.
If $B = 5$, then the bottom number is 55 and for the result to be positive, A could equal 6, 7, 8, or 9 (with the results being 1, 2, 3, and 4 respectively).
Thus, there are 4 possible positive results in this case.
If $B = 4$, then the bottom number is 45 and for the result to be positive, A could equal each of the digits from 1 to 9 with the exception of 4 (since $A \neq B$).
In this case, the results of subtracting the bottom number from the top are 6, 7, 8, 10, 11, 12, 13, and 14 respectively. Thus, there are 8 possible positive results when $B = 4$.
If $B = 3$, then the bottom number is 35 and for the result to be positive, A could equal 1, 2, 4, 5, 6, 7, 8, or 9 (with the results being 16, 17, 19, 20, 21, 22, 23, and 24 respectively).
Thus, there are 8 possible positive results when $B = 3$.
Similarly, when $B = 2$ there are 8 possible positive results and when $B = 1$ there are 8 possible positive results (and each of these results is different from any of the other results).
In total, the number of possible results that are positive is $4 + (8 \times 4) = 36$.

ANSWER: (A)

20. The table below shows the possible sums when two standard dice are rolled.
Each sum in bold is equal to a prime number.

Number on the First Die

Number on the Second Die	1	2	3	4	5	6
1	**2**	**3**	4	**5**	6	**7**
2	**3**	4	**5**	6	**7**	8
3	4	**5**	6	**7**	8	9
4	**5**	6	**7**	8	9	10
5	6	**7**	8	9	10	**11**
6	**7**	8	9	10	**11**	12

Looking at the table above, the total number of possible outcomes is $6 \times 6 = 36$.
The total number of outcomes for which the sum is a prime number is 15.
The probability that the sum of the numbers on the top faces is a prime number is $\frac{15}{36} = \frac{5}{12}$.

ANSWER: (A)

21. We begin by considering cases in which 1 is subtracted from numbers that start with one and are followed by smaller numbers of zeros.

$$\begin{array}{r} 10 \\ -1 \\ \hline 9 \end{array} \qquad \begin{array}{r} 100 \\ -1 \\ \hline 99 \end{array} \qquad \begin{array}{r} 1000 \\ -1 \\ \hline 999 \end{array} \qquad \begin{array}{r} 10\,000 \\ -1 \\ \hline 9999 \end{array} \qquad \begin{array}{r} 100\,000 \\ -1 \\ \hline 99\,999 \end{array}$$

In the examples above, each result consists of 9s only and the number of 9s is equal to the number of zeros in the original number. Can you explain why this pattern continues as we increase the number of zeros?

Since each of the digits in the result is a 9 and the sum of these digits is 252, then the number of 9s in the result is equal to $\frac{252}{9} = 28$.

The number of zeros in the original number equals the number of 9s in the result, which is 28.

ANSWER: (B)

22. The perimeter of Figure 1 consists of 4 rectangle side lengths of 10 cm (each of which is horizontal) and 4 rectangle side lengths of 5 cm (each of which is vertical).

Thus, the perimeter of Figure 1 is $(4 \times 10 \text{ cm}) + (4 \times 5 \text{ cm}) = 40 \text{ cm} + 20 \text{ cm} = 60 \text{ cm}$.

The perimeter of Figure 2 consists of 4 rectangle side lengths of 10 cm (each of which is horizontal) and 6 rectangle side lengths of 5 cm (each of which is vertical).

Thus, the perimeter of Figure 2 is $(4 \times 10 \text{ cm}) + (6 \times 5 \text{ cm}) = 40 \text{ cm} + 30 \text{ cm} = 70 \text{ cm}$.

The perimeter of Figure 3 consists of 4 rectangle side lengths of 10 cm (each of which is horizontal) and 8 rectangle side lengths of 5 cm (each of which is vertical).

Thus, the perimeter of Figure 3 is $(4 \times 10 \text{ cm}) + (8 \times 5 \text{ cm}) = 40 \text{ cm} + 40 \text{ cm} = 80 \text{ cm}$.

Each figure after Figure 1 is formed by joining two rectangles to the bottom of the previous figure.

The bottom edge of a figure (consisting of two 10 cm side lengths) is replaced by two 10 cm lengths when the two new rectangles are adjoined.

That is, the addition of two rectangles does not change the number of 10 cm side lengths contributing to the perimeter of the new figure and so the number of 10 cm lengths remains constant at 4 for each figure.

The addition of two new rectangles does not replace any of the previous 5 cm (vertical) side lengths.

Thus, the addition of the two rectangles does add two 5 cm vertical segments to the previous perimeter, increasing the perimeter of the previous figure by $2 \times 5 \text{ cm} = 10 \text{ cm}$.

That is, the perimeter of Figure 1 is 60 cm, and the perimeter of each new figure is 10 cm greater than the previous figure.

We need to add 10 cm 65 times to get a total of 710 cm (that is, $60 \text{ cm} + 10 \text{ cm} \times 65 = 710 \text{ cm}$).

Thus, Figure 66 has a perimeter of 710 cm, and so $n = 66$.

ANSWER: (C)

23. To encode a letter, James multiplies its corresponding number by 3 and then subtracts 5, continuing this process a total of n times.

To decode a number, the inverse operations must be performed in the opposite order.

The inverse operation of multiplication is division. The inverse operation of subtraction is addition.

Thus to decode a number, add 5 and then divide by 3, and continue this process a total of n times.

For example, when $n = 1$ the number 4 (corresponding to the letter D) is encoded to $4 \times 3 - 5 = 7$. The number 7 is decoded by adding 5 and then dividing by 3. We may check that this works by noting that $(7 + 5) \div 3 = 4$ (which corresponds to the letter D), as required.

Each letter of James' original message corresponds to a number from 1 to 26, inclusive.

To determine the value of n, we may begin with the four given encoded numbers $(367, 205, 853, 1339)$ and continue to apply the decoding process until the resulting numbers are each equal to a number from 1 to 26, inclusive (since each letter of the original message corresponds to a number from 1 to 26, inclusive).

We show this work in the table below.

	367	205	853	1339
$n = 1$	$(367 + 5) \div 3 = 124$	$(205 + 5) \div 3 = 70$	$(853 + 5) \div 3 = 286$	$(1339 + 5) \div 3 = 448$
$n = 2$	$(124 + 5) \div 3 = 43$	$(70 + 5) \div 3 = 25$	$(286 + 5) \div 3 = 97$	$(448 + 5) \div 3 = 151$
$n = 3$	16	10	34	52
$n = 4$	7	5	13	19
$n = 5$	4	$\frac{10}{3}$	6	8

From the table, the first value of n for which each of the four numbers is from 1 to 26 inclusive, is $n = 4$.

Further, we note that if $n = 5$ the original number corresponding to 205 is $\frac{10}{3}$, which is not possible.

Therefore, the value of n used by James was 4.

(Although the question did not ask for the original message, the letters corresponding to $7, 5, 13, 19$ are G, E, M, S.)

Answer: (C)

24. We begin by considering the prime factors (called the *prime factorization*) of each of the two numbers, 4 and 4620.

$$4 = 2 \times 2$$

$$4620 = 2 \times 2 \times 3 \times 5 \times 7 \times 11$$

Let a pair of positive whole numbers whose greatest common factor is 4 and whose lowest common multiple is 4620 be (a, b).

Since the pair (a, b) has a greatest common factor of 4, then each of a and b is a multiple of 4 and thus 2×2 is included in the prime factorization of each of a and b.

Further, each of a and b cannot have any other prime factors in common otherwise their greatest common factor would be greater than 4.

The lowest common multiple of a and b is 4620 and so each of a and b is less than or equal to 4620.

Further, if for example a has prime factors which are not prime factors of 4620 then 4620 is not a multiple of a.

That is, each of a and b must have prime factors which are chosen from $2, 2, 3, 5, 7$, and 11 only.

Summarizing, a is a positive whole number of the form $2 \times 2 \times m$ and b is a positive whole number of the form $2 \times 2 \times n$, where:

- m and n are positive whole numbers
- the prime factors of m are chosen from $3, 5, 7, 11$ only
- the prime factors of n are chosen from $3, 5, 7, 11$ only
- m and n have no prime factors in common

In the table below, we list the possible values for m and n which give the possible pairs a and b. To ensure that we don't double count the pairs (a, b), we assume that $a \le b$ and thus $m \le n$.

m	n	a	b	(a, b)
1	$3 \times 5 \times 7 \times 11$	$2 \times 2 \times 1$	$2 \times 2 \times 3 \times 5 \times 7 \times 11$	$(4, 4620)$
3	$5 \times 7 \times 11$	$2 \times 2 \times 3$	$2 \times 2 \times 5 \times 7 \times 11$	$(12, 1540)$
5	$3 \times 7 \times 11$	$2 \times 2 \times 5$	$2 \times 2 \times 3 \times 7 \times 11$	$(20, 924)$
7	$3 \times 5 \times 11$	$2 \times 2 \times 7$	$2 \times 2 \times 3 \times 5 \times 11$	$(28, 660)$
11	$3 \times 5 \times 7$	$2 \times 2 \times 11$	$2 \times 2 \times 3 \times 5 \times 7$	$(44, 420)$
3×5	7×11	$2 \times 2 \times 3 \times 5$	$2 \times 2 \times 7 \times 11$	$(60, 308)$
3×7	5×11	$2 \times 2 \times 3 \times 7$	$2 \times 2 \times 5 \times 11$	$(84, 220)$
3×11	5×7	$2 \times 2 \times 3 \times 11$	$2 \times 2 \times 5 \times 7$	$(132, 140)$

There are 8 different pairs of positive whole numbers having a greatest common factor of 4 and a lowest common multiple of 4620.

Answer: (D)

25. Since $12 \times 12 \times 12 = 1728$, Jonas uses each of his 1728 copies of a $1 \times 1 \times 1$ cube with the net shown to build the large cube.
The net contains the numbers 100 and c only, and so each of the numbers appearing on the exterior faces of the large cube is 100 or c.
Jonas builds the large cube in such a way that the sum of the numbers on the exterior faces is as large as possible.
Since $c < 100$, Jonas builds the large cube so that the number of 100s appearing on the exterior faces is as large as possible (and the number of c's appearing on the exterior faces is as small as possible).
The $1 \times 1 \times 1$ cubes which contribute to the numbers on the exterior faces of the large cube can be classified as one of three types.
We call these three types: corner, edge and inside. In the portion of the large $12 \times 12 \times 12$ cube shown in the diagram below, each of these three types is shown.

(i) A *corner cube* is shown in Figure 1. These are cubes that appear in one of the "corners" of the large cube and so there are 8 such corner cubes.

(ii) An *edge cube* is shown in Figure 2. These are cubes that appear along the edges but not in the corners of the large cube. A cube has 12 edges and each edge of the large cube contains 10 edge cubes, and so there are $10 \times 12 = 120$ such cubes.

(iii) An *inside cube* is shown in Figure 3. These are the remaining cubes that contribute to the numbers on the exterior faces of the large cube. A cube has 6 faces and each face of the large cube contains 10×10 inside cubes, and so there are $6 \times 10 \times 10 = 600$ such cubes.

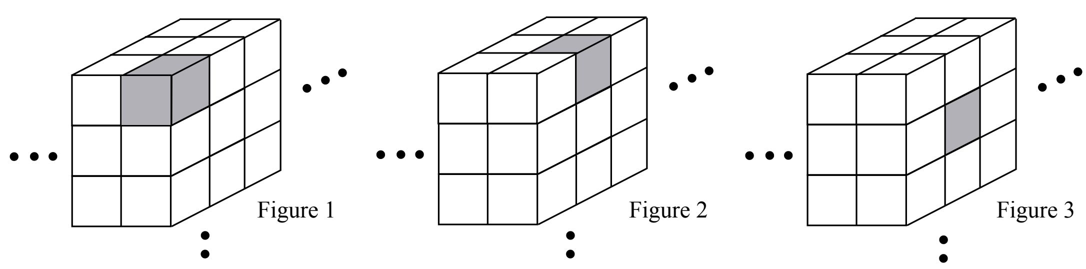


Figure 1 Figure 2 Figure 3

Let S be the sum of the numbers on the exterior faces of the large cube.
Each corner cube has 3 faces which contribute to S. For S to be as large as possible, 100 will

appear on exactly 1 of these 3 faces (there is exactly one 100 in the net of the $1 \times 1 \times 1$ cube), and c will appear on the remaining 2 faces.
Thus, the 8 corner cubes contribute $8 \times 100 + 8 \times 2 \times c$ or $800 + 16c$ to S.

Each edge cube has 2 faces which contribute to S.
For S to be as large as possible, 100 will appear on exactly 1 of these 2 faces and c will appear on the other face.
Thus, the 120 edge cubes contribute $120 \times 100 + 120 \times c$ or $12\,000 + 120c$ to S.

Finally, each inside cube has 1 face which contributes to S.
For S to be as large as possible, 100 will appear on this face.
Thus, the 600 inside cubes contribute 600×100 or $60\,000$ to S.

In total, we get $S = 800 + 16c + 12\,000 + 120c + 60\,000$ and so $S = 136c + 72\,800$.
Since we want S to be at least $80\,000$ and $80\,000 - 72\,800 = 7200$, then $136c$ is at least 7200.
Because $136 \times 52 = 7072$ and $136 \times 53 = 7208$, it must be the case that c is at least 53.
Since we want S to be at most $85\,000$ and $85\,000 - 72\,800 = 12\,200$, then $136c$ is at most $12\,200$.
Because $136 \times 90 = 12\,240$ and $136 \times 89 = 12\,104$, it must be the case that c is at most 89.
This means that c is a positive integer that is at least 53 and at most 89.
There are $89 - 52 = 37$ such integers. (Think of listing the integers from 1 to 89 and removing the integers from 1 to 52.)

Answer: (C)

Grade 8

1. Since 1000 is 1 more than 999, then $1000 + 1000 = 2000$ is 2 more than $999 + 999$.
Thus, $999 + 999 = 2000 - 2 = 1998$.

Answer: (C)

2. An equilateral triangle has 3 sides of equal length.
If the perimeter of an equilateral triangle is 15 m, then the length of each side is $\dfrac{15 \text{ m}}{3} = 5$ m.

Answer: (B)

3. Since $25 \times 4 = 100$, then 100 is a multiple of 4.
Therefore, the greatest multiple of 4 less than 100 is $24 \times 4 = 96$ (or alternately, $100 - 4 = 96$).

Answer: (B)

4. Points which lie to the right of the y-axis have x-coordinates which are positive.
Points which lie below the x-axis have y-coordinates which are negative.
Point $P(x, y)$ lies to the right of the y-axis and below the x-axis and thus the value of x is positive and the value of y is negative.

Answer: (B)

5. Substituting $x = -6$ into each expression and evaluating, we get

(A) $\begin{aligned} 2 + x &= 2 + (-6) \\ &= -4 \end{aligned}$ (B) $\begin{aligned} 2 - x &= 2 - (-6) \\ &= 2 + 6 \\ &= 8 \end{aligned}$ (C) $\begin{aligned} x - 1 &= -6 - 1 \\ &= -7 \end{aligned}$

(D) $x = -6$ (E) $\begin{aligned} x \div 2 &= (-6) \div 2 \\ &= -3 \end{aligned}$

Of these, $2 - x$ gives the greatest value when $x = -6$.

Answer: (B)

6. At this rate, it would take 6 seconds to fill a 500 mL bottle.
A 250 mL bottle has half the volume of a 500 mL bottle and so it will take half as long or 3 seconds to fill.

Answer: (C)

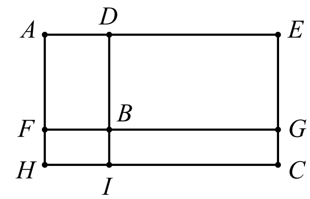
7. If the tens digit of a two-digit number is even, then when the digits are reversed the new number will have a units digit that is even and therefore the number will be even.
If a two-digit number is even, then it is divisible by 2 and so it cannot be a prime number.
Since 29, 23 and 41 each have a tens digit that is even, we may eliminate these three as possible answers.
When the digits of 53 are reversed, the result is 35.
Since 35 is divisible by 5, it is not a prime number.
Finally, when the digits of 13 are reversed, the result is 31.
Since 31 has no positive divisors other than 1 and 31, it is a prime number.

Answer: (D)

8. When 3 red beans are added to the bag, the number of red beans in the bag is $5 + 3 = 8$.
When 3 black beans are added to the bag, the number of black beans in the bag is $9 + 3 = 12$.
The number of beans now in the bag is $8 + 12 = 20$.
If one bean is randomly chosen from the bag, the probability that the bean is red is $\frac{8}{20} = \frac{2}{5}$.

Answer: (B)

9. We begin by labelling additional points, as shown.
Beginning at A, the ant may travel right (to D) or down (to F).



Assume the ant begins by travelling right, to D. From D, if the ant continues travelling right (to E) the path cannot pass through B (since the ant can travel only right or down).
From D, the ant must travel down to B. From B, there are two paths that end at C, one travelling right to G and then down to C, and another travelling down to I and then right to C.
Therefore, there are two paths in which the ant begins by travelling right: $A - D - B - G - C$ and $A - D - B - I - C$.

Assume the ant begins by travelling down, to F. From F, if the ant continues travelling down (to H) the path cannot pass through B (since the ant can travel only right or down).
Thus, from F, the ant must travel right to B. From B, there are two paths that end at C, as discussed above.
Therefore, there are two paths in which the ant begins by travelling down: $A - F - B - G - C$ and $A - F - B - I - C$.

There are 4 different paths from A to C that pass through B.

Answer: (C)

10. By assigning the largest digits to the largest place values, we form the largest possible four-digit number.
The largest four-digit number that can be formed by rearranging the digits of 2021 is 2210.
By assigning the smallest digits to the largest place values, we form the smallest possible four-digit number.
The smallest four-digit number (greater than 1000) that can be formed by rearranging the digits of 2021 is 1022.
Thus the largest possible difference between two such four-digit numbers is $2210 - 1022 = 1188$.

Answer: (A)

11. *Solution 1*
PQ and RS intersect at T. Thus, $\angle PTR$ and $\angle STQ$ are vertically opposite angles and so $\angle PTR = \angle STQ = 140^\circ$.
Since $\angle PTR = \angle PTU + \angle RTU$, then

$$\begin{aligned} \angle RTU &= \angle PTR - \angle PTU \\ &= 140^\circ - 90^\circ \\ &= 50^\circ \end{aligned}$$

and so the measure of $\angle RTU$ is 50°.

Solution 2
RS is a line segment and so $\angle RTQ + \angle STQ = 180^\circ$ or $\angle RTQ = 180^\circ - 140^\circ = 40^\circ$.
PQ is a line segment and so $\angle RTQ + \angle RTU + \angle PTU = 180^\circ$ or $\angle RTU = 180^\circ - 40^\circ - 90^\circ = 50^\circ$.

Answer: (C)

12. Given three consecutive integers, the smallest integer is one less than the middle integer and the largest integer is one more than the middle integer.
So, the sum of the smallest and largest integers is twice the middle integer.
Then, the sum of three consecutive integers is equal to 3 times the middle integer and so the sum of three consecutive integers is a multiple of 3.
Of the answers given, the only number that is a multiple of 3 is 21.
Alternately, we could use trial and error to solve this problem to find that $6 + 7 + 8 = 21$.

Answer: (D)

13. Reading from the bar graph, there are 8 yellow shirts, 4 red shirts, 2 blue shirts, and 2 green shirts. In total, the number of shirts is $8 + 4 + 2 + 2 = 16$.
Thus, 8 yellow shirts represents $\frac{8}{16}$ or $\frac{1}{2}$ of the total number of shirts.
The only circle graph showing that approximately half of the shirts are yellow is (E) and thus it probably best represents the information in the bar graph.
We may confirm that this circle graph also shows that approximately $\frac{4}{16} = \frac{1}{4}$ of the shirts are red, approximately $\frac{2}{16} = \frac{1}{8}$ of the shirts are green and approximately $\frac{1}{8}$ of the shirts are blue.

Answer: (E)

14. Let the unknown whole number be n.
Since 16 is a factor of n, then each positive factor of 16 is also a factor of n.
That is, the positive factors of n include 1, 2, 4, 8, 16, and n.
Since 16 is a factor of n, then n is a positive multiple of 16.
The smallest whole number multiple of 16 is 16.
However, if $n = 16$, then n has exactly 5 positive factors, namely 1, 2, 4, 8, and 16.
The next smallest whole number multiple of 16 is 32.
If $n = 32$, then n has exactly 6 positive factors, namely 1, 2, 4, 8, 16, and 32, as required.

Answer: (B)

15. *Solution 1*
The sum of the three interior angles in a triangle is 180°.
A triangle's three interior angles are in the ratio $1 : 4 : 7$ and so the smallest of the angles measures $\frac{1}{1+4+7} = \frac{1}{12}$ of the sum of the three interior angles.
Thus, the smallest angle in the triangle meausres $\frac{1}{12}$ of 180° or $\frac{180^\circ}{12} = 15^\circ$.
The measure of the next largest angle is 4 times the measure of the smallest angle or $4 \times 15^\circ = 60^\circ$.
The measure of the largest angle is 7 times the measure of the smallest angle or $7 \times 15^\circ = 105^\circ$.
The measures of the interior angles are 15°, 60° and 105°.

Solution 2
The sum of the three interior angles in a triangle is 180°.
Working backward from the possible answers, we may eliminate (B) and (C) since the sum of the three given angles is not 180°.
The measures of the smallest and largest angles in the triangle are in the ratio $1 : 7$.
Since $7 \times 12^\circ = 84^\circ$ and not 120°, we may eliminate (A).
Since $7 \times 14^\circ = 98^\circ$ and not 110°, we may eliminate (E).
The remaining possibility is (D) and we may confirm that 15°, 60° and 105° have a sum of 180° and are in the ratio $1 : 4 : 7$.

Answer: (D)

16. The seven numbers $1, 2, 5, 10, 25, 50, 100$ repeat to form the pattern shown.
Thus, the 7^{th} number in the pattern is 100, the 14^{th} number in the pattern is 100, the 21^{st} number in the pattern is 100, and so on.
The 14^{th} number in the pattern is 100 and so the 15^{th} number in the pattern is 1, the 16^{th} is 2, the 17^{th} is 5, and the 18^{th} number in the pattern is 10.
Since 70 is a multiple of 7, the 70^{th} number in the pattern is also 100 and so the 71^{st} number in the pattern is 1, the 72^{nd} is 2, the 73^{rd} is 5, the 74^{th} is 10, and the 75^{th} number in the pattern is 25.
The sum of the 18^{th} and 75^{th} numbers in the pattern is $10 + 25 = 35$.
ANSWER: (E)

17. *Solution 1*
Gaussville's soccer team won 40% of their first 40 games.
Thus they won $0.40 \times 40 = 16$ games.
After winning the next n games in a row, they had won $16 + n$ games and had played $40 + n$ games.
At this point, they had won 50% or $\frac{1}{2}$ of their games.
This means that the number of games won, $16+n$, was $\frac{1}{2}$ of the number of games played, $40+n$.
For which of the possible answers is $16 + n$ equal to $\frac{1}{2}$ of $40 + n$?
Substituting each of the five possible answers, we get that $16 + 8 = 24$ is $\frac{1}{2}$ of $40 + 8 = 48$, and so the value of n is 8.

Solution 2
Gaussville's soccer team won 40% of their first 40 games.
Thus they had $0.40 \times 40 = 16$ wins and $40 - 16 = 24$ non-wins (losses or ties) in their first 40 games.
At this point, the team went on a winning streak which means they did not accumulate any additional non-wins.
Thus, their 24 non-wins represent 50% of the final total, and so the final number of wins is 24.
Therefore, Gaussville's soccer team won $n = 24 - 16 = 8$ games in a row.
ANSWER: (D)

18. The fraction of the area of the larger circle that is not shaded does not depend on the actual radius of either circle, and so we begin by letting the radius of the smaller circle be 1 and thus the radius of the larger circle is 3.
In this case, the area of the smaller circle is $\pi(1)^2 = \pi$.
The area of the larger circle is $\pi(3)^2 = 9\pi$.
The area of the larger circle that is not shaded is $9\pi - \pi = 8\pi$.
Therefore, the fraction of the area of the larger circle that is not shaded is $\frac{8\pi}{9\pi} = \frac{8}{9}$.
(Alternately, we could note that the fraction of the area of the larger circle that is shaded is $\frac{\pi}{9\pi} = \frac{1}{9}$ and so the fraction of the area of the larger circle that is not shaded is $1 - \frac{1}{9} = \frac{8}{9}$.)
ANSWER: (A)

19. We proceed to work backward from the final sum, 440, 'undoing' each of the three operations to determine the sum of their two numbers before any operations were performed.
The final operation performed by each of Asima and Nile was to multiply their number by 4.
Multiplying each of their numbers by 4 increases the sum of the two numbers by a factor of 4.
That is, the final sum of their two numbers was 440, and so the sum of their two numbers immediately before the last operation was performed was $440 \div 4 = 110$.
The second operation performed by each of Asima and Nile was to subtract 10 from their

number.
Subtracting 10 from each of their numbers decreases the sum of the two numbers by 20.
That is, the sum of their two numbers immediately following the second operation was 110, and so the sum of their two numbers immediately before the second operation was performed was $110 + 20 = 130$.
Finally, the first operation performed by each of Asima and Nile was to double their number.
Doubling each of their numbers increases the sum of the two numbers by a factor of 2.
That is, the sum of their two numbers immediately following the first operation was 130, and so the sum of their two numbers before the first operation was performed was $130 \div 2 = 65$.
Each of their original integers is greater than 0 and the two integers have a sum of 65.
Therefore, Asima's original integer could be any integer from 1 to 64, inclusive.
Thus, there are 64 possibilities for Asima's original integer.

ANSWER: (A)

20. *Solution 1*
The table below shows the possible differences between the number on Ruby's roll and the number on Sam's roll.

Number on Ruby's Roll

Number on Sam's Roll	1	2	3	4	5	6
1	0	1	2	3	4	5
2	−1	0	1	2	3	4
3	−2	−1	0	1	2	3
4	−3	−2	−1	0	1	2
5	−4	−3	−2	−1	0	1
6	−5	−4	−3	−2	−1	0

Looking at the table above, the total number of possible outcomes is $6 \times 6 = 36$.
The total number of outcomes for which the difference is negative is 15.
The probability that the result from subtracting the number on Sam's roll from the number on Ruby's roll is negative is $\frac{15}{36} = \frac{5}{12}$.

Solution 2
Ruby and Sam each have 6 possible outcomes when they roll the dice, and so the total number of possible outcomes is $6 \times 6 = 36$.
Of these 36 possible outcomes, there are 6 outcomes in which Sam and Ruby each roll the same number and thus the difference between the numbers rolled is 0.
For the remaining $36 - 6 = 30$ possible outcomes, it is equally probable that the number on Ruby's roll is greater than the number on Sam's roll as it is that the number on Sam's roll is greater than the number on Ruby's roll.
That is, one half of these 30, or 15 possible outcomes have a result that is negative and 15 have a result that is positive.

Therefore, the probability that the result from subtracting the number on Sam's roll from the number on Ruby's roll is negative is $\frac{15}{36} = \frac{5}{12}$.

Answer: (B)

21. If n is a positive integer, 10^n is equal to 1 followed by n zeros, when evaluated.
For example, $10^4 = 10\,000$ and 10^{2021} is equal to $1\underbrace{00...0}_{2021\text{ 0s}}$.
The result of adding 1 to the positive integer consisting of only n 9s is 10^n.
For example, $1 + 9999 = 10\,000 = 10^4$ and $1 + \underbrace{99...9}_{2021\text{ 9s}} = 1\underbrace{00...0}_{2021\text{ 0s}} = 10^{2021}$.
Let S be the integer equal to $10^{2021} - 2021$.
Since $10^{2021} = 1 + \underbrace{99...9}_{2021\text{ 9s}}$, then $S = 1 + \underbrace{99...9}_{2021\text{ 9s}} - 2021 = \underbrace{99...9}_{2021\text{ 9s}} - 2020 = \underbrace{99...9}_{2017\text{ 9s}}7979$.
The sum of the digits of the integer equal to $10^{2021} - 2021$ is $2019 \times 9 + 2 \times 7 = 18\,185$.

Answer: (E)

22. We begin by listing the prime numbers up to and including 31.
We choose to end the list at 31 since $30 \times 30 = 900$ and so 31×37 is greater than 900.
This list of prime numbers is

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31$$

Considering consecutive pairs of prime numbers from the list above, we get 10 positive integers less than 900 that can be written as a product of two consecutive prime numbers. These are

$$2 \times 3 = 6 \qquad 3 \times 5 = 15 \qquad 5 \times 7 = 35 \qquad 7 \times 11 = 77 \qquad 11 \times 13 = 143$$

$$13 \times 17 = 221 \qquad 17 \times 19 = 323 \qquad 19 \times 23 = 437 \qquad 23 \times 29 = 667 \qquad 29 \times 31 = 899$$

There are 3 positive integers less than 900 that can be written as a product of three consecutive prime numbers. These are

$$2 \times 3 \times 5 = 30 \qquad 3 \times 5 \times 7 = 105 \qquad 5 \times 7 \times 11 = 385$$

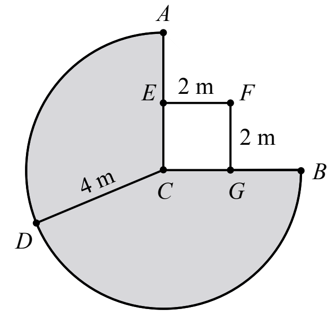
Since each positive integer greater than 1 is either a prime number or can be written as a unique product of prime numbers, it must be that these 3 numbers, 30, 105, 385, are different than those obtained from a product of two consecutive prime numbers. (Alternatively, we could check that 30, 105 and 385 do not appear in the previous list.)
Further, we note that the next smallest positive integer that can be written as a product of three consecutive prime numbers is $7 \times 11 \times 13 = 1001$ which is greater than 900.
Exactly one positive integer less than 900 can be written as a product of four consecutive prime numbers. This number is

$$2 \times 3 \times 5 \times 7 = 210$$

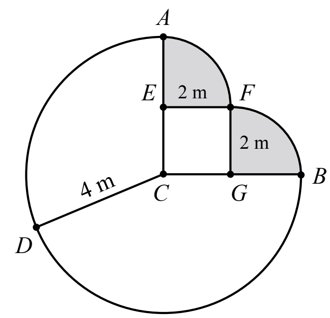
The next smallest positive integer that can be written as a product of four consecutive prime numbers is $3 \times 5 \times 7 \times 11 = 1155$ which is greater than 900.
There are no positive integers that can be written as a product of five or more consecutive prime numbers since $2 \times 3 \times 5 \times 7 \times 11 = 2310$ which is greater than 900.
In total, the number of positive integers less than 900 that can be written as a product of two or more consecutive prime numbers is $10 + 3 + 1 = 14$.

Answer: (A)

23. We begin by labelling some additional points as shown.
With the leash extended the full 4 m, the dog can reach points A, B and D where $AC = BC = DC = 4$ m and thus the dog is able to play anywhere inside the shaded area shown.
The doghouse $EFGC$ is a square and so $\angle ECG = 90^\circ$.
Therefore, the shaded figure is $\frac{3}{4}$ of a circle centred at C, has radius $CD = 4$ m, and thus has area $\frac{3}{4}\pi(4\text{ m})^2 = 12\pi\text{ m}^2$.



There is additional area outside the doghouse in which the dog can play, as shaded in the diagram.
Since $AC = 4$ m and $EC = 2$ m, then $AE = 2$ m. Since $EC + EF = 2\text{ m} + 2\text{ m} = 4$ m, the length of the leash, then the dog can reach F.
Similarly, $BG = 2$ m and $GC + GF = 2\text{ m} + 2\text{ m} = 4$ m, the length of the leash, and so the dog can also reach F by travelling around sides GC and GF of the doghouse.
Since $\angle AEF = \angle FGB = 90^\circ$, then each of these two shaded figures is $\frac{1}{4}$ of a circle (centred at E and G respectively), has radius 2 m, and thus each has area $\frac{1}{4}\pi(2)^2 = \pi\text{ m}^2$. (For example, when the dog is above and to the right of E, it can stretch to at most 2 m of rope and so can form a quarter circle of radius 2 m coming from E.)



The area outside of the doghouse in which the dog can play is $12\pi\text{ m}^2 + 2 \times \pi\text{ m}^2 = 14\pi\text{ m}^2$.

ANSWER: (A)

24. Let the sum of the numbers on the exterior faces of the $n \times n \times n$ cube be S.
To determine the smallest value of n for which $S > 1500$, we choose to position the $1 \times 1 \times 1$ cubes within the large cube in such a way that S is as large as possible.
The $1 \times 1 \times 1$ cubes which contribute to the numbers on the exterior faces of the large cube can be classified as one of three types.
We call these three types: corner, edge and inside.
In the portion of the large $n \times n \times n$ cube shown in the diagram below, each of these three types is shown.

(i) A *corner cube* is shown in Figure 1. These are cubes that appear in one of the "corners" of the large cube and so there are 8 such corner cubes.

(ii) An *edge cube* is shown in Figure 2. These are cubes that appear along the edges but not in the corners of the large cube.
A cube has 12 edges and each edge of the large cube contains $n - 2$ edge cubes, and so there are $12 \times (n - 2)$ such cubes.

(iii) An *inside cube* is shown in Figure 3. These are the remaining cubes that contribute to the numbers on the exterior faces of the large cube.
A cube has 6 faces and each face of the large cube contains $(n - 2) \times (n - 2)$ inside cubes, and so there are $6 \times (n - 2)^2$ such cubes.

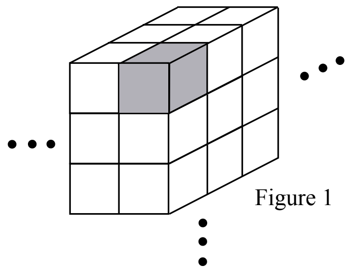


Figure 1

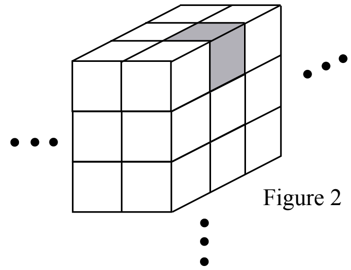


Figure 2

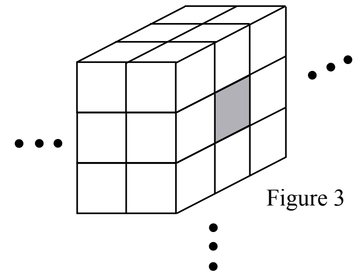


Figure 3

Each corner cube has 3 faces which contribute to S.
For S to be as large as possible, the $1 \times 1 \times 1$ cube must be positioned in such a way that the sum of the 3 external faces is as large as possible.
We may determine from the given net that the three faces meeting at a vertex of the $1 \times 1 \times 1$ cube may contain the numbers: $-1, 0, 1$ or $-1, 2, 0$ or $-2, 2, 0$ or $-2, 1, 0$.
The sums of these three faces are $0, 1, 0$, and -1 respectively.
To make S as large as possible, we choose to place each corner cube so that the numbers appearing on the exterior faces of the large cube are $-1, 2$ and 0, and thus have the largest possible sum of 1.
Therefore, the 8 corner cubes contribute $8 \times 1 = 8$ to S.

Each edge cube has 2 faces which contribute to S. For S to be as large as possible, the $1 \times 1 \times 1$ cube must be positioned in such a way that the sum of the 2 external faces is as large as possible.
We may determine from the given net that two faces which share an edge of the $1 \times 1 \times 1$ cube contain the numbers: $-1, 0$ or $-1, 2$ or $-1, 1$ or $2, 0$ or $1, 0$ or $-2, 0$ or $-2, 1$ or $-2, 2$.
The sums of these two adjacent faces are $-1, 1, 0, 2, 1, -2, -1$, and 0 respectively.
To make S as large as possible, we choose to place each edge cube so that the numbers appearing on the exterior faces of the large cube are 2 and 0, and thus have the largest possible sum of 2.
Therefore, the $12 \times (n-2)$ edge cubes contribute $2 \times 12 \times (n-2)$ or $24 \times (n-2)$ to S.

Finally, each inside cube has 1 face which contributes to S. For S to be as large as possible, we position each of these cubes so that 2 will appear on this face (since 2 is the largest number in the given net).
Thus, the $6 \times (n-2)^2$ inside cubes contribute $2 \times 6 \times (n-2)^2$ or $12 \times (n-2)^2$ to S.

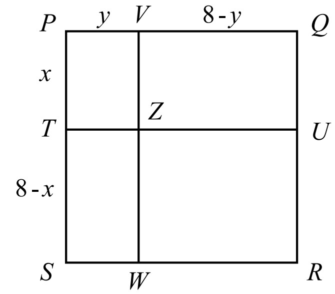
In total, we get $S = 8 + 24 \times (n-2) + 12 \times (n-2)^2$.
We want the smallest value of n for which $S > 1500$ or $8 + 24 \times (n-2) + 12 \times (n-2)^2 > 1500$.
Using trial and error with the given answers, we get the following:

- When $n = 9$, $S = 8 + 24 \times (n-2) + 12 \times (n-2)^2 = 8 + 24 \times 7 + 12 \times 7^2 = 764$ which is less than 1500.
- When $n = 11$, $S = 8 + 24 \times 9 + 12 \times 9^2 = 1196$ which is less than 1500.
- When $n = 12$, $S = 8 + 24 \times 10 + 12 \times 10^2 = 1448$ which is less than 1500.
- When $n = 13$, $S = 8 + 24 \times 11 + 12 \times 11^2 = 1724$ which is greater than 1500.

The smallest value of n for which the sum of the exterior faces of the $n \times n \times n$ cube is greater than 1500 is 13.

ANSWER: (D)

25. We begin by drawing a well-labelled diagram, as shown.
Since $PS = 8$, by letting $PT = x$ we get $TS = 8 - x$. Similarly if $PV = y$, then $VQ = 8 - y$.
The value of x, where $0 < x < 8$, uniquely determines the position of line segment TU (which is parallel to PQ).
The value of y, where $0 < y < 8$, uniquely determines the position of line segment VW (which is parallel to QR).
Thus, the value of N is equal to the number of ordered pairs (x, y) which give integer areas for each of the four rectangles $PVZT$, $TZWS$, $VQUZ$, and $ZURW$.



Since the areas of rectangles $PVZT$ and $VQUZ$ are integers, then the area of rectangle $PQUT$ is given by the sum of two integers and thus is also an integer.
The area of $PQUT$ is equal to $(PQ)(PT) = 8x$ and so $8x$ is an integer.
Similarly, the area of $PVWS$ is an integer and so $8y$ is an integer.
For what values of x is $8x$ an integer?
If x is an integer, then $8x$ is an integer and so x can equal any integer from 1 to 7 inclusive.
Similarly, y can equal any integer from 1 to 7 inclusive.
For each of the 7 possible values of x, there are 7 possible values for y, and so in this case where each of x and y is an integer, there are $7 \times 7 = 49$ ordered pairs (x, y) which give integer areas for each of the four rectangles $PVZT$, $TZWS$, $VQUZ$, and $ZURW$.
As an example, consider $(x, y) = (2, 3)$.
In this case, $8 - x = 6$, $8 - y = 5$ and the areas of the four rectangles $PVZT$, $TZWS$, $VQUZ$, and $ZURW$ are the integers $2 \times 3 = 6$, $6 \times 3 = 18$, $2 \times 5 = 10$, and $6 \times 5 = 30$, respectively.

Having considered the cases in which both x and y are integers, next we will consider the possible cases in which exactly one of x and y is an integer.
Specifically, we will assume that y is an integer and x is not an integer.

Are there non-integer values of x for which $8x$ is an integer?
When $x = \frac{1}{2}$, $8x = 8 \times \frac{1}{2} = 4$ and thus there exist fractional values of x for which $8x$ is an integer.
Let $x = \frac{a}{b}$ where a and b are positive integers and a and b have no factors in common (that is, $\frac{a}{b}$ is in lowest terms).
Since $8x = 8 \times \frac{a}{b}$ is an integer exactly when b is a positive divisor of 8, then b could equal $1, 2, 4$, or 8.
Next, we consider each of these possible values for b.

When $b = 1$, $x = \frac{a}{b} = \frac{a}{1} = a$ and so x is an integer.
Integer values of both x and y is the case previously considered in which we determined there are 49 ordered pairs (x, y) that satisfy the given conditions.

Next, we consider the cases for which $b = 2$.
Since a and b have no common factors, then a must be odd.
Further, $0 < \frac{a}{b} < 8$ and so a can be $1, 3, 5, 7, 9, 11, 13$, or 15 (note that $\frac{17}{2} > 8$).
When x is equal to $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}$, or $\frac{15}{2}$, what are the possible values of y?
Recall that we are considering cases in which exactly one of x and y is an integer and thus we require y to be an integer.
The area of rectangle $PVZT$ is an integer and so $(PV)(PT) = xy$ is an integer. (We note that the area of each of the other 3 rectangles is also an integer.)
When x is equal to a fraction of the form $\frac{a}{2}$ (where a is odd and comes from the list above),

and xy is an integer, then 2 is a factor of y and so y is an even integer.
Since $0 < y < 8$, then y can be $2, 4$ or 6.
For each of the 8 choices for x, there are 3 choices for y and so in this case there are $8 \times 3 = 24$ ordered pairs (x, y) which give integer areas for each of the four rectangles $PVZT$, $TZWS$, $VQUZ$, and $ZURW$.
As an example, consider $(x, y) = (\frac{3}{2}, 6)$. In this case, $8 - x = \frac{13}{2}$, $8 - y = 2$ and the areas of the four rectangles $PVZT$, $TZWS$, $VQUZ$, and $ZURW$ are the integers $\frac{3}{2} \times 6 = 9$, $\frac{13}{2} \times 6 = 39$, $\frac{3}{2} \times 2 = 3$, and $\frac{13}{2} \times 2 = 13$, respectively.
Recall from earlier that $8y$ is also an integer.
Thus, each of the possible values of x is a possible value for y, and vice versa.
That is, y can be equal to $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}$, or $\frac{15}{2}$, which gives x equal to 2,4 or 6, and 24 additional ordered pairs (x, y).

To summarize to this point, if both x and y are integers, there are 49 ordered pairs (x, y) which satisfy the given conditions.
If x is a fraction of the form $\frac{a}{2}$ for odd integers $1 \leq a \leq 15$ and y is an integer, there are 24 ordered pairs (x, y) which satisfy the given conditions.
If y is a fraction of the form $\frac{a}{2}$ for odd integers $1 \leq a \leq 15$ and x is an integer, there are 24 ordered pairs (x, y) which satisfy the given conditions.

Next, we consider the cases for which $b = 4$.
Since a and b have no common factors, then a must be odd.
Further, $0 < \frac{a}{b} < 8$ and so a can be $1, 3, 5, \ldots, 27, 29$, or 31 (note that $\frac{33}{4} > 8$).
When x is equal to $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \ldots \frac{27}{4}, \frac{29}{4}$, or $\frac{31}{4}$, what are the possible values of y?
Recall that we are considering cases in which exactly one of x and y is an integer and thus we require y to be an integer.
Since xy is an integer and x is equal to a fraction of the form $\frac{a}{4}$, then 4 is a factor of y.
Since $0 < y < 8$, then $y = 4$.
For each of the 16 choices for x, there is 1 choice for y and so in this case there are $16 \times 1 = 16$ ordered pairs (x, y) which give integer areas for each of the four rectangles $PVZT$, $TZWS$, $VQUZ$, and $ZURW$.
As an example, consider $(x, y) = (\frac{5}{4}, 4)$.
In this case, $8 - x = \frac{27}{4}$, $8 - y = 4$ and the areas of the four rectangles $PVZT$, $TZWS$, $VQUZ$, and $ZURW$ are the integers $\frac{5}{4} \times 4 = 5$, $\frac{27}{4} \times 4 = 27$, $\frac{5}{4} \times 4 = 5$, and $\frac{27}{4} \times 4 = 27$, respectively.
When y is equal to $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \ldots \frac{27}{4}, \frac{29}{4}$, or $\frac{31}{4}$, and $x = 4$, there are 16 additional ordered pairs (x, y).

To complete the cases for which exactly one of x and y is an integer, we consider $b = 8$.
Since xy is an integer and x is equal to a fraction of the form $\frac{a}{8}$, then 8 is a factor of y.
However, $y < 8$ and thus cannot have a factor of 8 and so there are no ordered pairs (x, y) when $b = 8$.

Finally, we consider cases in which both x and y are not integers.
As previously determined, if x is not an integer, then it is of the form $\frac{a}{2}$ or $\frac{a}{4}$ for some positive odd integers a.
If y is not an integer, then it is similarly of the form $\frac{a}{2}$ or $\frac{a}{4}$ for some positive odd integers a.
However, if x and y are each of this form, then their product xy is of the form $\frac{c}{4}$, $\frac{c}{8}$ or $\frac{c}{16}$ where c is the product of two odd integers and thus is odd.
Since it is not possible for 4, 8 or 16 to be a factor of an odd number, then xy cannot be an integer and so the area of rectangle $PVZT$ cannot be an integer in this case.
Thus, there are no cases for which both x and y are not integers.

Therefore, the value of N is equal to $49 + (2 \times 24) + (2 \times 16) = 49 + 48 + 32 = 129$, and so the remainder when $N^2 = 16\,641$ is divided by 100 is 41.

ANSWER: (D)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2020 Gauss Contests

(Grades 7 and 8)

Wednesday, May 13, 2020
(in North America and South America)

Thursday, May 14, 2020
(outside of North America and South America)

Solutions

Centre for Education in Mathematics and Computing Faculty and Staff

Ed Anderson
Jeff Anderson
Terry Bae
Jacquelene Bailey
Shane Bauman
Jenn Brewster
Ersal Cahit
Diana Castañeda Santos
Sarah Chan
Ashely Congi
Serge D'Alessio
Fiona Dunbar
Mike Eden
Sandy Emms
Barry Ferguson
Steve Furino
John Galbraith
Lucie Galinon
Robert Garbary
Rob Gleeson
Sandy Graham
Conrad Hewitt
Valentina Hideg
Angie Hildebrand
Carrie Knoll
Christine Ko
Judith Koeller
Laura Kreuzer
Bev Marshman
Paul McGrath
Jen Nelson
Ian Payne
J.P. Pretti
Alexandra Rideout
Nick Rollick
Kim Schnarr
Carolyn Sedore
Ashley Sorensen
Ian VanderBurgh
Troy Vasiga
Heather Vo
Bonnie Yi

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Robert Wong, Edmonton, AB
Chris Wu, Ledbury Park E. and M.S., North York, ON
Lori Yee, William Dunbar P.S., Pickering, ON

Grade 7

1. One pens costs \$2. The cost of 10 pens is $10 \times \$2 = \$20$.

   Answer: (E)

2. Beginning at the origin $(0, 0)$, point P is located right 2 units (in the positive x direction) and up 4 units (in the positive y direction).
   Thus the coordinates of point P are $(2, 4)$.

   Answer: (E)

3. Since 99 is close to 100 and 9 is close to 10, the value of 99×9 is approximately equal to $100 \times 10 = 1000$.
   Of the given answers, this means that the integer that is closest to 99×9 is most likely 1000.
   Multiplying, we get that the value of 99×9 is 891.
   Of the given answers, 99×9 is indeed closest to 1000.

   Answer: (D)

4. The increase in temperature equals the warmer afternoon temperature minus the colder morning temperature.
   The difference between 5 and -3 is $5 - (-3) = 5 + 3 = 8$.
   The temperature increased by 8°C.

   Answer: (A)

5. In April, Alexis averaged $243\,000 \div 30 = 8100$ steps per day.

   Answer: (B)

6. *Solution 1*
   One complete rotation equals 360°.
   Since 90° is $\frac{90^\circ}{360^\circ} = \frac{1}{4}$ of one complete rotation, then $\frac{1}{4}$ of all students chose juice.
   Therefore, the remaining $1 - \frac{1}{4} = \frac{3}{4}$ of all students chose milk.
   Since $\frac{3}{4}$ is $3 \times \frac{1}{4}$, then $3 \times 80 = 240$ students chose milk.

   Solution 2
   As in Solution 1, $\frac{1}{4}$ of all students chose juice.
   Since the 80 students who chose juice represent $\frac{1}{4}$ of the total number of students, then the total number of students is $4 \times 80 = 320$.
   Therefore, $320 - 80 = 240$ students chose milk.

   Answer: (C)

7. Since the third and fourth numbers in the list are consecutive and add to 11, then the third number in the list is 5 and the fourth is 6.
   The fifth number in the list is 7, and so the sixth number in the list is 8.

   Answer: (D)

8. *Solution 1*
   Between 0 and 1.0, the number line is divided by tick marks (at P, Q, R) into 4 equal lengths.
   Thus, the distance between adjacent tick marks is $\frac{1.0-0}{4} = 0.25$.
   Since R is the 3^{rd} tick mark moving right from 0, the value of R is $3 \times 0.25 = 0.75$.
   Since U is the 6^{th} tick mark moving right from 0, the value of U is $6 \times 0.25 = 1.5$.
   The value of R divided by the value of U equals $\frac{0.75}{1.5} = 0.5$.

Solution 2
The number R is located at the 3^{rd} tick mark to the right of 0.
The number U is located at the 6^{th} tick mark to the right of 0.
The tick marks are equally spaced along the number line, and so the value of R must be half the value of U (since 3 is half of 6).
The value of R divided by the value of U equals $\frac{1}{2}$ or 0.5.

ANSWER: (B)

9. The triangle is isosceles, and so the missing side length is also 12 cm.
The perimeter of the triangle is $14 + 12 + 12 = 38$ cm.
The perimeter of the rectangle is given by $x + 8 + x + 8 = 2x + 16$ cm.
The perimeter of the triangle is equal to the perimeter of the rectangle, and so $2x + 16 = 38$.
Since $22 + 16 = 38$, then $2x = 22$ and so $x = 11$.

ANSWER: (C)

10. In the table below, we list the divisors of each of the given answers (other than the number itself) and determine their sum.

Given Answers	Divisors	Sum of the Divisors
8	$1, 2, 4$	$1 + 2 + 4 = 7$
10	$1, 2, 5$	$1 + 2 + 5 = 8$
14	$1, 2, 7$	$1 + 2 + 7 = 10$
18	$1, 2, 3, 6, 9$	$1 + 2 + 3 + 6 + 9 = 21$
22	$1, 2, 11$	$1 + 2 + 11 = 14$

The sum of the divisors of each of $8, 10, 14$, and 22 is less than the number itself.
Each of these four answers is not an abundant number. (These numbers are called "deficient".)
The sum of the divisors of 18 is 21, which is greater than 18.
Thus 18 is an abundant number.

ANSWER: (D)

11. Each of 7 boxes contains exactly 10 cookies, and so the total number of cookies is $7 \times 10 = 70$.
If the cookies are shared equally among 5 people, then each person receives $70 \div 5 = 14$ cookies.

ANSWER: (A)

12. Abdul is 9 years older than Susie, and Binh is 2 years older than Susie, and so Abdul is $9 - 2 = 7$ years older than Binh.
For example, if Susie is 10 years old, then Abdul is $9 + 10 = 19$, Binh is $2 + 10 = 12$, and Abdul is $19 - 12 = 7$ years older than Binh.

ANSWER: (E)

13. The y-coordinates of points $P(15, 55)$ and $Q(26, 55)$ are equal.
Therefore, the distance between P and Q is equal to the positive difference between their x-coordinates, or $26 - 15 = 11$.
Similary, the x-coordinates of points $R(26, 35)$ and $Q(26, 55)$ are equal.
Therefore, the distance between R and Q is equal to the positive difference between their y-coordinates, or $55 - 35 = 20$.
Since $PQ = 11$ and $RQ = 20$, the area of rectangle $PQRS$ is $11 \times 20 = 220$.

ANSWER: (C)

14. Before Jack eats any jelly beans, the box contains $15 + 20 + 16 = 51$ jelly beans.
After Jack eats 2 jelly beans, there are $51 - 2 = 49$ jelly beans remaining in the box.
One of the jelly beans that Jack ate was green, and the other was blue.
Thus after eating the 2 jelly beans, there are still 15 red jelly beans in the box.
If each of the remaining jelly beans is equally likely to be chosen, the probability that Jack chooses a red jelly bean next is $\frac{15}{49}$.

ANSWER: (C)

15. *Solution 1*

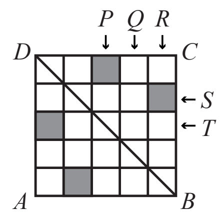
There are 60 minutes in 1 hour, and so there are $60 + 52 = 112$ minutes in 1 hour 52 minutes.
If Emil's race time was 54 minutes, then Olivia's race time was 4 minutes more, or 58 minutes.
In this case, their race times total $54 + 58 = 112$ minutes, as required.
Therefore, it took Olivia 58 minutes to run the race.

Solution 2

As in Solution 1, the total of their race times is 112 minutes.
If Emil's race time was 4 minutes more, then his race time would be equal to Olivia's, and the total of their race times would be $112 + 4 = 116$ minutes.
If their race times are equal and total 116 minutes, then they each finished the race in $116 \div 2 = 58$ minutes.
Therefore, it took Olivia 58 minutes to run the race.

Answer: (C)

16. For BD to be a line of symmetry of $ABCD$, the squares labelled P and S should be shaded.



To see why this is, consider that the reflection in BD of vertex A is vertex C, and so the reflection in BD of side DA is side DC.
Further, the reflection in BD of the first column of square $ABCD$ is the first row of square $ABCD$.
Thus, the reflection in BD of the shaded square in row 3, column 1 is the square in row 1, column 3 (the square labelled P).
More generally, the reflection in BD of the square in row r, column c is the square in row c, column r.
Thus, the reflection in BD of the shaded square in row 5, column 2 is the square in row 2, column 5 (the square labelled S).

Answer: (A)

17. If Rosie deposits \$30 each month for m months, she will save $30m$ dollars.
Rosie has \$120 in her account today, and so after m deposits Rosie's total savings (in dollars) is best represented by the expression $120 + 30m$.

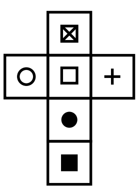
Answer: (E)

18. Isosceles triangles have two equal angles, and so the possibilities for these two triangles are:
1) The two equal angles are each equal to $70°$, or
2) The two equal angles are each not equal to $70°$.
(We note that a triangle can not have three angles measuring $70°$ since the sum of the three angles would be $3 \times 70° = 210°$, which is greater than $180°$.)
If the two equal angles are each equal to $70°$, then the measure of the third angle is $180° - 2 \times 70° = 180° - 140° = 40°$.
If the two equal angles are each not equal to $70°$, then the sum of the measures of the two equal angles is $180° - 70° = 110°$, and so the measure of each of the equal angles is half of $110°$ or $55°$.

We note that in the first triangle, the measure of each of the two remaining angles (70° and 40°) is even, and in the second triangle, the measure of each of the two remaining angles (55° and 55°) is odd.
In the first triangle, the sum of the two equal angles is $S = 70^\circ + 70^\circ = 140^\circ$.
In the second triangle, the sum of the two equal angles is $T = 55^\circ + 55^\circ = 110^\circ$.
The value of $S + T$ is $140^\circ + 110^\circ = 250^\circ$.

ANSWER: (B)

19. We begin by recognizing that there are 6 different symbols, and so each face of the cube contains a different symbol.
From left to right, let us number the views of the cube 1, 2 and 3.
Views 1 and 2 each show a face containing the symbol ⊠ .
What symbol is on the face opposite to the face containing ⊠?
In view 1, □ and ○ are on faces adjacent to the face containing ⊠ , and so neither of these can be the symbol that is on the face opposite ⊠ .
In view 2, ■ and + are on faces adjacent to the face containing ⊠ , and so neither of these can be the symbol that is on the face opposite ⊠ .
There is only one symbol remaining, and so ● must be the symbol that is on the face opposite ⊠, and vice versa.
A net of the cube is shown below.



ANSWER: (C)

20. On each of her four tosses of the coin, Jane will either move up one dot or she will move right one dot.
Since Jane has two possible moves on each of her four tosses of the coin, she has a total of $2 \times 2 \times 2 \times 2 = 16$ different paths that she may take to arrive at one of P, Q, R, S, or T.
If we denote a move up one dot by U, and a move right one dot by R, these 16 paths are: $UUUU$, $UUUR$, $UURU$, $UURR$, $URUU$, $URUR$, $URRU$, $URRR$, $RUUU$, $RUUR$, $RURU$, $RURR$, $RRUU$, $RRUR$, $RRRU$, $RRRR$.
The probability of tossing a head (and thus moving up one dot) is equal to the probability of tossing a tail (and thus moving right one dot).
That is, it is equally probable that Jane will take any one of these 16 paths.
Therefore, the probability that Jane will finish at dot R is equal to the number of paths that end at dot R divided by the total number of paths, 16.
How many of the 16 paths end at dot R?
Beginning at A, a path ends at R if it has two moves up (two U's), and two moves right (two R's).
There are 6 such paths: $UURR$, $URUR$, $URRU$, $RUUR$, $RURU$, $RRUU$.
(We note that each of the other 10 paths will end at one of the other 4 dots, P, Q, S, T.)
After four tosses of the coin, the probability that Jane will be at dot R is $\frac{6}{16} = \frac{3}{8}$.

ANSWER: (B)

21. Each four-digit number must be greater than 2000, and so the smallest two-digit number that may be repeated is 20 (to give 2020).
Each four-digit number must be less than 10 000, and so the largest two-digit number that may be repeated is 99 (to give 9999).
Each two-digit number between 20 and 99 may be repeated to give a four-digit number between 2000 and 10 000.
In total, there are $99 - 20 + 1 = 80$ numbers that satisfy the given conditions.

Answer: (A)

22. Celyna spent \$5.00 on candy A and \$7.00 on candy B, or \$12.00 in total.
The average price of all the candy that she purchased was \$1.50 per 100 grams.
This means that if Celyna bought 100 grams of candy, she would have spent \$1.50.
If she bought 200 grams of candy, she would have spent $2 \times \$1.50 = \$3.00$.
How many grams of candy would Celyna need to buy to spend \$12.00?
Since $8 \times \$1.50 = \$12.00$ (or $\$12.00 \div \$1.50 = 8$), then she would need to buy a total of 800 grams of candy.
Celyna bought 300 grams of candy A, and so she must have purchased $800 - 300 = 500$ grams of candy B. The value of x is 500.

Answer: (C)

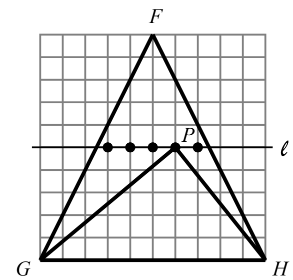
23. If the first positive integer in the list is a and the second is b, then the third integer is $a + b$, the fourth is $b + (a + b)$ or $a + 2b$, and the fifth is $(a + b) + (a + 2b)$ or $2a + 3b$.
Thus, we are asked to find the number of pairs of positive integers a and b, where a is less than b (since the list is increasing), and for which $2a + 3b = 124$.
What is the largest possible value for b?
If $b = 42$, then $3b = 3 \times 42 = 126$ which is too large since $2a + 3b = 124$. (Note that a larger value of b makes $3b$ even larger.)
If $b = 41$, then $3b = 3 \times 41 = 123$.
However in this case, we get that $2a = 124 - 123 = 1$, which is not possible since a is a positive integer.
If $b = 40$, then $3b = 3 \times 40 = 120$ and so $2a = 4$ or $a = 2$.
Thus, the largest possible value for b is 40.
What is the smallest value for b?
If $b = 26$, then $3b = 3 \times 26 = 78$ and so $2a = 124 - 78 = 46$ or $a = 23$.
If $b = 25$, then $3b = 3 \times 25 = 75$.
However in this case, we get that $2a = 124 - 75 = 49$, which is not possible since a is a positive integer.
If $b = 24$, then $3b = 3 \times 24 = 72$ and so $2a = 124 - 72 = 52$ or $a = 26$.
However, if the first integer in the list is 26, then the second integer can not equal 24 since the list is increasing.
Smaller values of b will give larger values of a, and so the smallest possible value of b is 26.
From the values of b attempted thus far, we notice that when b is an odd integer, $3b$ is also odd (since the product of two odd integers is odd), and $124 - 3b$ is odd (since the difference between an even integer and an odd integer is odd).
So when b is odd, $124 - 3b$ is odd, and so $2a$ is odd (since $2a = 124 - 3b$).
However, $2a$ is even for every choice of the integer a and so b cannot be odd.
Conversely, when b is even, $124 - 3b$ is even (as required), and so all even integer values of b from 26 to 40 inclusive will satisfy the requirements.
These values of b are $26, 28, 30, 32, 34, 36, 38, 40$, and so there are 8 such lists of five integers that have 124 as the fifth integer.

Here are the 8 lists: $2, 40, 42, 82, 124$; $5, 38, 43, 81, 124$; $8, 36, 44, 80, 124$; $11, 34, 45, 79, 124$; $14, 32, 46, 78, 124$; $17, 30, 47, 77, 124$; $20, 28, 48, 76, 124$; $23, 26, 49, 75, 124$.

ANSWER: (E)

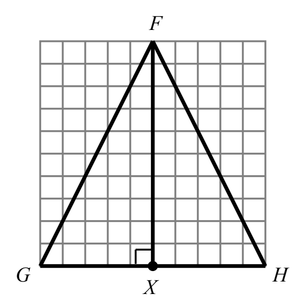
24. We begin by determining the area of $\triangle FGH$.
The base GH has length 10, the perpendicular height of the triangle from GH to F is also 10, and so the area of $\triangle FGH$ is $\frac{1}{2} \times 10 \times 10 = 50$.
We wish to determine which of the 41 points is a possible location for P so that $\triangle FPG$ or $\triangle GPH$ or $\triangle HPF$ has area $\frac{1}{2} \times 50 = 25$.

We begin by considering the possible locations for P so that $\triangle GPH$ has area 25.
The base GH has length 10, and so the perpendicular height from GH to P must be 5 (since $\frac{1}{2} \times 10 \times 5 = 25$).
Since the distance between two parallel lines remains constant, any point P lying on a line that is parallel to GH and that is 5 units from GH will give a $\triangle GPH$ whose area is 25.
The line labelled ℓ is parallel to GH and lies 5 units above GH, and so any point that lies on ℓ is a distance of 5 units from base GH.

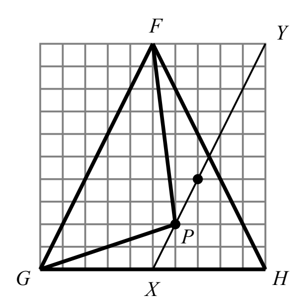


There are 5 points that lie on ℓ (that are at the intersection of gridlines) and that are inside $\triangle FGH$. These 5 points and one of the 5 possibilities for $\triangle GPH$ are shown.

Next, we consider the possible locations for P so that $\triangle FPG$ has area 25.
Consider the point X on GH so that FX is perpendicular to GH.
Since $\triangle FGH$ is isosceles, then FX divides the area of $\triangle FGH$ in half and so $\triangle FXG$ has area 25.
However, if X lies on GH, then X does not lie *inside* $\triangle FGH$.
For this reason, X is not a possible location for P, however it does provide some valuable information and insight.
If we consider the base of $\triangle FXG$ to be FG, then the perpendicular distance from X to FG is equal to the height required from base FG so that $\triangle FXG$ has area 25.



Any point P that lies on a line that is parallel to FG and that is the same distance from FG as X will give a $\triangle FPG$ whose area is also 25.
(This is the same property that we saw previously for $\triangle GPH$, with base GH.)
How do we create a line that passes through X and that is parallel to FG?
Beginning at G, if we move 5 units right and 10 units up we arrive at F.
Begin at X, move 5 units right and 10 units up, and call this point Y.
Can you explain why the line segment XY is parallel to FG?
Any point that lies on XY is a distance from base FG equal to the required height of $\triangle FPG$.
There are 2 points that lie on XY (that are at the intersection of gridlines) and that are inside $\triangle FGH$.



(We note that we may move right 5 and up 10 by moving in 'steps' of right 1 and up 2 to arrive at each of these 2 points.)
These 2 points and one possibility for $\triangle FPG$ are shown.

Finally, we consider the possible locations for P so that $\triangle HPF$ has area 25.
As a result of symmetry, this case is identical to the previous case.

Thus, there are 2 possible locations for P so that $\triangle HPF$ has area 25.
In total, there are $5+2+2=9$ triangles that have an area that is exactly half of the area of $\triangle FGH$.

ANSWER: (E)

25. Bus A takes 12 minutes to complete one round trip that begins and ends at P.
Since $PX = XS$, it takes Bus A $12 \div 4 = 3$ minutes to travel from P to X, 6 minutes to travel from X to S to X (3 minutes from X to S and 3 minutes from S to X), and 6 minutes to travel from X to P to X.
That is, Bus A first arrives at X at 1:03 and then continues to return to X every 6 minutes.
We write times that Bus A arrives at X in the table below.

Bus A	1:03	1:09	1:15	1:21	1:27	1:33	1:39	1:45	1:51	1:57	2:03

Notice that Bus A arrives at X at 1:03 and exactly one hour later at 2:03.
This makes sense since Bus A returns to X every 6 minutes and 60 minutes (one hour) is divisible by 6.
This tells us that Bus A will continue to arrive at the same number of minutes past each hour, or 2:03, 2:09, 2:15, ..., 3:03, 3:09, ..., 5:03, 5:09, ..., 9:03, 9:09, ..., 9:51, 9:57.

Bus B takes 20 minutes to complete one round trip that begins and ends at Q.
Since $QX = XT$, it takes Bus B $\frac{20}{4} = 5$ minutes to travel from Q to X, 10 minutes to travel from X to T to X (5 minutes from X to T and 5 minutes from T to X), and 10 minutes to travel from X to Q to X.
That is, Bus B first arrives at X at 1:05 and then continues to return to X every 10 minutes.
We write times that Bus B arrives at X in the table below.

Bus B	1:05	1:15	1:25	1:35	1:45	1:55	2:05

Notice that Bus B arrives at X at 1:05 and exactly one hour later at 2:05.
This makes sense since Bus B returns to X every 10 minutes and 60 minutes (one hour) is divisible by 10.
This tells us that Bus B will continue to arrive at the same number of minutes past each hour, or 2:05, 2:15, 2:25, ..., 3:05, 3:15, ..., 5:05, 5:15, ..., 9:05, 9:15, ..., 9:45, 9:55.
From the two tables above, we see that Bus A and Bus B both arrive at X at 15 minutes and 45 minutes past each hour.
Thus between 5:00 p.m. and 10:00 p.m., these two buses will meet $2 \times 5 = 10$ times at X.
These times are: 5:15, 5:45, 6:15, 6:45, 7:15, 7:45, 8:15, 8:45, 9:15, and 9:45.

Bus C takes 28 minutes to complete one round trip that begins and ends at R.
Since $RX = XU$, it takes Bus C $\frac{28}{4} = 7$ minutes to travel from R to X, $2 \times 7 = 14$ minutes to travel from X to U to X, and 14 minutes to travel from X to R to X.
That is, Bus C first arrives at X at 1:07 and then continues to return to X every 14 minutes.
Unlike Bus A and Bus B, Bus C will not arrive at X at consistent times past each hour since 60 is not divisible by 14.
What is the first time after 5:00 p.m. that Bus C arrives at X?
Since 238 is a multiple of 14 ($14 \times 17 = 238$), Bus C will arrive at X 238 minutes after first arriving at X at 1:07 p.m.
Since 238 minutes is 2 minutes less than 4 hours ($4 \times 60 = 240$), Bus C will arrive at X at 5:05 p.m. (This is the first time after 5:00 p.m. that Bus C arrives at X.)

Bus B also arrives at X at 5:05 p.m.
Are there other times after 5:05 p.m. (and before 10:00 p.m.) that Bus B and Bus C arrive at X at the same time?
Bus B arrives at X every 10 minutes and Bus C arrives at X every 14 minutes.
Since the lowest common multiple of 10 and 14 is 70, then Bus B and Bus C will each arrive at X every 70 minutes after 5:05 p.m., or at 6:15 p.m., 7:25 p.m., 8:35 p.m., and at 9:45 p.m.
Next we determine if there are times when Bus A and Bus C arrive at X at the same time.
Bus C arrives at X every 14 minutes after 5:05 p.m., or 5:19 p.m., 5:33 p.m., and so on.
Bus A also arrives at X at 5:33 p.m.
Are there other times after 5:33 p.m. (and before 10:00 p.m.) that Bus A and Bus C arrive at X at the same time?
Bus A arrives at X every 6 minutes and Bus C arrives at X every 14 minutes.
Since the lowest common multiple of 6 and 14 is 42, then Bus A and Bus C will each arrive at X every 42 minutes after 5:33 p.m., or at 6:15 p.m., 6:57 p.m., 7:39 p.m., 8:21 p.m., 9:03 p.m., and at 9:45 p.m.
The times when each pair of buses meet at X at the same time between 5:00 p.m. and 10:00 p.m. are listed below.
Bus A and Bus B: 15 and 45 minutes past each hour
Bus B and Bus C: 5:05 p.m., 6:15 p.m., 7:25 p.m., 8:35 p.m., 9:45 p.m.
Bus A and Bus C: 5:33 p.m., 6:15 p.m., 6:57 p.m., 7:39 p.m., 8:21 p.m., 9:03 p.m., 9:45 p.m.
Finally, we determine the number of different times that two or more buses arrive at X at the same time.
Bus A and Bus B arrive at X at 10 different times.
Bus B and Bus C arrive at X at 5 different times; however 2 of these times (6:15 p.m. and 9:45 p.m.) have already been counted, so there are 3 new times.
Bus A and Bus C arrive at X at 7 different times; however 2 of these times (6:15 p.m. and 9:45 p.m.) have already been counted, so there are 5 new times.
The number of times that two or more buses arrive at X between 5:00 p.m. and 10:00 p.m. is $10 + 3 + 5 = 18$.

Answer: (A)

Grade 8

1. Including 1 with the four given numbers and ordering the list, we get $-0.2, 0.03, 0.76, 1, 1.5$. Thus, there are 3 numbers in the list which are less than 1 (these are $-0.2, 0.03$ and 0.76).

   ANSWER: (D)

2. If the total cost of 4 one-litre cartons of milk is \$4.88, then the cost of 1 one-litre carton of milk is $\$4.88 \div 4 = \$1.22$.

   ANSWER: (E)

3. Of the answers given, $\frac{12}{2} = 6$ is the only fraction which is equal to a whole number.

   ANSWER: (E)

4. Since $x + y = 0$, then x and y must be opposites.
   Since $x = 4$, then $y = -4$, and we may check that $4 + (-4) = 0$.

   ANSWER: (E)

5. The length of the base is the distance from the origin to the point $(4, 0)$ or 4.
   The height is the distance from the origin to the point $(0, 6)$ or 6.
   Thus, the area of the triangle is $\frac{1}{2} \times 4 \times 6 = 12$.
   (We note that the base and height are perpendicular.)

   ANSWER: (A)

6. The whole numbers between 2 and 20 whose square root is a whole number are: $4, 9, 16$ (since $\sqrt{4} = 2, \sqrt{9} = 3, \sqrt{16} = 4$).
   We may note that perfect squares are equal to the square of an integer.
   That is, the smallest five positive perfect squares are $1^2 = 1, 2^2 = 4, 3^2 = 9, 4^2 = 16$ and $5^2 = 25$.
   Of these five, there are three that are between 2 and 20.

   ANSWER: (D)

7. *Solution 1*
   For each of the 4 different notebooks that Yvonn may choose, there are 5 different pens that may be chosen. Thus, there are $4 \times 5 = 20$ possible combinations of notebooks and pens that he could bring to class.

   Solution 2
   We begin by naming the 4 notebooks A, B, C, D, and numbering the 5 pens $1, 2, 3, 4, 5$.
   If Yvonn chooses notebook A and pen number 1, then we may call this combination $A1$.
   If Yvonn chooses notebook A, he may instead choose one of the pens numbered $2, 3, 4$, or 5.
   If Yvonn chooses notebook A, he has a total of 5 possible combinations: $A1, A2, A3, A4, A5$.
   Similarly, Yvonn has 5 pen choices for each choice of notebook.
   These remaining possible combinations are: $B1, B2, B3, B4, B5, C1, C2, C3, C4, C5, D1, D2, D3, D4, D5$.
   Thus, there are 20 possible combinations of notebooks and pens that Yvonn could bring to class.

   ANSWER: (C)

8. We begin by recognizing that a right angle is marked in the pie chart.
   Since 90° is $\frac{1}{4}$ of a full (360°) rotation, then $\frac{1}{4}$ of the students chose apples as their favourite fruit. If $\frac{1}{4}$ of the students chose apples, then the remaining $1 - \frac{1}{4} = \frac{3}{4}$ of the students chose bananas. If 168 students represent $\frac{3}{4}$ of all students, then $\frac{1}{4}$ of all students is $168 \div 3 = 56$ students. Thus, 56 students chose apples as their favourite fruit.

   ANSWER: (B)

9. There are 8 letters in the bag and 2 of these letters are B's.
If Elina randomly chooses one of the 8 letters, then the probability that she chooses a B is $\frac{2}{8} = \frac{1}{4}$.

Answer: (A)

10. Balil's result, b, is 5 more than Vita's number.
Cali's result, c, is 5 less than Vita's number.
Thus the difference between Balil's result and Cali's result, $b - c$, is 10.
For example, if Vita chooses 8, then $b = 8 + 5 = 13$ and $c = 8 - 5 = 3$ and $b - c = 13 - 3 = 10$.

Answer: (E)

11. The bus stops at the library at 1:00 p.m., 1:20 p.m., and 1:40 p.m..
The bus stops at the library at 2:00 p.m., 2:20 p.m., and 2:40 p.m..
Similarly, the bus stops at the library at x:00 p.m., x:20 p.m., and x:40 p.m. for $x = 3, 4, 5$.
Finally, the bus stops at 6:00 p.m..
In total, the bus stops at the library $3 \times 5 + 1 = 16$ times.

Answer: (A)

12. *Solution 1*
Beginning with the units (ones) column, the units digit of the sum $R + R$ is 2.
Thus, $R = 1$ or $R = 6$ are the only possibilities ($1 + 1 = 2$ and $6 + 6 = 12$ have units digit 2).
If $R = 1$, then there is no 'carry' from the units column to the tens column.
In this case, the units digit of the sum $Q + Q$ is 1.
This is not possible since $Q + Q = 2Q$ which is an even number for all possible digits Q.
Thus, $R = 6$, which gives $R + R = 12$, and so the 'carry' from the units column to the tens column is 1.
In the tens column the units digit of the sum $Q + Q + 1$ is 1, and so the units digit of $Q + Q$ is 0. Thus, $Q = 0$ or $Q = 5$ are the only possibilities ($0 + 0$ and $5 + 5$ have units digit 0).
If $Q = 0$, then there is no 'carry' from the tens column to the hundreds column.
This is not possible since the sum of the hundreds column is 10 and P is a digit and so cannot be 10.
Thus, $Q = 5$, which gives $Q + Q + 1 = 11$, and so the 'carry' from the tens column to the hundreds column is 1.
The sum of the hundreds column is 10, and so $P + 1 = 10$ or $P = 9$.
The value of $P + Q + R$ is $9 + 5 + 6 = 20$.

Solution 2
QR is a 2-digit number and thus less than 100.
The sum of PQR and QR is greater than 1000, and so PQR must be greater than 900 which means that $P = 9$.
Since the sum of the hundreds column is 10, the 'carry' from the tens column must be 1.
Considering the tens and units (ones) columns together, we see that the result of $QR + QR$ has units (ones) digit 2, tens digit 1, and hundreds digit 1 (since there is a 'carry' of 1 to the hundreds column). That is, $QR + QR = 112$ and so $QR = 56$.
The value of $P + Q + R$ is $9 + 5 + 6 = 20$.

Answer: (E)

13. *Solution 1*
There are 60 minutes in 1 hour, and so there are $60 + 52 = 112$ minutes in 1 hour 52 minutes.
If Emil's race time was 54 minutes, then Olivia's race time was 4 minutes more, or 58 minutes.
In this case, their race times total $54 + 58 = 112$ minutes, as required.
Therefore, it took Olivia 58 minutes to run the race.

Solution 2
As in Solution 1, the total of their race times is 112 minutes.
If Emil's race time was 4 minutes more, then his race time would be equal to Olivia's, and the total of their race times would be $112 + 4 = 116$ minutes.
If their race times are equal and total 116 minutes, then they each finished the race in $116 \div 2 = 58$ minutes. Therefore, it took Olivia 58 minutes to run the race.

ANSWER: (C)

14. In $\triangle ABC$, $\angle ABC = 90^\circ$ and so by the Pythagorean Theorem, $BC^2 = 34^2 - 16^2 = 1156 - 256$ or $BC^2 = 900$ and so $BC = \sqrt{900} = 30$ m.
The perimeter of $ABCD$ is $16 + 30 + 16 + 30 = 92$ m.

ANSWER: (D)

15. If Francesca first chooses -4, then there is no integer that she may choose second to give a sum of 3 (the largest integer in the list is 6, and $-4 + 6 = 2$).
If she first chooses -3, she may then choose 6 to give a sum of 3.
If she first chooses -2, she may then choose 5 to give a sum of 3.
If she first chooses -1, she may then choose 4 second to give a sum of 3.
If she first chooses 0, she may then choose 3 to give a sum of 3.
If she first chooses 1, she may then choose 2 to give a sum of 3.
If Francesca first chooses 2 (or any integer larger than 2), then there is no integer that she may choose second to give a sum of 3 (the second integer must be larger than the first, and so the sum will be larger than 5).
There are 5 such pairs of integers that Francesca can choose so that the sum is 3.

ANSWER: (B)

16. Since $\triangle QRS$ is an isosceles right-angled triangle with $QR = SR$, then $\angle RQS = \angle RSQ = 45^\circ$.
Opposite angles are equal in measure, and so $\angle SUV = \angle PUQ = y^\circ$ and $\angle SVU = \angle RVT = y^\circ$.
In $\triangle SVU$, $\angle VSU + \angle SUV + \angle SVU = 180^\circ$ or $45^\circ + y^\circ + y^\circ = 180^\circ$ or $2y = 135$ and so $y = 67.5$.

ANSWER: (C)

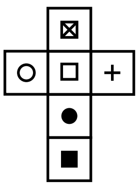
17. In a list of five numbers ordered from smallest to largest, the median is equal to the third (middle) number in the list.
Since x and y must be integers, the values of x and y must belong to exactly one of the following three possibilities:

- each of x and y is less than or equal to 11, or
- each of x and y is greater than or equal to 13, or
- at least one of x or y is equal to 12.

If each of x and y is less than or equal to 11, then the median number in the list is 11.
(In this case, the ordered list could be $x, y, 11, 12, 13$ or $y, x, 11, 12, 13$.)
If each of x and y is greater than or equal to 13, then the median number in the list is 13.
(In this case, the ordered list could be $11, 12, 13, x, y$ or $11, 12, 13, y, x$.)
If at least one of x or y is equal to 12, then the list includes $11, 12, 12, 13$ and one other number.
When the list is ordered, the two 12s will either be the 2nd and 3rd numbers in the list, or the 3rd and 4th numbers in the list depending on whether the unknown number is less than or equal to 12 or greater than or equal to 12.
In either case, the median is 12.
Thus, there are three different possible medians for Mark's five point totals.

ANSWER: (C)

18. We begin by recognizing that there are 6 different symbols, and so each face of the cube contains a different symbol.
From left to right, let us number the views of the cube 1, 2 and 3.
Views 1 and 2 each show a face containing the symbol ⊠ .
What symbol is on the face opposite to the face containing ⊠?
In view 1, □ and ○ are on faces adjacent to the face containing ⊠ , and so neither of these can be the symbol that is on the face opposite ⊠ .
In view 2, ■ and + are on faces adjacent to the face containing ⊠ , and so neither of these can be the symbol that is on the face opposite ⊠ .
There is only one symbol remaining, and so ● must be the symbol that is on the face opposite ⊠, and vice versa.
A net of the cube is shown below.



Answer: (C)

19. X is 20% of 50, and so $X = 0.20 \times 50 = 10$.
20% of 100 is 20, and so 20% of 200 is 40. Thus $Y = 200$.
40 is Z% of 50, and so $Z = \frac{40}{50} \times 100 = 80$.
(We may check that 80% of 50 is indeed $0.80 \times 50 = 40$.)
Therefore, $X + Y + Z = 10 + 200 + 80$ or $X + Y + Z = 290$.

Answer: (D)

20. We begin by expressing $\frac{20}{19}$ in a form that is similar to the right side of the given equation.
Converting $\frac{20}{19}$ to a mixed fraction we get, $\frac{20}{19} = 1\frac{1}{19} = 1 + \frac{1}{19}$.
Since $\frac{20}{19} = 1 + \dfrac{1}{1 + \frac{a}{b}}$ and $\frac{20}{19} = 1 + \frac{1}{19}$, then $1 + \dfrac{1}{1 + \frac{a}{b}} = 1 + \frac{1}{19}$ and so $\dfrac{1}{1 + \frac{a}{b}} = \frac{1}{19}$.
The numerators of $\dfrac{1}{1 + \frac{a}{b}}$ and $\frac{1}{19}$ are each equal to 1, and since these fractions are equal to one another, their denominators must also be equal.
That is, $1 + \frac{a}{b} = 19$ and so $\frac{a}{b} = 18$.
Since a and b are positive integers, then the fractions $\frac{a}{b}$ which are equal to 18 are $\frac{18}{1}$, $\frac{36}{2}$, $\frac{54}{3}$, and so on.
Thus, the least possible value of $a + b$ is $18 + 1 = 19$.

Answer: (B)

21. Originally, the ratio of green balls to yellow balls in the bag was $3 : 7$.
This means that for every 3 green balls in the bag, there were 7 yellow balls.
Equivalently, if there were $3n$ green balls, then there were $7n$ yellow balls where n is a positive integer.
After 9 balls of each colour are removed, the number of green balls in the bag is $3n - 9$ and the number of yellow balls is $7n - 9$.
At this point, the ratio of green balls to yellow balls is $1 : 3$, and so 3 times the number of green balls is equal to the number of yellow balls.
Multiplying the number of green balls by 3, we get $3 \times 3n - 3 \times 9$ or $9n - 27$ green balls.
Solving the equation $9n - 27 = 7n - 9$, we get $9n - 7n = 27 - 9$ or $2n = 18$, and so $n = 9$.

Originally, there were $3n$ green balls and $7n$ yellow balls, for a total of $3n + 7n = 10n$ which is $10 \times 9 = 90$ balls.
Note: If there were 90 balls, then 27 were green and 63 were yellow (since $27 : 63 = 3 : 7$ and $27 + 63 = 90$). After 9 balls of each colour are removed, the ratio of green balls to yellow balls becomes $18 : 54 = 1 : 3$, as required.

ANSWER: (B)

22. A number is divisible by 6 if it is divisible by both 2 and 3.
To be divisible by 2, the three-digit number that is formed must be even and so the ones digit must be 0 or 2.
To be divisible by 3, the sum of the digits of the number must be a multiple of 3.
Consider the possible tens and hundreds digits when the ones digit is 0.
In this case, the sum of the tens and hundreds digits must be a multiple of 3 (since the ones digit does not add anything to the sum of the digits).
We determine the possible sums of the tens and hundreds digits in the table below.
The sums which are a multiple of 3 are circled.

The Tens Digit

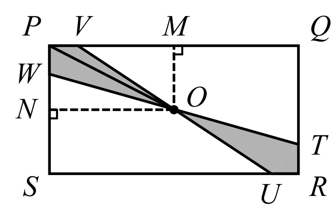
The Hundreds Digit

100s \ 10s	5	6	7	8
1	6	7	8	9
2	7	8	9	10
3	8	9	10	11
4	9	10	11	12

When the ones digit is 0, the possible three-digit numbers are: $150, 180, 270, 360, 450$, and 480.
Consider the possible tens and hundreds digits when the ones digit is 2.
In this case, the sum of the tens and hundreds digits must be 2 less than a multiple of 3 (since the ones digit adds 2 to the sum of the digits).
When the ones digit is 2, the possible three-digit numbers are: $162, 252, 282, 372$, and 462.
The number of three-digit numbers that can be formed that are divisible by 6 is 11.

ANSWER: (A)

23. We begin by joining the centre of the rectangle, O, to vertex P.
We also draw OM perpendicular to side PQ and ON perpendicular to side PS.
Since O is the centre of the rectangle, then M is the midpoint of side PQ and so $PM = \frac{1}{2} \times 4 = 2$.
Similarly, N is the midpoint of PS and so $PN = \frac{1}{2} \times 2 = 1$.



$\triangle PVO$ has base $PV = a$ and height $OM = 1$, and so has area $\frac{1}{2} \times a \times 1 = \frac{1}{2}a$.
$\triangle PWO$ has base $PW = a$ and height $ON = 2$, and so has area $\frac{1}{2} \times a \times 2 = a$.
Thus, quadrilateral $PWOV$ has area equal to the sum of the areas of these two triangles, or $\frac{1}{2}a + a = \frac{3}{2}a$.

Similarly, we can show that quadrilateral $RTOU$ also has area $\frac{3}{2}a$ and so the total area of the shaded region is $2 \times \frac{3}{2}a = 3a$.
The area of rectangle $PQRS$ is $4 \times 2 = 8$ and since the area of the shaded region is $\frac{1}{8}$ the area of $PQRS$, then $3a = \frac{1}{8} \times 8$ or $3a = 1$ and so $a = \frac{1}{3}$.

Answer: (D)

24. Bus A takes 12 minutes to complete one round trip that begins and ends at P.
Since $PX = XS$, it takes Bus A $12 \div 4 = 3$ minutes to travel from P to X, 6 minutes to travel from X to S to X (3 minutes from X to S and 3 minutes from S to X), and 6 minutes to travel from X to P to X.
That is, Bus A first arrives at X at 1:03 and then continues to return to X every 6 minutes.
We write times that Bus A arrives at X in the table below.

Bus A	1:03	1:09	1:15	1:21	1:27	1:33	1:39	1:45	1:51	1:57	2:03

Notice that Bus A arrives at X at 1:03 and exactly one hour later at 2:03.
This makes sense since Bus A returns to X every 6 minutes and 60 minutes (one hour) is divisible by 6.
This tells us that Bus A will continue to arrive at the same number of minutes past each hour, or 2:03, 2:09, 2:15, ..., 3:03, 3:09, ..., 5:03, 5:09, ..., 9:03, 9:09, ..., 9:51, 9:57.

Bus B takes 20 minutes to complete one round trip that begins and ends at Q.
Since $QX = XT$, it takes Bus B $\frac{20}{4} = 5$ minutes to travel from Q to X, 10 minutes to travel from X to T to X (5 minutes from X to T and 5 minutes from T to X), and 10 minutes to travel from X to Q to X.
That is, Bus B first arrives at X at 1:05 and then continues to return to X every 10 minutes.
We write times that Bus B arrives at X in the table below.

Bus B	1:05	1:15	1:25	1:35	1:45	1:55	2:05

Notice that Bus B arrives at X at 1:05 and exactly one hour later at 2:05.
This makes sense since Bus B returns to X every 10 minutes and 60 minutes (one hour) is divisible by 10.
This tells us that Bus B will continue to arrive at the same number of minutes past each hour, or 2:05, 2:15, 2:25, ..., 3:05, 3:15, ..., 5:05, 5:15, ..., 9:05, 9:15, ..., 9:45, 9:55.
From the two tables above, we see that Bus A and Bus B both arrive at X at 15 minutes and 45 minutes past each hour.
Thus between 5:00 p.m. and 10:00 p.m., these two buses will meet $2 \times 5 = 10$ times at X.
These times are: 5:15, 5:45, 6:15, 6:45, 7:15, 7:45, 8:15, 8:45, 9:15, and 9:45.

Bus C takes 28 minutes to complete one round trip that begins and ends at R.
Since $RX = XU$, it takes Bus C $\frac{28}{4} = 7$ minutes to travel from R to X, $2 \times 7 = 14$ minutes to travel from X to U to X, and 14 minutes to travel from X to R to X.
That is, Bus C first arrives at X at 1:07 and then continues to return to X every 14 minutes.
Unlike Bus A and Bus B, Bus C will not arrive at X at consistent times past each hour since 60 is not divisible by 14.
What is the first time after 5:00 p.m. that Bus C arrives at X?
Since 238 is a multiple of 14 ($14 \times 17 = 238$), Bus C will arrive at X 238 minutes after first arriving at X at 1:07 p.m.
Since 238 minutes is 2 minutes less than 4 hours ($4 \times 60 = 240$), Bus C will arrive at X at 5:05 p.m. (This is the first time after 5:00 p.m. that Bus C arrives at X.)

Bus B also arrives at X at 5:05 p.m.
Are there other times after 5:05 p.m. (and before 10:00 p.m.) that Bus B and Bus C arrive at X at the same time?
Bus B arrives at X every 10 minutes and Bus C arrives at X every 14 minutes.
Since the lowest common multiple of 10 and 14 is 70, then Bus B and Bus C will each arrive at X every 70 minutes after 5:05 p.m., or at 6:15 p.m., 7:25 p.m., 8:35 p.m., and at 9:45 p.m.
Next we determine if there are times when Bus A and Bus C arrive at X at the same time.
Bus C arrives at X every 14 minutes after 5:05 p.m., or 5:19 p.m., 5:33 p.m., and so on.
Bus A also arrives at X at 5:33 p.m.
Are there other times after 5:33 p.m. (and before 10:00 p.m.) that Bus A and Bus C arrive at X at the same time?
Bus A arrives at X every 6 minutes and Bus C arrives at X every 14 minutes.
Since the lowest common multiple of 6 and 14 is 42, then Bus A and Bus C will each arrive at X every 42 minutes after 5:33 p.m., or at 6:15 p.m., 6:57 p.m., 7:39 p.m., 8:21 p.m., 9:03 p.m., and at 9:45 p.m.
The times when each pair of buses meet at X at the same time between 5:00 p.m. and 10:00 p.m. are listed below.
Bus A and Bus B: 15 and 45 minutes past each hour
Bus B and Bus C: 5:05 p.m., 6:15 p.m., 7:25 p.m., 8:35 p.m., 9:45 p.m.
Bus A and Bus C: 5:33 p.m., 6:15 p.m., 6:57 p.m., 7:39 p.m., 8:21 p.m., 9:03 p.m., 9:45 p.m.
Finally, we determine the number of different times that two or more buses arrive at X at the same time.
Bus A and Bus B arrive at X at 10 different times.
Bus B and Bus C arrive at X at 5 different times; however 2 of these times (6:15 p.m. and 9:45 p.m.) have already been counted, so there are 3 new times.
Bus A and Bus C arrive at X at 7 different times; however 2 of these times (6:15 p.m. and 9:45 p.m.) have already been counted, so there are 5 new times.
The number of times that two or more buses arrive at X between 5:00 p.m. and 10:00 p.m. is $10 + 3 + 5 = 18$.

Answer: (A)

25. The property of an integer being either even or odd is called its *parity*.
If two integers are both even or they are both odd, then we say that the two integers have the *same parity*.
If one integer is even and a second integer is odd, then we say that the two integers have *different parity*.
The result of adding two integers that have the same parity is an even integer.
The result of adding two integers that have different parity is an odd integer.
The parity of each term of an FT sequence (after the second term) is determined by the parity of the first two terms in the sequence.
For example, if each of the first two terms of an FT sequence is odd, then the third term is even (since odd plus odd is even), the fourth term is odd (since odd plus even is odd), the fifth term is odd (since even plus odd is odd), and so on.
There are 4 possibilities for the parities of the first two terms of an FT sequence.
The sequence could begin odd, odd, or even, even, or odd, even, or even, odd.
In the table below, we write the parity of the first few terms of the FT sequences that begin in each of the 4 possible ways.

Term Number	1	2	3	4	5	6	7	8	9	10
Parity #1	odd	odd	even	odd	odd	even	odd	odd	even	odd
Parity #2	even	even	even	even	even	even	even	even	even	even
Parity #3	odd	even	odd	odd	even	odd	odd	even	odd	odd
Parity #4	even	odd	odd	even	odd	odd	even	odd	odd	even

The FT sequence beginning odd, odd (Parity #1) continues to repeat odd, odd, even.
Since the parity of each term is dependent on the parity of the two terms preceding it, this odd, odd, even pattern will continue throughout the entire sequence.
That is, in each successive group of three terms beginning at the first term, one out of three terms will be even and two out of three terms will be odd.
The odd, odd, even pattern ends at term numbers that are multiples of 3 (the even-valued terms are terms 3, 6, 9, 12, and so on).
Since 2019 is a multiple of 3 ($2019 = 3 \times 673$), $\frac{1}{3}$ of the first 2019 terms will be even-valued and $\frac{2}{3}$ will be odd-valued, and so there are twice as many odd-valued terms as there are even-valued terms in the first 2019 terms.
The 2020^{th} term is odd (since the pattern begins with an odd-valued term), and so there are more than twice as many odd-valued terms as there are even-valued terms in every FT sequence that begins odd, odd.
This is exactly the required condition for the FT sequences that we are interested in.
How many FT sequences begin with two odd-valued terms, each of which is a positive integer less than $2m$?
There are $2m-1$ positive integers less than $2m$ (these are $1, 2, 3, 4, \ldots, 2m-1$).
Since m is a positive integer, $2m$ is always an even integer and so $2m-1$ is always odd.
Thus, the list of integers from 1 to $2m-1$ begins and ends with an odd integer, and so the list contains m odd integers and $m-1$ even integers.
The first term in the sequence is odd-valued and so there are m choices for it.
Similarly, the second term in the sequence is also odd-valued and so there are also m choices for it.
Thus, there are a total of $m \times m$ or m^2 FT sequences that begin with two odd-valued terms.

Do any of the other 3 types of FT seqences satisfy the required condition that there are more than twice as many odd-valued terms as there are even-valued terms?
Clearly the FT sequence beginning even, even (Parity #2) does not satisfy the required condition since every term in the sequence is even-valued.
The FT sequence beginning odd, even (Parity #3) continues to repeat odd, even, odd.
That is, in each successive group of three terms beginning at the first term, one out of three terms will be even and two out of three terms will be odd.
As we saw previously, 2019 is a multiple of 3 and so $\frac{1}{3}$ of the first 2019 terms are even-valued and $\frac{2}{3}$ are odd-valued.
Thus there are twice as many odd-valued terms as there are even-valued terms in the first 2019 terms.
The 2020^{th} term is odd (since the pattern begins with an odd-valued term), and so there are more than twice as many odd-valued terms as there are even-valued terms in every FT sequence that begins odd, even.
How many FT sequences begin with an odd-valued first term and an even-valued second term, each being a positive integer less than $2m$?
As we showed previously, the list of integers from 1 to $2m-1$ begins and ends with an odd-valued integer, and so the list contains m odd-valued integers and $m-1$ even-valued integers.
The first term in the sequence is odd-valued and so there are m choices for it.

The second term in the sequence is even-valued and so there are $m-1$ choices for it.
Thus, there are a total of $m \times (m-1)$ FT sequences that begin with an odd-valued term followed by an even-valued term.

Finally, we consider the FT sequences that begin with an even-valued term followed by an odd-valued term (Parity #4).
Again, there are exactly twice as many odd-valued terms as there are even-valued terms in the first 2019 terms (since the pattern repeats even, odd, odd).
However in this case, the 2020^{th} term is even and so there are fewer than twice as many odd-valued terms as there are even-valued terms.

Thus, there are $m^2 + m \times (m-1)$ FT sequences that satisfy the required conditions.
Since there are 2415 such FT sequences, we may solve $m^2 + m \times (m-1) = 2415$ by trial and error.
Evaluating $m^2 + m \times (m-1)$ when $m = 30$, we get $30^2 + 30 \times 29 = 1770$, and so m is greater than 30.
When $m = 33$, we get $33^2 + 33 \times 32 = 2145$.
When $m = 34$, we get $34^2 + 34 \times 33 = 2278$.
When $m = 35$, we get $35^2 + 35 \times 34 = 2415$, as required.

ANSWER: (D)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2019 Gauss Contests

(Grades 7 and 8)

Wednesday, May 15, 2019
(in North America and South America)

Thursday, May 16, 2019
(outside of North America and South America)

Solutions

Centre for Education in Mathematics and Computing Faculty and Staff

Ed Anderson
Jeff Anderson
Terry Bae
Jacquelene Bailey
Shane Bauman
Jenn Brewster
Ersal Cahit
Sarah Chan
Serge D'Alessio
Rich Dlin
Fiona Dunbar
Mike Eden
Barry Ferguson
Brian Fernandes
Judy Fox
Carley Funk
Steve Furino
John Galbraith
Lucie Galinon
Robert Garbary
Melissa Giardina
Rob Gleeson
Sandy Graham
Conrad Hewitt
Angie Hildebrand
Carrie Knoll
Judith Koeller
Laura Kreuzer
Paul Leistra
Bev Marshman
Josh McDonald
Paul McGrath
Mike Miniou
Carol Miron
Dean Murray
Jen Nelson
Ian Payne
Anne Petersen
J.P. Pretti
Kim Schnarr
Carolyn Sedore
Ashley Sorensen
Ian VanderBurgh
Troy Vasiga
Heather Vo

Gauss Contest Committee

Ashley Sorensen (Chair), University of Waterloo, Waterloo, ON
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Kora Lee Gallant, Madeline Symonds M.S., Hammonds Plains, NS
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John McLoughlin, University of New Brunswick, Fredericton, NB
David Switzer, Scott Central P.S., Richmond Hill, ON
Rachael Verbruggen, University of Waterloo, Waterloo, ON
Laurissa Werhun, Parkdale C.I., Toronto, ON
Robert Wong, Vernon Barford School, Edmonton, AB
Chris Wu, Ledbury Park E. and M.S., Toronto, ON
Lori Yee, E.B. Phin P.S., Pickering, ON

Grade 7

1. Erin receives \$3 a day. To receive a total of \$30, it will take Erin $\frac{\$30}{\$3} = 10$ days.

   Answer: (E)

2. Beginning at the origin $(0,0)$, the point $(2,4)$ is located right 2 units and up 4 units.
   The point $(2,4)$ is located at D.

   Answer: (D)

3. One of four identical squares is shaded and so the fraction of square $PQRS$ that is shaded is $\frac{1}{4}$ which is equivalent to 25%.

   Answer: (C)

4. Adding, we get $0.9 + 0.09 = 0.99$.

   Answer: (D)

5. The mode is the amount of rainfall that occurs most frequently.
   Reading from the graph, the daily rainfall amounts for Sunday through to Saturday are 6 mm, 15 mm, 3 mm, 6 mm, 3 mm, 3 mm, and 9 mm.
   The mode for the amount of rainfall for the week is 3 mm.

   Answer: (C)

6. When $x = 3$,

$$\begin{array}{rcccc} 2x & = & 2\times 3 & = & 6 \\ 3x-1 & = & 3\times 3-1 & = & 8 \\ x+5 & = & 3+5 & = & 8 \\ 7-x & = & 7-3 & = & 4 \\ 6+2x & = & 6+2\times 3 & = & 12 \end{array}$$

   Of the given answers, $3x - 1 = 8$ is the only one of the given equations which is true when $x = 3$.

   Answer: (B)

7. When -37 and 11 are added, the result is $-37 + 11 = -26$. The correct answer is (A).
   Alternately, the result can be found by subtracting, $-26 - 11 = -37$.

   Answer: (A)

8. *Solution 1*
   One third of 396 is $396 \div 3 = 132$. Thus, Joshua has read 132 pages of the book.
   To finish the book, Joshua has $396 - 132 = 264$ pages left to read.
   Solution 2
   Joshua has read the first third of the book only, and so he has $1 - \frac{1}{3} = \frac{2}{3}$ of the book left to read.
   Two thirds of 396 is $396 \times \frac{2}{3} = \frac{396\times 2}{3} = \frac{792}{3} = 264$.
   Joshua has 264 pages left to read.

   Answer: (A)

9. One complete rotation equals 360°.
   Therefore, $k^\circ + 90^\circ = 360^\circ$ and so $k = 360 - 90 = 270$.

   Answer: (D)

10. *Solution 1*

The mean of the numbers $20, 30, 40$ is $\dfrac{20+30+40}{3} = \dfrac{90}{3} = 30$.
Since each of the given answers has three numbers, then for the mean to equal 30, the sum of the three numbers must also equal $30 \times 3 = 90$.
Of the given answers, only (D) has numbers whose sum is 90 ($23 + 30 + 37 = 90$).

Solution 2

Since 20 is 10 less than 30 and 40 is 10 more than 30, the mean of the numbers $20, 30, 40$ is 30.
In each of the given answers, 30 is the middle number in the list of three numbers.
Thus, for the mean of the three numbers to equal 30, the first and last numbers must be equal "distances" away from 30 (with one number being less than 30 and the other greater than 30).
Looking at answer (D), 23 is 7 less than 30 and 37 is 7 more than 30 and so the mean of these three numbers is 30. (We may check that this isn't the case for each of the other four answers.)

Answer: (D)

11. Evaluating, $\sqrt{81} = 9$ and $9 = 3^2$, so $\sqrt{81} = 3^2$.

Answer: (B)

12. The width of rectangle $PQRS$ is the horizontal distance between points P and Q (or points S and R), since these points have equal y-coordinates.
This distance is equal to the difference between their x-coordinates or $4 - (-4) = 8$.
Similarly, the height of rectangle $PQRS$ is equal the vertical distance between points S and P (or points R and Q), since these points have equal x-coordinates.
This distance is equal to the difference between their y-coordinates or $2 - (-2) = 4$.
The area of rectangle $PQRS$ is $8 \times 4 = 32$.

Answer: (B)

13. The repeating pattern of ABCDEFG has 7 white keys.
Since the first white key is A, the pattern repeats after each number of keys that is a multiple of 7.
Since 28 is a multiple of 7, then the 28^{th} white key is G and so the 29^{th} white key is A, the 30^{th} white key is B, the 31^{st} white key is C, the 32^{nd} white key is D, and the 33^{rd} white key is E.

Answer: (E)

14. On the spinner given, the prime numbers that are odd are $3, 5$ and 7.
Since the spinner is divided into 8 equal sections, the probability that the arrow stops in a section containing a prime number that is odd is $\frac{3}{8}$.

Answer: (C)

15. Barry's 12 coins include at least one of each of the 5 coins of different values.
The total value of these 5 coins is $\$2.00 + \$1.00 + \$0.25 + \$0.10 + \$0.05 = \$3.40$.
Barry has the smallest total amount of money that he could have if each of his remaining $12 - 5 = 7$ coins has value $\$0.05$ (the smallest possible value of a coin).
Thus, the smallest total amount of money that Barry could have is

$$\$3.40 + 7 \times \$0.05 = \$3.40 + \$0.35 = \$3.75.$$

Answer: (A)

16. *Solution 1*

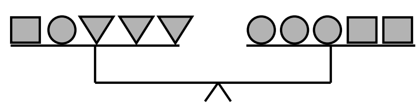
The are 10 palindromes between 100 and 200: $101, 111, 121, 131, 141, 151, 161, 171, 181$, and 191.
The are 10 palindromes between 200 and 300: $202, 212, 222, 232, 242, 252, 262, 272, 282$, and 292.
Similary, there are 10 palindromes between each of 300 and 400, 400 and 500, 500 and 600, 600 and 700, 700 and 800, 800 and 900, 900 and 1000.
That is, there are 10 palindromes between each of the 9 pairs of consecutive multiples of 100 from 100 to 1000.
The number of palindromes between 100 and 1000 is $10 \times 9 = 90$.

Solution 2

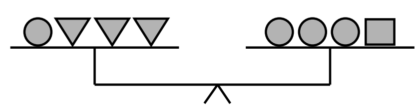
Each palindrome between 100 and 1000 is a 3-digit number of the form aba, where a is a digit from 1 to 9 inclusive and b is a digit from 0 to 9 inclusive, and a and b are not necessarily different digits.
Thus, there are 9 choices for the first digit a and each of those choices determines the third digit which must also be a.
There are 10 choices for the second digit b, and so there are $9 \times 10 = 90$ choices for a and b.
Thus, there are 90 palindromes between 100 and 1000.

Answer: (B)

17. The first scale shows that ○○○ has the same mass as ▽▽.
The second scale shows that □○▽ has the same mass as □□.
Therefore, the sum of the masses of ▽▽ and □○▽ is equal to the sum of the masses of ○○○ and □□.
That is, the equal-arm scale shown below is balanced.



The left and right sides of this scale each contain a □, and the scale remains balanced if one □ is removed from each side of the scale (since they are equal in mass).
That is, the equal-arm scale shown below is balanced.



Therefore of the answers given, ○▽▽▽ has the same mass as ○○○□.

Answer: (D)

18. The area of the rectangle with length x and width y is $x \times y$.
The area of the triangle with base 16 and height x is $\frac{1}{2} \times 16 \times x$ or $8 \times x$.
The area of the rectangle is equal to the area of the triangle, or $x \times y = 8 \times x$, and so $y = 8$.

Answer: (C)

19. *Solution 1*

Each of the four fractions is equal to the other three fractions, and each fraction has numerator 1, so then each denominator must be equal to the other three denominators.
That is, $a - 2 = b + 2 = c + 1 = d - 3$.
If we let $b = 4$, then $a - 2 = 4 + 2 = c + 1 = d - 3$ or $a - 2 = 6 = c + 1 = d - 3$, and so

$a = 8, c = 5$, and $d = 9$.
Thus, the correct ordering is $b < c < a < d$.

Solution 2
Each of the four fractions is equal to the other three fractions, and each fraction has numerator 1, so then each denominator must be equal to the other three denominators.
That is, $a-2 = b+2 = c+1 = d-3$, and by adding 3 to each, we get $a+1 = b+5 = c+4 = d$.
Since $d = a + 1$, then d is one more than a and so $a < d$.
Since $a + 1 = c + 4$, then a is 3 more than c and so $c < a$.
Since $c + 4 = b + 5$, then c is 1 more than b and so $b < c$.
Thus, the correct ordering is $b < c < a < d$.

Answer: (E)

20. Each of 14 and 21 is a divisor of n.
Since $14 = 2 \times 7$, then each of 2 and 7 is also a divisor of n.
Since $21 = 3 \times 7$, then 3 is a divisor of n (as is 7 which we already noted).
So far, the positive divisors of n are: $1, 2, 3, 7, 14$, and 21.
Since 2 and 3 are divisors of n, then their product $2 \times 3 = 6$ is a divisor of n.
Since $2, 3$ and 7 are divisors of n, then their product $2 \times 3 \times 7 = 42$ is a divisor of n.
The positive divisors of n are: $1, 2, 3, 6, 7, 14, 21$, and 42.
We are given that n has exactly 8 positive divisors including 1 and n, and so we have found them all, and thus $n = 42$.
The sum of these 8 positive divisors is $1 + 2 + 3 + 6 + 7 + 14 + 21 + 42 = 96$.

Answer: (D)

21. We begin by separating the given information, as follows:

- Kathy owns more cats than Alice
- Kathy owns more dogs than Bruce
- Alice owns more dogs than Kathy
- Bruce owns more cats than Alice

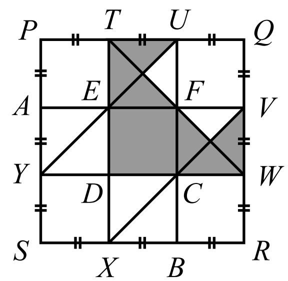
From bullets 2 and 3, we can conclude that Alice owns more dogs than both Kathy and Bruce.
From bullet 4, we can conclude that answer (A) is not true.
From bullets 1 and 4, we can conclude that both Kathy and Bruce own more cats than Alice.
However, we cannot determine if Kathy owns more cats than Bruce, or vice versa.
Therefore, we cannot conclude that (B) or (C) *must* be true.
From bullet 2, we can conclude that (E) is not true.
Thus the statement which *must* be true is (D).

Answer: (D)

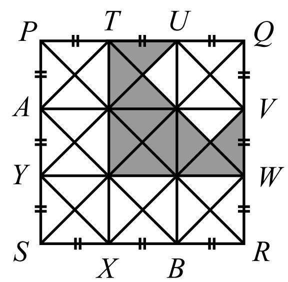
22. The single-digit divisors of 36 are: $1, 2, 3, 4, 6$, and 9.
The groups of 3 of these digits whose product is 36 are: $1, 4, 9$; $1, 6, 6$; $2, 2, 9$; $2, 3, 6$, and $3, 3, 4$.
Next, we count the number of ways to arrange each of these 5 groups of digits.
The digits $1, 4, 9$ can be arranged to form: $149, 194, 419, 491, 914, 941$.
The digits $1, 6, 6$ can be arranged to form: $166, 616, 661$.
The digits $2, 2, 9$ can be arranged to form: $229, 292, 922$.
The digits $2, 3, 6$ can be arranged to form: $236, 263, 326, 362, 623, 632$.
The digits $3, 3, 4$ can be arranged to form: $334, 343, 433$.
The number of 3-digit positive integers having digits whose product is 36 is $6+3+3+6+3 = 21$.

Answer: (A)

23. Begin by constructing AV perpendicular to TX and UB perpendicular to YW, as shown.
The four segments TX, UB, AV, and YW divide $PQRS$ into 9 identical squares.
Label the intersections of the perpendicular pairs of these four segments as points C, D, E, and F.
Segment UY is a diagonal of square $PUCY$ and so UY passes through E, the centre of square $PUCY$.
Segment UE is a diagonal of square $TUFE$.



Segment TW is a diagonal of square $TQWD$ and TF is a diagonal of square $TUFE$.
The diagonals in any square divide the square into 4 identical triangles.
For example, the diagonals TF and UE divide the square $TUFE$ into 4 identical triangles, 3 of which are shaded.
Similarly, we can show that diagonals FW and VC divide square $FVWC$ into 4 identical triangles, 3 of which are shaded.
We may construct the missing diagonals in each of the 9 squares, as shown.
These diagonals divide square $PQRS$ into $9 \times 4 = 36$ identical triangles.
Since 10 of these triangles are shaded, then $\frac{10}{36} = \frac{5}{18}$ of square $PQRS$ is shaded.



Answer: (A)

24. *Solution 1*
The ten moves have lengths 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
If the first move is vertical, then the five vertical moves have lengths 1, 3, 5, 7, 9 and the five horizontal moves have lengths 2, 4, 6, 8, 10.
If the first move is horizontal, then the five horizontal moves have lengths 1, 3, 5, 7, 9 and the five vertical moves have lengths 2, 4, 6, 8, 10.
If a horizontal move is to the right, then the length of the move is added to the x-coordinate.
If a horizontal move is to the left, then the length of the move is subtracted from the x-coordinate.
If a vertical move is up, then the length of the move is added to the y-coordinate.
If a vertical move is down, then the length of the move is subtracted from the y-coordinate.
Therefore, once the ten moves have been made, the change in one of the coordinates is a combination of adding and subtracting 1, 3, 5, 7, 9 and the change in the other coordinate is a combination of adding and subtracting 2, 4, 6, 8, 10.
For example, if the dot moves right 1, down 2, right 3, up 4, right 5, down 6, right 7, up 8, left 9, and up 10, then its final x-coordinate is

$$20 + 1 + 3 + 5 + 7 - 9 = 27$$

and its final y-coordinate is

$$19 - 2 + 4 - 6 + 8 + 10 = 33$$

making its final location $(27, 33)$, which is choice (A). This means that choice (A) is not the answer.
We note that, in the direction with the moves of even length, the final change in coordinate

will be even, since whenever we add and subtract even integers, we obtain an even integer.
In the other direction, the final change in coordinate will be odd, since adding or subtracting an odd number of odd integers results in an odd integer. (Since odd plus odd is even and odd minus odd is even, then after two moves of odd length, the change to date is even, and after four moves of odd length, the change to date is still even, which means that the final change after the fifth move of odd length is completed must be odd, since even plus or minus odd is odd.)
In the table below, these observations allow us to determine in which direction to put the moves of odd length and in which direction to put the moves of even length.

Choice	Change in x	Change in y	Horizontal moves	Vertical moves
(A) $(27,33)$	7	14	$1+3+5+7-9=7$	$-2+4-6+8+10=14$
(B) $(30,40)$	10	21	$2-4-6+8+10=10$	
(C) $(21,21)$	1	2	$1-3+5+7-9=1$	$2+4-6-8+10=2$
(D) $(42,44)$	22	25	$2-4+6+8+10=22$	$1+3+5+7+9=25$
(E) $(37,37)$	17	18	$-1-3+5+7+9=17$	$2+4-6+8+10=18$

Since each of the locations (A), (C), (D), and (E) is possible, then the location that is not possible must be (B).
We note that in the case of (B), it is the change in the y-coordinate that is not possible to make.
In other words, we cannot obtain a total of 21 by adding and subtracting 1, 3, 5, 7, 9.
Can you see why?

Solution 2
Let a be the horizontal change from the initial to the final position of the point and let b be the vertical change from the initial to the final position.
For example, if $a=-5$ and $b=6$, the point's final position is 5 units to the left and 6 units up from its original position of $(20,19)$.
Notice that if (x,y) is the final position of the point, then a and b can be calculated as $a+20=x$ or $a=x-20$ and $b+19=y$ or $b=y-19$.
First, assume the initial move is in the horizontal direction.
This means the second move will be in the vertical direction, the third will be horizontal, and so on.
In all, the first, third, fifth, seventh, and ninth moves will be horizontal and the others will be vertical.
Also, the first move is by one unit, the second is by two units, the third is by three units, and so on, so the horizontal moves are by 1, 3, 5, 7, and 9 units.
Each of these moves is either to the left or the right.
If all of the horizontal moves are to the right, then $a=1+3+5+7+9=25$.
If the point moves, left on the first move, right on the third, then left on the fifth, seventh, and ninth moves, a will be $-1+3-5-7-9=-19$.
In this case, the final position is 19 units to the left of the initial position (and has potentially moved up or down, as well).
In each of the five horizontal moves, the point either moves to the left or to the right.
This means there are two choices (left or right) for each of the five horizontal moves, so there are $2\times2\times2\times2\times2=32$ possible configurations similar to the examples above.
Going through these carefully, the table below computes all possible values of a:

$$
\begin{array}{rcr@{\qquad\qquad}rcr}
1+3+5+7+9 &=& 25 & 1+3+5+7-9 &=& 7 \\
-1+3+5+7+9 &=& 23 & -1+3+5+7-9 &=& 5 \\
1-3+5+7+9 &=& 19 & 1-3+5+7-9 &=& 1 \\
-1-3+5+7+9 &=& 17 & -1-3+5+7-9 &=& -1 \\
1+3-5+7+9 &=& 15 & 1+3-5+7-9 &=& -3 \\
-1+3-5+7+9 &=& 13 & -1+3-5+7-9 &=& -5 \\
1-3-5+7+9 &=& 9 & 1-3-5+7-9 &=& -9 \\
-1-3-5+7+9 &=& 7 & -1-3-5+7-9 &=& -11 \\
1+3+5-7+9 &=& 11 & 1+3+5-7-9 &=& -8 \\
-1+3+5-7+9 &=& 9 & -1+3+5-7-9 &=& -9 \\
1-3+5-7+9 &=& 5 & 1-3+5-7-9 &=& -13 \\
-1-3+5-7+9 &=& 3 & -1-3+5-7-9 &=& -15 \\
1+3-5-7+9 &=& 1 & 1+3-5-7-9 &=& -17 \\
-1+3-5-7+9 &=& -1 & -1+3-5-7-9 &=& -19 \\
1-3-5-7+9 &=& -5 & 1-3-5-7-9 &=& -23 \\
-1-3-5-7+9 &=& -7 & -1-3-5-7-9 &=& -25
\end{array}
$$

Some numbers appear more than once, but after inspecting the list, we see that when the first move is in the horizontal direction, a can be any odd number between -25 and 25 inclusive except for -21 and 21.

When the first move is horizontal, the second, fourth, sixth, eighth, and tenth moves will be vertical and have lengths 2, 4, 6, 8, and 10 units.
Using the same idea as the previous case, all 32 possible values of b when the first move is horizontal are calculated as follows:

$$
\begin{array}{rcr@{\qquad\qquad}rcr}
2+4+6+8+10 &=& 30 & 2+4+6+8-10 &=& 10 \\
-2+4+6+8+10 &=& 26 & -2+4+6+8-10 &=& 6 \\
2-4+6+8+10 &=& 22 & 2-4+6+8-10 &=& 2 \\
-2-4+6+8+10 &=& 18 & -2-4+6+8-10 &=& -2 \\
2+4-6+8+10 &=& 18 & 2+4-6+8-10 &=& -2 \\
-2+4-6+8+10 &=& 14 & -2+4-6+8-10 &=& -6 \\
2-4-6+8+10 &=& 10 & 2-4-6+8-10 &=& -10 \\
-2-4-6+8+10 &=& 6 & -2-4-6+8-10 &=& -14 \\
2+4+6-8+10 &=& 14 & 2+4+6-8-10 &=& -6 \\
-2+4+6-8+10 &=& 10 & -2+4+6-8-10 &=& -10 \\
2-4+6-8+10 &=& 6 & 2-4+6-8-10 &=& -14 \\
-2-4+6-8+10 &=& 2 & -2-4+6-8-10 &=& -18 \\
2+4-6-8+10 &=& 2 & 2+4-6-8-10 &=& -18 \\
-2+4-6-8+10 &=& -2 & -2+4-6-8-10 &=& -22 \\
2-4-6-8+10 &=& -6 & 2-4-6-8-10 &=& -26 \\
-2-4-6-8+10 &=& -10 & -2-4-6-8-10 &=& -30
\end{array}
$$

Again, some numbers appear more than once, but examination of the table shows that when the first move is horizontal, the possible values of b are the even numbers between -30 and 30 inclusive which are not multiples of 4.
We now begin to inspect the possible answers. The point in (A) is $(27, 33)$.
In this case, we have that $a = 27 - 20 = 7$ and $b = 33 - 19 = 14$.
These values for a and b fit the descriptions above, so $(27, 33)$ is a possible final position.

The point $(21, 21)$ in (C) has $a = 1$ and $b = 2$, which means $(21, 21)$ is also a possible final position for the point.
The point $(37, 37)$ in (E) has $a = 17$ and $b = 18$, so this point is also a possible final position.
We have shown that the answer must be either (B) or (D).
Note that in both cases we have that a is even and b is odd, which means that if either of these points is a possible final position, the first move cannot have been horizontal, which means it must have been vertical.
If the first move is vertical, then the third, fifth, seventh, and ninth moves are also vertical, and the other moves are horizontal.
Going through similar analysis as before, we will see that the restrictions on a and b have switched.
That is, a must be an even number between -30 and 30 inclusive which is not a multiple of 4, and b must be an odd number between -25 and 25 inclusive other than -21 and 21.
The point $(42, 44)$ in (D) has $a = 22$ and $b = 25$, which is possible with a vertical first move.
However, the point $(30, 40)$ in (B) has $a = 10$ and $b = 21$.
Since 21 is not a possible value for b, then $(30, 40)$ is not a possible final position.

ANSWER: (B)

25. The rectangular prism has two faces whose area is $8 \times 8 = 64$, and four faces each of whose area is $8 \times n$.
Therefore, A is $2 \times 64 + 4 \times 8 \times n = 128 + 32n$.
The prism is made up of $8 \times 8 \times n = 64 \times n$ cubes, each of which has dimensions $1 \times 1 \times 1$.
Each $1 \times 1 \times 1$ cube has surface area 6 because each has 6 faces which are all 1×1 squares.
Therefore, $B = 6 \times 64 \times n = 384n$.
Thus, we get

$$\frac{B}{A} = \frac{384n}{128 + 32n}.$$

This expression can be simplified by recognizing that each of $384, 128$ and 32 is divisible by 32.
After dividing the numerator and denominator by 32, we get $\frac{B}{A} = \frac{12n}{4+n}$. We require that $\dfrac{B}{A}$ be equal to some integer, so we will determine which integers $\dfrac{12n}{4+n}$ can be.

First, note that n is positive, so both $12n$ and $4+n$ are positive, which means $\dfrac{12n}{4+n}$ is positive.
This means $\dfrac{12n}{4+n}$ is a positive integer, so we determine which positive integers $\dfrac{12n}{4+n}$ can equal.
If $\dfrac{12n}{4+n} = 1$, then $12n = 4 + n$ which can be rearranged to give $11n = 4$ or $n = \frac{4}{11}$.
Since n must be an integer, we conclude that $\dfrac{12n}{4+n}$ cannot be equal to 1.
What if $\dfrac{12n}{4+n} = 2$? In this case, we need $12n$ to be twice as large as $4 + n$, or $12n = 8 + 2n$.
This can be rearranged to give $10n = 8$ or $n = \frac{8}{10}$.
Again, this value of n is not an integer, so we conclude that $\dfrac{12n}{4+n}$ is not 2.

Following this reasoning, if $\dfrac{12n}{4+n}$ is 3, we find that n must be $\frac{4}{3}$, which is also not an integer, so $\dfrac{12n}{4+n}$ is not equal to 3.
If $\dfrac{12n}{4+n}$ is equal to 4, we have that $12n$ is four times $4 + n$, or $12n = 16 + 4n$.

Rearranging this gives $8n = 16$ which means $n = 2$. Therefore, $\dfrac{12n}{4+n}$ can be 4, and it happens when $n = 2$.

We continue in this way for all possible positive integer values of $\dfrac{12n}{4+n}$ up to and including $\dfrac{12n}{4+n} = 11$.

The results are summarized in the table below.

$\dfrac{12n}{n+4}$	n	n is an integer
1	$\frac{4}{11}$	×
2	$\frac{8}{10}$	×
3	$\frac{4}{3}$	×
4	2	✓
5	$\frac{20}{7}$	×
6	4	✓
7	$\frac{28}{5}$	×
8	8	✓
9	12	✓
10	20	✓
11	44	✓

According to the table, $\dfrac{12n}{4+n}$ can be any of the integers $4, 6, 8, 9, 10$, and 11 and these occur when n is equal to $2, 4, 8, 12, 20$, and 44, respectively.

We now consider what happens when $\dfrac{12n}{4+n}$ is 12 or greater.

If $\dfrac{12n}{4+n} = 12$, then $12n$ is 12 times as large as $4+n$, or $12n = 48 + 12n$.

Since $48 + 12n$ is always greater than $12n$, there is no value of n for which $12n = 48 + 12n$.

Similarly, since n is a positive integer, there is no value of n for which $\dfrac{12n}{4+n}$ is 13 or greater.

We conclude that the only possible positive integer values of $\dfrac{12n}{4+n}$ are those in the table, so the only values of n which make $\dfrac{12n}{4+n}$ an integer are $2, 4, 8, 12, 20$, and 44.

The sum of these numbers is $2 + 4 + 8 + 12 + 20 + 44 = 90$.

Answer: (B)

Grade 8

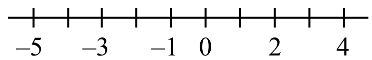
1. As a percentage, one half of a muffin is equivalent to $\frac{1}{2} \times 100\% = 50\%$ of a muffin.

   Answer: (E)

2. The sum of the angles in a triangle is 180°.
   Thus $x + x + x = 180$ or $3x = 180$, and so $x = \frac{180}{3} = 60$.

   Answer: (B)

3. Place zero and the given answers in their correct locations on the number line, as shown.
   The integer closest to 0 is -1.

   Answer: (A)

4. A number gives a remainder of 3 when divided by 5 if it is 3 more than a multiple of 5.
   Since 88 is 3 more than 85, and 85 is a multiple of 5, then 88 gives a remainder of 3 when divided by 5.
   We may check that this is the only answer which gives a remainder of 3 when divided by 5.

   Answer: (D)

5. Recall that a prime number is an integer greater than 1 whose only divisors are 1 and itself.
   The prime numbers between 10 and 20 are: $11, 13, 17$, and 19.
   Thus, there are 4 integers between 10 and 20 which are prime numbers.

   Answer: (E)

6. Reading from the graph, 15 vehicles had an average speed from 80 km/h to 89 km/h, 30 vehicles had an average speed from 90 km/h to 99 km/h, and 5 vehicles had an average speed from 100 km/h to 109 km/h.
   The average speed of each of the remaining vehicles was less than 80 km/h.
   The number of vehicles that had an average speed of at least 80 km/h was $15 + 30 + 5 = 50$.

   Answer: (E)

7. Any positive integer that is divisible by both 3 and 7 is divisible by their product, $3 \times 7 = 21$.
   That is, we are being asked to count the number of positive integers less than 100 that are multiples of 21.
   These integers are: $21, 42, 63$, and 84.
   We note that the next multiple of 21 is $21 \times 5 = 105$, which is greater than 100.
   There are 4 positive integers less than 100 that are divisible by both 3 and 7.

   Answer: (C)

8. The circumference, C, of a circle is given by the formula $C = \pi \times d$, where d is the diameter of the circle.
   If the circumference is 100, then $100 = \pi \times d$ and so $d = \dfrac{100}{\pi}$.

   Answer: (C)

9. The area, A, of a triangle is given by the formula $A = \dfrac{b \times h}{2}$, where b is the length of the base and h is the perpendicular height of the triangle.
   If the area of a triangle is 6, then substituting we get $6 = \dfrac{b \times h}{2}$, and so $b \times h = 12$.
   Consider the base, b, of the triangle to be PQ and so $b = 4$.
   Since $b \times h = 12$ and $b = 4$, then the perpendicular height, h, is equal to 3.
   The point which lies a perpendicular distance of 3 units above PQ is E.

   Answer: (E)

10. Barry's 12 coins include at least one of each of the 5 coins of different values.
The total value of these 5 coins is $\$2.00 + \$1.00 + \$0.25 + \$0.10 + \$0.05 = \$3.40$.
Barry has the smallest total amount of money that he could have if each of his remaining $12 - 5 = 7$ coins has value $\$0.05$ (the smallest possible value of a coin).
Thus, the smallest total amount of money that Barry could have is

$$\$3.40 + 7 \times \$0.05 = \$3.40 + \$0.35 = \$3.75.$$

Answer: (A)

11. An isosceles triangle has two sides of equal length.
Since we are given that two of the side lengths in an isosceles triangle are 6 and 8, then it is possible that the three side lengths are 6, 6 and 8.
In this case, the perimeter of the triangle is $6 + 6 + 8 = 20$, but this is not one of the given answers.
The only other possibility is that the side lengths are $6, 8$ and 8.
In this case, the perimeter is $6 + 8 + 8 = 22$.

Answer: (C)

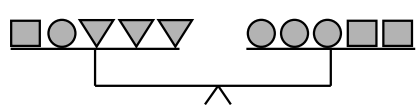
12. The angles measuring 60° and $(y+5)^\circ$ lie along the line segment PQ, and so they add to 180°.
Thus, $60 + y + 5 = 180$ or $y + 65 = 180$, and so $y = 180 - 65 = 115$.
The angles measuring $(4x)^\circ$ and $(y+5)^\circ$ are opposite angles and so $4x = y + 5$.
Since $y = 115$, we get $4x = 120$ and so $x = 30$.
Therefore, $x + y = 30 + 115 = 145$.

Answer: (A)

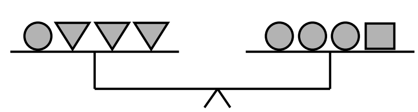
13. For the five numbers $12, 9, 11, 16, x$ to have a unique mode, x must equal one of the existing numbers in the list: $12, 9, 11$, or 16.
If $x = 9$, then the mode is 9, but the mean of the five numbers $9, 9, 11, 12, 16$ is greater than 9.
If $x = 16$, then the mode is 16, but the mean of the five numbers $9, 11, 12, 16, 16$ is less than 16.
Therefore, x must either be equal to 11 or 12.
If $x = 11$, the mean of the numbers is $\frac{9+11+11+12+16}{5} = \frac{59}{5}$ which is not equal to the mode 11.
If $x = 12$, the mean of the numbers is $\frac{9+11+12+12+16}{5} = \frac{60}{5} = 12$ which is equal to the mode 12.
And finally we confirm that if $x = 12$, then the median (the middle number in the ordered list $9, 11, 12, 12, 16$) is also equal to 12.

Answer: (C)

14. The first scale shows that ○○○ has the same mass as ▽▽.
The second scale shows that □○▽ has the same mass as □□.
Therefore, the sum of the masses of ▽▽ and □○▽ is equal to the sum of the masses of ○○○ and □□.
That is, the equal-arm scale shown below is balanced.



The left and right sides of this scale each contain a □, and the scale remains balanced if one □ is removed from each side of the scale (since they are equal in mass).
That is, the equal-arm scale shown below is balanced.



Therefore of the answers given, ○▽▽▽ has the same mass as ○○○□.

Answer: (D)

15. *Solution 1*

After the arrow is spun twice, the possible outcomes are: red, red; red, blue; red, green; blue, red; blue, blue; blue, green, green, red; green, blue; green, green.
That is, there are 9 possible outcomes.
Of these 9 outcomes, 3 have the same colour appearing twice: red, red; blue, blue; green, green.
The probability that the arrow lands on the same colour twice is $\frac{3}{9} = \frac{1}{3}$.

Solution 2

First, we count the total number of possible outcomes given that the arrow is spun twice.
When the arrow is spun the first time, there are 3 possible outcomes (red, blue, green).
When the arrow is spun the second time, there are again 3 possible outcomes.
Thus when the arrow is spun twice, there are $3 \times 3 = 9$ possible outcomes.
Next, we count the number of outcomes in which the arrow lands on the same colour twice.
When the arrow is spun the first time, there are 3 possible outcomes (red, blue, green).
When the arrow is spun the second time, there is 1 possible outcome (since the colour must match the colour that the arrow landed on after the first spin).
Thus when the arrow is spun twice, there are $3 \times 1 = 3$ possible outcomes in which the arrow lands on the same colour twice.
The probability that the arrow lands on the same colour twice is $\frac{3}{9} = \frac{1}{3}$.

Solution 3

The first spin will land on one of the three colours.
For the arrow to land on the same colour twice, the second spin must land on the colour matching the first spin.
Since exactly 1 of the 3 colours matches the first spin, the probability that the arrow lands on the same colour twice is $\frac{1}{3}$.

ANSWER: (D)

16. The lightbulb is used for exactly 2 hours every day and it will work for 24 999 hours.
Thus, the lightbulb will work for $24\,999 \div 2 = 12\,499.5$ days, which means that it will stop working on the $12\,500^{th}$ day.
How many weeks are there in 12 500 days?
Since $12\,500 \div 7 \approx 1785.71$, there are between 1785 and 1786 weeks in 12 500 days.
Specifically, $12\,500 - 7 \times 1785 = 12\,500 - 12\,495 = 5$, and so there are 1785 complete weeks and 5 additional days in 12 500 days.
The lightbulb starts being used on a Monday and so the last day of 1785 complete weeks will be a Sunday.
The lightbulb will stop working 5 days past Sunday, which is a Friday.

ANSWER: (B)

17. Since $w + x = 45$ and $x + y = 51$, then $x + y$ is 6 more than $w + x$ (since 51 is 6 more than 45).
Both $x + y$ and $w + x$ contain x, and so the difference between these two expressions, 6, must be the difference between y and w.
That is, y is 6 more than w (or $y = 6 + w$).
In the final equation given, $y + z = 28$, y is 6 more than w, and so 6 more than w added to z equals 28.
Since $6 + 22 = 28$, then $w + z = 22$.

ANSWER: (B)

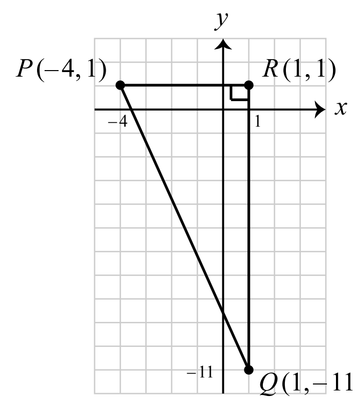
18. We begin by separating the given information, as follows:

    - Kathy owns more cats than Alice
    - Kathy owns more dogs than Bruce
    - Alice owns more dogs than Kathy
    - Bruce owns more cats than Alice

    From bullets 2 and 3, we can conclude that Alice owns more dogs than both Kathy and Bruce.
    From bullet 4, we can conclude that answer (A) is not true.
    From bullets 1 and 4, we can conclude that both Kathy and Bruce own more cats than Alice. However, we cannot determine if Kathy owns more cats than Bruce, or vice versa.
    Therefore, we cannot conclude that (B) or (C) *must* be true.
    From bullet 2, we can conclude that (E) is not true.
    Thus the statement which *must* be true is (D).

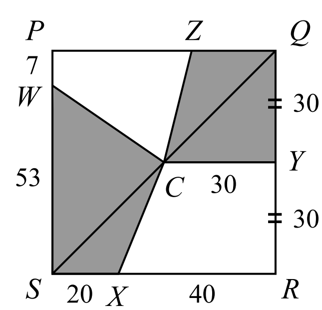
    ANSWER: (D)

19. The horizontal line through P intersects the vertical line through Q at $R(1,1)$, as shown.
    Joining P, Q, R creates right-angled $\triangle PQR$, with hypotenuse PQ.
    The x-coordinate of P is -4 and the x-coordinate of R is 1, and so PR has length $1-(-4)=5$ (since P and R have equal y-coordinates).
    The y-coordinate of Q is -11 and the y-coordinate of R is 1, and so QR has length $1-(-11)=12$ (since Q and R have equal x-coordinates).
    Using the Pythagorean Theorem, $PQ^2 = PR^2 + QR^2$ or $PQ^2 = 5^2 + 12^2 = 25 + 144 = 169$, and so $PQ = \sqrt{169} = 13$ (since $PQ > 0$).
    (Alternatively, we could have drawn the vertical line through P and the horizontal line through Q which meet at $(-4,-11)$.)



    ANSWER: (A)

20. *Solution 1*
    Square $PQRS$ has side length 60, and so its area is $60^2 = 3600$.
    We require the total area of the shaded regions to equal the total area of the non-shaded regions, and so the total area of the shaded regions is to be half the area of the square, or 1800.
    Since C is the centre of the square, its distance to each of the sides of the square is $\frac{1}{2}(60) = 30$.
    To determine the area of each of the shaded regions, we begin by labelling each of the given lengths (as well as those that we can determine), and constructing diagonal SQ, as shown.



    Quadrilateral $WCXS$ is made up of two triangles: $\triangle SCX$ and $\triangle SCW$.
    In $\triangle SCX$, the base $SX = 20$ and the perpendicular height from C to SR is 30 (since C is the centre of the square $PQRS$).
    Thus, the area of $\triangle SCX$ is $\frac{1}{2}(20)(30) = 300$.
    In $\triangle SCW$, the base $SW = 53$ and the perpendicular height from C to SW is 30.
    The area of $\triangle SCW$ is $\frac{1}{2}(53)(30) = 795$.

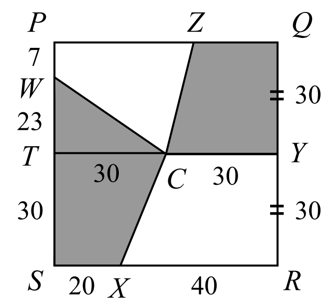
Therefore, the area of quadrilateral $WCXS$ is $300 + 795 = 1095$.

Quadrilateral $ZQYC$ is made up of two triangles: $\triangle CQY$ and $\triangle CQZ$.
The midpoint of QR is Y and C is the centre of the square, so CY is perpendicular to QY.
In $\triangle CQY$, the base $CY = 30$, the perpendicular height $YQ = 30$, and so the area of $\triangle CQY$ is $\frac{1}{2}(30)(30) = 450$.
In $\triangle CQZ$, if the base is ZQ, then the perpendicular height from C to PQ is 30.
The area of $\triangle CQZ$ is $\frac{1}{2}(ZQ)(30) = 15(ZQ)$.
Therefore, the area of quadrilateral $ZQYC$ is $15 \times ZQ + 450$.

Finally, adding the areas of the shaded regions, we get $1095 + 15 \times ZQ + 450 = 1800$ or $15 \times ZQ = 1800 - 1095 - 450 = 255$, and so $ZQ = \frac{255}{15} = 17$.

Solution 2

As in Solution 1, we require the total area of the shaded regions to equal 1800.
We begin by labelling T, the midpoint of side PS.
Since Y is the midpoint of QR, then TY passes through C and is perpendicular to each of PS and QR.
Since C is the centre of the square, its distance to each of the sides of the square is $\frac{1}{2}(60) = 30$.
We label each of the given lengths (as well as those that we can determine), as shown.



Quadrilateral $WCXS$ is made up of $\triangle WCT$ and trapezoid $TCXS$ (TC is parallel SX and thus $TCXS$ is a trapezoid).
In $\triangle WCT$, base $WT = WS - TS = 53 - 30 = 23$ and the perpendicular height CT is 30.
Thus, the area of $\triangle WCT$ is $\frac{1}{2}(23)(30) = 345$.
In trapezoid $TCXS$, the parallel sides have lengths $TC = 30$ and $SX = 20$, and the perpendicular height is $TS = 30$.
The area of trapezoid $TCXS$ is $\frac{1}{2}(30)(30 + 20) = 750$.
Therefore, the area of quadrilateral $WCXS$ is $345 + 750 = 1095$.

Quadrilateral $ZQYC$ is also a trapezoid (since ZQ is parallel to CY).
The area of trapezoid $ZQYC$ is $\frac{1}{2}(QY)(ZQ + 30) = 15(ZQ + 30)$.

Adding the areas of the shaded regions, we get $1095 + 15(ZQ + 30) = 1800$ or $15(ZQ + 30) = 1800 - 1095 = 705$ or $ZQ + 30 = \frac{705}{15} = 47$, and so $ZQ = 17$.

ANSWER: (E)

21. Let the number of teams in Jen's baseball league be n.
Each of these n teams plays 6 games against each of the $n - 1$ other teams in the league.
Since there are 2 teams in each of these games, the total number of games played is $\dfrac{6n(n-1)}{2}$.
The total number of games played is 396, so $\dfrac{6n(n-1)}{2} = 396$ or $3n(n - 1) = 396$ or $n(n - 1) = 132$.
The numbers n and $n-1$ differ by 1, and so we are looking for two consecutive positive integers whose product is 132.
Since $12 \times 11 = 132$, the number of teams in Jen's league is 12.

ANSWER: (A)

22. Let Rich's 4-digit positive integer be $abcd$, where d is the units (ones) digit, c is the tens digit, b is the hundreds digit, and a is the thousands digit of the original number .
We begin by determining which of the 4 digits is erased.
If Rich erases the digit a, then the 3-digit integer that remains is bcd, and the sum of the two integers is $abcd + bcd$.
The units digit of this sum is determined by adding the units digits of $abcd$ and bcd, or $d+d = 2d$.
In this case, the units digit of the sum is even since $2d$ is even.
The sum of these two integers is 6031 which has an odd units digit, and so the digit that is erased cannot be a.
Using a similar argument, the digit that is erased cannot be b or c, and thus must be d.
Next, we determine the digits a, b, c, d so that

$$\begin{array}{rr} & a\,bcd \\ + & abc \\ \hline & 6\,031 \end{array}$$

The hundreds column of this sum gives that $b + a$ added to the 'carry' from the tens column has units digit 0. That is, this sum is either 0, 10 or 20.
If the sum is 0, then $a = 0$, but looking to the thousands column we see that a cannot be 0.
If the sum is 20, then $a + b = 18$ (since any digit can be at most 9 and the carry from the tens column can be at most 2).
If $a + b = 18$, then $a = 9$, but looking to the thousands column we see that a cannot be 9.
Therefore, $b + a$ added to the carry from the tens column is 10, and so the carry from the hundreds column to the thousands column is 1.
The thousands column then gives $a + 1 = 6$ and so $a = 5$.

$$\begin{array}{rr} & 5\,bcd \\ + & 5bc \\ \hline & 6\,031 \end{array}$$

The hundreds column gives $b + 5$ added to the carry from the tens column is 10.
If the carry from the tens column is 0, then $b + 5 = 10$ and so $b = 5$.
In the tens column, if $b = 5$ then $c + b = c + 5$ is greater than 3 and so must be 13.
This tells us that the carry from the tens colunn cannot be 0.
If the carry from the tens column is 1, then $b + 5 + 1 = 10$ and so $b = 4$.
(Note that sum of the tens column has units digit 3 and so the carry cannot be 2.)

$$\begin{array}{rr} & 5\,4cd \\ + & 54c \\ \hline & 6\,031 \end{array}$$

The units digit of the sum in the tens column is 3, and so $c + 4$ added to the carry from the units column is 13 (it can't be 3 or 23).
If there is no carry from the units column, then $c + 4 = 13$ and so $c = 9$.
But if $c = 9$ then the units column gives that $d + 9$ must be 11 and so there is carry from the units column. Hence c is not 9.
If the carry from the units column is 1, then $c + 4 + 1 = 13$ and so $c = 8$ and $d = 3$.
The final sum is shown here:

$$\begin{array}{rr} & 5\,483 \\ + & 548 \\ \hline & 6\,031 \end{array}$$

The sum of the digits of the original 4-digit number is $a + b + c + d = 5 + 4 + 8 + 3 = 20$.

ANSWER: (B)

23. We begin by recognizing that each of the given answers has a common numerator of (20!)(19!). Since 20! is equal to the product of the integers from 1 to 20 inclusive, we can consider that 20! is equal to the product of the integers from 1 to 19 inclusive, multiplied by 20.
However, the product of the integers from 1 to 19 inclusive is equal to 19!, and so $20! = 19! \times 20$.
That is, the common numerator (20!)(19!) can be rewritten as $(19! \times 20)(19!)$ or $(19!)^2 \times 20$.
Next, we consider the result after dividing $(19!)^2 \times 20$ by each of the denominators:

$$\frac{(20!)(19!)}{1} = \frac{(19!)^2 \times 20}{1} = (19!)^2 \times 20$$
$$\frac{(20!)(19!)}{2} = \frac{(19!)^2 \times 20}{2} = (19!)^2 \times 10$$
$$\frac{(20!)(19!)}{3} = \frac{(19!)^2 \times 20}{3} = (19!)^2 \times \tfrac{20}{3}$$
$$\frac{(20!)(19!)}{4} = \frac{(19!)^2 \times 20}{4} = (19!)^2 \times 5$$
$$\frac{(20!)(19!)}{5} = \frac{(19!)^2 \times 20}{5} = (19!)^2 \times 4$$

Since $(19!)^2$ is the square of an integer (19!), it is a perfect square.
The product of a perfect square and some positive integer factor f, is equal a perfect square only if f is a perfect square.
An answer satisfying this condition is (E).
Why is $(19!)^2 \times 4$ a perfect square?
Rewriting, $(19!)^2 \times 4 = (19!)^2 \times 2^2 = (19! \times 2)^2$, which is the square of the integer $19! \times 2$ and thus is a perfect square.
Can you explain why each of the other four answers is not equal to a perfect square?

Answer: (E)

24. First, notice that the given list of 10 numbers has a sum of $0+1+2+3+4+5+6+7+8+9 = 45$.
If n is the number of groups and m is the total in each group, then we must have $mn = 45$.
This means 45 is a multiple of the number of groups.
We also require that the number of groups be at least 2, so the number of groups is one of $3, 5, 9, 15, 45$.
If the number of groups is 9, then the total in each group is $\frac{45}{9} = 5$, but this is not possible since one of the groups must contain 9 and therefore cannot have a sum of 5.
Similarly, if the number of groups is 15 or 45, the sum of each group must be 3 or 1 respectively, which is too small.
Therefore, there are either 5 groups or 3 groups.

Let's first assume that there are five groups.
In this case, the total in each group must be $\frac{45}{5} = 9$.
Since 0 does not contribute anything to the sum, we will ignore it for now.
Since the total in each group must be 9, we are forced to have 9 in a group by itself.
The number 8 must be paired with 1 since adding any other positive integer to 8 will give a sum greater than 9.
That is, one of the groups has the numbers 1 and 8. We will denote this group by $\{1, 8\}$.
The number 7 cannot be in a group with any number larger than 2, and 1 is already paired with 8, so another group must be $\{2, 7\}$.
Continuing with this reasoning, we get that $\{3, 6\}$ must be a group, and $\{4, 5\}$ must also be a group.

Therefore, if there are 5 groups, they must be $\{9\}, \{1,8\}, \{2,7\}, \{3,6\}$, and $\{4,5\}$.
As mentioned before, 0 does not contribute anything to the sum, so it can be placed in any of the 5 groups without changing the sum.
There are 5 ways to do this: $\{0,9\}, \{1,8\}, \{2,7\}, \{3,6\}, \{4,5\}$, or $\{9\}, \{0,1,8\}, \{2,7\}, \{3,6\}, \{4,5\}$, and so on.
We have shown that there are 5 ways to separate the numbers 0 through 9 into 5 groups so that each group has the same sum.

Let's now assume that there are three groups.
As we did when there were 5 groups, we ignore 0 for now.
Since there are three groups, the sum in each group is $\frac{45}{3} = 15$.
We now find all groups which add up to 15.
There are 17 of them, so we will label them using A through Q:

A	$\{6,9\}$	G	$\{2,5,8\}$	M	$\{1,3,4,7\}$
B	$\{1,5,9\}$	H	$\{3,4,8\}$	N	$\{4,5,6\}$
C	$\{2,4,9\}$	I	$\{1,2,4,8\}$	O	$\{1,3,5,6\}$
D	$\{1,2,3,9\}$	J	$\{2,6,7\}$	P	$\{2,3,4,6\}$
E	$\{7,8\}$	K	$\{3,5,7\}$	Q	$\{1,2,3,4,5\}$
F	$\{1,6,8\}$	L	$\{1,2,5,7\}$		

Of these 17 groups, only A, B, C, and D contain the number 9.
Therefore, any way of separating the integers into three groups must use exactly one of these four groups.

Assume $A = \{6,9\}$ is one of the groups. This means the other two groups do not contain 6 or 9.
The groups which lack both 6 and 9 are E, G, H, I, K, L, M, and Q.
If $E = \{7,8\}$ is one of the groups, we will have used $6, 7, 8$, and 9, so the remaining group must be $Q = \{1,2,3,4,5\}$.
If $G = \{2,5,8\}$ is one of the groups, we will have used $2, 5, 6, 8$, and 9.
This forces the third group to be $M = \{1,3,4,7\}$.
Similarly, if H is one of the groups, so is L, and if I is one of the groups, so is K.
At this point, we can stop checking.
This is because, for example, if we assume M is one of the groups, we are forced to take G as the third group, but we have already identified the case where we separate into A, M, and G.
To summarize, if A is one of the groups, the possible configurations are

$$A, E, Q \qquad A, G, M \qquad A, H, L \qquad A, I, K$$

If we assume B is one of the groups, the other two groups must not contain any of 1, 5, and 9.
The groups satisfying this condition are E, H, J, and P.
By the same reasoning as in the previous paragraph, the configurations including B are

$$B, E, P \qquad B, H, J$$

The groups which have no members in common with C are E, F, K, and O, and the configurations which use C are

$$C, E, O \qquad C, F, K$$

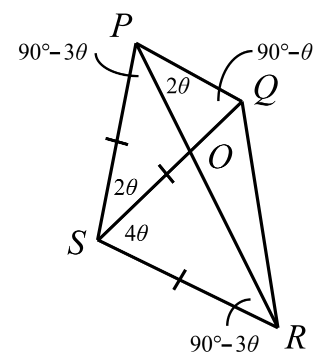
The only groups which have no members in common with D are E and N, so the only configuration using D is

$$D, E, N.$$

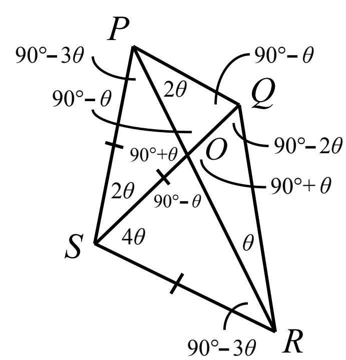
In total, we have found that there are 9 possible ways to separate the numbers from 1 through 9 into three groups each having the same sum.
As before, placing 0 in a group has no effect on the sum of that group. Since there are three groups, there are three possible ways to include 0 for each configuration, so there are $9 \times 3 = 27$ ways to separate the integers from 0 to 9 into three groups where each group has the same sum.
Adding this to the five from earlier, the number of ways to separate the list $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ into at least two groups so that the sum of the numbers in each group is the same is $27+5 = 32$.

ANSWER: (E)

25. We begin by labeling $\angle QSP = 2\theta$. Since $\angle QSR = 2\angle QSP$, $\angle QSR = 2 \times 2\theta = 4\theta$.
We will now find the measures of all 12 angles in terms of θ.
Let $x = \angle SPR$. Since $\triangle SPR$ has $SP = SR$, it is isosceles which means $\angle SRP = \angle SPR = x$.
Also, $\angle PSR = 2\theta + 4\theta = 6\theta$, so $180° = 6\theta + x + x$.
Rearranging this equation, we have $2x = 180° - 6\theta$, so $x = 90° - 3\theta$.
In $\triangle SPQ$, we know $\angle SPQ + \angle SQP + \angle PSQ = 180°$, so rearranging and using $\angle PSQ = 2\theta$ gives $\angle SPQ + \angle SQP = 180° - 2\theta$.



Since $SP = SR$, $\triangle SPQ$ is isosceles which means $\angle SPQ = \angle SQP$.
Substituting this into the above equation, we have $2\angle SQP = 180° - 2\theta$ or $\angle SQP = 90° - \theta$.
Note that this means $\angle SPQ = 90° - \theta$ as well, so $\angle SPR + \angle RPQ = 90° - \theta$.
Substituting $\angle SPR = 90° - 3\theta$, we get that $90° - 3\theta + \angle RPQ = 90° - \theta$, so $\angle RPQ = 2\theta$.
Since $\angle POQ + \angle PQO + \angle OPQ = 180°$, substitution gives $\angle POQ + (90° - \theta) + 2\theta = 180°$.
Solving this, we get $\angle POQ = 90° - \theta$.
Angles $\angle POQ$ and $\angle ROS$ are opposite, which means $\angle ROS = \angle POQ = 90° - \theta$. Also, $\angle POQ + \angle QOR = 180°$, so $\angle QOR = 180° - \angle POQ = 180° - (90° - \theta) = 90° + \theta$.
Since $\angle POS$ and $\angle ROQ$ are opposite, they are equal, so $\angle POS = 90° + \theta$.
We now set $\angle SQR = y$.
Since $SQ = SR$, $\triangle SQR$ is isosceles, which means $\angle SRQ = \angle SQR = y$.



Substituting this and $\angle QSR = 4\theta$ into $\angle QSR + \angle SRQ + \angle SQR = 180°$ gives $4\theta + 2y = 180°$, so $2y = 180° - 4\theta$ or $y = 90° - 2\theta$.
Finally, we have $\angle ROQ + \angle OQR + \angle QRO = 180°$, so $\angle QRO = 180° - (90° + \theta) - (90° - 2\theta) = \theta$.

In summary, the 12 angles in terms of θ are

$$\begin{array}{ccccccc} & & \angle QSP & = & \angle RPQ & = & 2\theta \\ & & & & \angle QSR & = & 4\theta \\ & & \angle SRP & = & \angle RPS & = & 90° - 3\theta \\ & & & & \angle QRP & = & \theta \\ & & & & \angle RQS & = & 90° - 2\theta \\ \angle SQP & = & \angle POQ & = & \angle ROS & = & 90° - \theta \\ & & \angle POS & = & \angle QOR & = & 90° + \theta \end{array}$$

We are given that the measure of each angle in degrees is an integer.
In particular, since $\angle QRP = \theta$, we get that θ must be an integer number of degrees.
We also know that PR and QS intersect inside $PQRS$, so it must be that $6\theta = \angle PSR < 180^\circ$.
This means $\theta < 30^\circ$.

Since 4θ is even (and greater than 2), it cannot be a prime number for any integer θ.
The integers 2θ and 3θ are only prime numbers when $\theta = 1^\circ$.
In this case, the twelve angles are

$$2^\circ, 2^\circ, 4^\circ, 87^\circ, 87^\circ, 1^\circ, 88^\circ, 89^\circ, 89^\circ, 89^\circ, 91^\circ, 91^\circ$$

of which five are prime numbers: two copies of 2° and three copies of 89°.
Therefore, $\theta \neq 1^\circ$.

By factoring, we have that $90^\circ - 2\theta = 2(45^\circ - \theta)$ which is even.
Therefore, this number can only be prime when $45^\circ - \theta = 1$ or $\theta = 44^\circ$.
We know that $\theta < 30^\circ$, so $90^\circ - 2\theta$ must be composite.
Similarly, $90^\circ - 3\theta = 3(30^\circ - \theta)$ can only be a prime number when $30^\circ - \theta = 1^\circ$ or $\theta = 29^\circ$.
When $\theta = 29^\circ$, the twelve angles are

$$58^\circ, 58^\circ, 116^\circ, 3^\circ, 3^\circ, 29^\circ, 32^\circ, 61^\circ, 61^\circ, 61^\circ, 119^\circ, 119^\circ$$

and exactly 6 of these are prime: 29°, the two copies of 3°, and the three copies of 61°.

When $\theta < 30^\circ$ is different from 1° and 29°, each of the six angles

$$2\theta, 4\theta, 90^\circ - 3\theta, 90^\circ - 2\theta, 2\theta, 90^\circ - 3\theta$$

is composite.
In order to satisfy the condition that the measures of exactly 6 of the angles in degrees is a prime number, we therefore require that each of θ, $90^\circ - \theta$, and $90^\circ + \theta$ is prime.
The prime numbers less than 30 are $2, 3, 5, 7, 11, 13, 17, 19, 23$, and 29.
We have already investigated what happens when $\theta = 29^\circ$, so the values for the angles $\theta, 90^\circ + \theta$, and $90^\circ - \theta$ in the other cases are calculated in the table:

θ	$90^\circ - \theta$	$90^\circ + \theta$	All 3 angles prime?
2°	88°	92°	$\times$
3°	87°	93°	$\times$
5°	85°	95°	$\times$
7°	83°	97°	✓
11°	79°	101°	✓
13°	77°	103°	$\times$
17°	73°	107°	✓
19°	71°	109°	✓
23°	67°	113°	✓

Of the angles that appear in the second two columns, the numbers which are prime are

$$67^\circ, 71^\circ, 73^\circ, 79^\circ, 83^\circ, 97^\circ, 101^\circ, 103^\circ, 107^\circ, 109^\circ, 113^\circ.$$

Therefore, all three angles are prime when θ is one of $7^\circ, 11^\circ, 17^\circ, 19^\circ, 23^\circ$.
Combining these with $\theta = 29^\circ$ from earlier, this gives a total of 6 quadrilaterals with the given properties.

Answer: (D)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2018 Gauss Contests

(Grades 7 and 8)

Wednesday, May 16, 2018
(in North America and South America)

Thursday, May 17, 2018
(outside of North America and South America)

Solutions

Centre for Education in Mathematics and Computing Faculty and Staff

Ed Anderson
Jeff Anderson
Terry Bae
Jacquelene Bailey
Grace Bauman
Shane Bauman
Ersal Cahit
Serge D'Alessio
Rich Dlin
Jennifer Doucet
Fiona Dunbar
Mike Eden
Barry Ferguson
Judy Fox
Steve Furino
John Galbraith
Robert Garbary
Rob Gleeson
Sandy Graham
Conrad Hewitt
Angie Hildebrand
Carrie Knoll
Judith Koeller
Laura Kreuzer
Bev Marshman
Mike Miniou
Dean Murray
Jen Nelson
J.P. Pretti
Kim Schnarr
Carolyn Sedore
Kevin Shonk
Ashley Sorensen
Ian VanderBurgh
Troy Vasiga
Christine Vender
Heather Vo

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Laurissa Werhun, Parkdale C.I., Toronto, ON
Chris Wu, Ledbury Park E. and M.S., Toronto, ON
Lori Yee, William Dunbar P.S., Pickering, ON

Grade 7

1. Since $21 - 13 = 8$, the number that should be subtracted from 21 to give 8 is 13.

   Answer: (B)

2. Reading from the pie chart, 20% of 100 students chose banana.
   Since 20% of 100 is 20, then 20 students chose banana.

   Answer: (D)

3. There are 30 minutes between 8:30 a.m. and 9:00 a.m.
   There are 5 minutes between 9:00 a.m. and 9:05 a.m.
   Therefore, the length of the class is $30 + 5 = 35$ minutes.

   Answer: (C)

4. The side length of a square having an area of 144 cm^2 is $\sqrt{144}$ cm or 12 cm.

   Answer: (D)

5. The cost of nine \$1 items and five \$2 items is $(9 \times \$1) + (5 \times \$2)$, which is $\$9 + \$10$ or \$19.
   The correct answer is (C).
   (We may check that each of the remaining four answers gives a cost that is less than \$18.)

   Answer: (C)

6. Converting each of the improper fractions to a mixed fraction, we get $\frac{5}{2} = 2\frac{1}{2}$, $\frac{11}{4} = 2\frac{3}{4}$, $\frac{11}{5} = 2\frac{1}{5}$, $\frac{13}{4} = 3\frac{1}{4}$, and $\frac{13}{5} = 2\frac{3}{5}$.
   Of the five answers given, the number that lies between 3 and 4 on a number line is $3\frac{1}{4}$ or $\frac{13}{4}$.

   Answer: (D)

7. Exactly 2 of the $2 + 3 + 4 = 9$ seeds are sunflower seeds.
   Therefore, the probability that Carrie chooses a sunflower seed is $\frac{2}{9}$.

   Answer: (A)

8. Since $x = 4$, then $y = 3 \times 4 = 12$.

   Answer: (A)

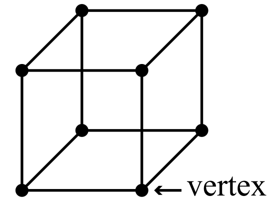
9. The sum of the three angles in any triangle is 180°.
   If one of the angles in an isosceles triangle measures 50°, then the sum of the measures of the two unknown angles in the triangle is $180^\circ - 50^\circ = 130^\circ$.
   Since the triangle is isosceles, then two of the angles in the triangle have equal measure.
   If the two unknown angles are equal in measure, then they each measure $130^\circ \div 2 = 65^\circ$.
   However, 65° and 65° is not one of the given answers.
   If the measure of one of the unknown angles is equal to the measure of the given angle, 50°, then the third angle in the triangle measures $180^\circ - 50^\circ - 50^\circ = 80^\circ$.
   Therefore, the measures of the other angles in this triangle could be 50° and 80°.

   Answer: (C)

10. Moving 3 letters clockwise from W, we arrive at the letter Z.
    Moving 1 letter clockwise from the letter Z, the alphabet begins again at A.
    Therefore, the letter that is 4 letters clockwise from W is A.
    Moving 4 letters clockwise from I, we arrive at the letter M.
    Moving 4 letters clockwise from N, we arrive at the letter R.
    The ciphertext of the message WIN is AMR.

    Answer: (C)

11. Every cube has exactly 8 vertices, as shown in the diagram.



Answer: (E)

12. The area of the 2 cm by 2 cm base of the rectangular prism is $2 \times 2 = 4$ cm^2.
The top face of the prism is identical to the base and so its area is also 4 cm^2.
Each of the 4 vertical faces of the prism has dimensions 2 cm by 1 cm, and thus has area $2 \times 1 = 2$ cm^2.
Therefore the surface area of the rectangular prism is $2 \times 4 + 4 \times 2 = 16$ cm^2.

Answer: (E)

13. Since 11 410 kg of rice is distributed into 3260 bags, then each bag contains $11\,410 \div 3260 = 3.5$ kg of rice.
Since a family uses 0.25 kg of rice each day, then it would take this family $3.5 \div 0.25 = 14$ days to use up one bag of rice.

Answer: (D)

14. Since Dalia's birthday is on a Wednesday, then any exact number of weeks after Dalia's birthday will also be a Wednesday.
Therefore, exactly 8 weeks after Dalia's birthday is also a Wednesday.
Since there are 7 days in each week, then $7 \times 8 = 56$ days after Dalia's birthday is a Wednesday.
Since 56 days after Dalia's birthday is a Wednesday, then 60 days after Dalia's birthday is a Sunday (since 4 days after Wednesday is Sunday).
Therefore, Bruce's birthday is on a Sunday.

Answer: (E)

15. *Solution 1*
Since each emu gets 2 treats and each chicken gets 4 treats, each of Karl's 30 birds gets *at least* 2 treats .
If Karl begins by giving his 30 birds exactly 2 treats each, then Karl will have given out $30 \times 2 = 60$ of the treats.
Since Karl has 100 treats to hand out, then he has $100 - 60 = 40$ treats left to give.
However, each emu has already received their 2 treats (since all 30 birds were given 2 treats).
So the remaining 40 treats must be given to chickens.
Each chicken is to receive 4 treats and has already received 2 treats.
Therefore, each chicken must receive 2 more treats.
Since there are 40 treats remaining, and each chicken receives 2 of these treats, then there are $40 \div 2 = 20$ chickens.
(We may check that if there are 20 chickens, then there are $30 - 20 = 10$ emus, and Karl would then give out $4 \times 20 + 2 \times 10 = 100$ treats.)

Solution 2
Using a trial and error approach, if Karl had 5 emus and $30 - 5 = 25$ chickens, then he would need to hand out $5 \times 2 + 25 \times 4 = 110$ treats.
Since Karl hands out 100 treats, we know that Karl has more emus than 5 (and fewer chickens than 25).
We show this attempt and continue with this approach in the table below.

Number of Emus	Number of Chickens	Number of Emu Treats	Number of Chicken Treats	Total Number of Treats
5	$30 - 5 = 25$	$5 \times 2 = 10$	$25 \times 4 = 100$	$10 + 100 = 110$
7	$30 - 7 = 23$	$7 \times 2 = 14$	$23 \times 4 = 92$	$14 + 92 = 106$
10	$30 - 10 = 20$	$10 \times 2 = 20$	$20 \times 4 = 80$	$20 + 80 = 100$

Therefore, Karl has 20 chickens.

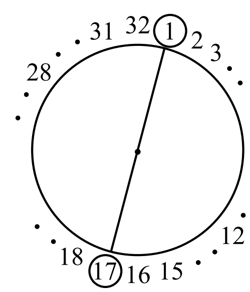
ANSWER: (D)

16. *Solution 1*

The integers 1 to 32 are spaced evenly and in order around the outside of a circle.
Consider drawing a first straight line that passes through the centre of the circle and joins any one pair of these 32 numbers.
This leaves $32 - 2 = 30$ numbers still to be paired.
Since this first line passes through the centre of the circle, it divides the circle in half.
In terms of the remaining 30 unpaired numbers, this means that 15 of these numbers lie on each side of the line drawn between the first pair.
Let the number that is paired with 12 be n.
If we draw the line through the centre joining 12 and n, then there are 15 numbers that lie between 12 and n (moving in either direction, clockwise or counter-clockwise).
Beginning at 12 and moving in the direction of the 13, the 15 numbers that lie between 12 and n are the numbers $13, 14, 15, \ldots, 26, 27$.
Therefore, the next number after 27 is the number n that is paired with 12.
The number paired with 12 is 28.

Solution 2

We begin by placing the integers 1 to 32, spaced evenly and in order, clockwise around the outside of a circle.
As in Solution 1, we recognize that there are 15 numbers on each side of the line which joins 1 with its partner.
Moving in a clockwise direction from 1, these 15 numbers are $2, 3, 4, \ldots, 15, 16$, and so 1 is paired with 17, as shown.
Since 2 is one number clockwise from 1, then the partner for 2 must be one number clockwise from 17, which is 18.
Similarly, 12 is 11 numbers clockwise from 1, so the partner for 12 must be 11 numbers clockwise from 17.
Therefore, the number paired with 12 is $17 + 11 = 28$.



ANSWER: (A)

17. We may begin by assuming that the area of the smallest circle is 1.

The area of the shaded middle ring is 6 times the area of the smallest circle, and thus has area 6.
The area of the unshaded outer ring is 12 times the area of the smallest circle, and thus has area 12.
The area of the largest circle is the sum of the areas of the smallest circle, the shaded middle ring, and the unshaded outer ring, or $1 + 6 + 12 = 19$.
Therefore, the area of the smallest circle is $\frac{1}{19}$ of the area of the largest circle.
Note: We assumed the area of the smallest circle was 1, however we could have assumed it to have any area. For example, assume the area of the smallest circle is 5 and redo the question. What is your final answer?

ANSWER: (E)

18. For the product of two integers to equal 1, the two integers must both equal 1 or must both equal -1.
Similarly, if the product of six integers is equal to 1, then each of the six integers must equal 1 or -1.
For the product of six integers, each of which is equal to 1 or -1, to equal 1, the number of -1s must be even, because an odd number of -1s would give a product that is negative.
That is, there must be zero, two, four or six -1s among the six integers.
We summarize these four possibilities in the table below.

Number of -1s	Product of the six integers	Sum of the six integers
0	$(1)(1)(1)(1)(1)(1) = 1$	$1+1+1+1+1+1=6$
2	$(-1)(-1)(1)(1)(1)(1) = 1$	$(-1)+(-1)+1+1+1+1=2$
4	$(-1)(-1)(-1)(-1)(1)(1) = 1$	$(-1)+(-1)+(-1)+(-1)+1+1=-2$
6	$(-1)(-1)(-1)(-1)(-1)(-1) = 1$	$(-1)+(-1)+(-1)+(-1)+(-1)+(-1)=-6$

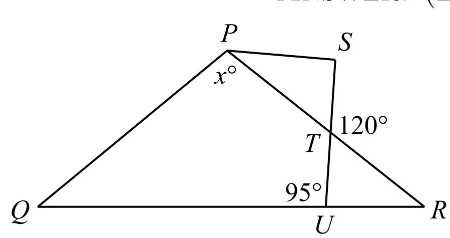
Of the answers given, the sum of such a group of six integers cannot equal 0.

ANSWER: (C)

19. Since the heights of the 4 athletes on the team are all different, then if Laurissa's height is different than each of these, there is no single mode height.
Therefore, Laurissa's height must be equal to the height of one of the 4 athletes on the team for there to be a single mode.
If Laurissa's height is 135 cm, then the median height of the 5 athletes is 160 cm which is not possible, since the median does not equal the mode.
Similarly, if Laurissa's height is 175 cm, then the median height of the 5 athletes is 170 cm which is not possible.
Therefore, Laurissa's height must equal 160 cm or 170 cm, since in either case the median height of the 5 athletes will equal Laurissa's height, which is the mode.
If Laurissa's height is 170 cm, then the mean height of the 5 athletes is
$\dfrac{135+160+170+170+175}{5} = 162$ cm.
If Laurissa's height is 160 cm, then the mean height of the 5 athletes is
$\dfrac{135+160+160+170+175}{5} = 160$ cm.
When Laurissa's height is 160 cm, the heights of the 5 athletes (measured in cm) are: 135, 160, 160, 170, 175.
In this case, each of the mode, median and mean height of the 5 athletes equals 160 cm.

ANSWER: (B)

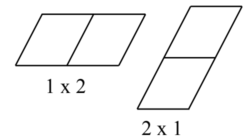
20. We begin by labelling points S, T and U, as shown.



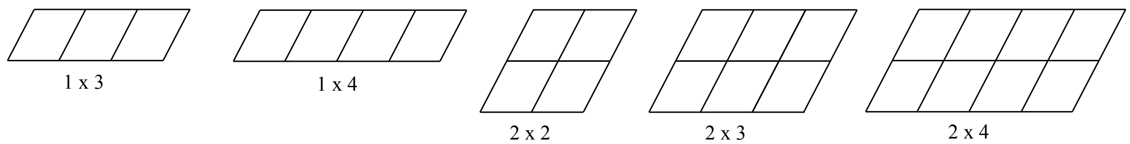
Since S, T, U lie on a straight line, $\angle STU$ measures 180°.
Therefore, $\angle RTU = 180^\circ - \angle STR = 180^\circ - 120^\circ = 60^\circ$.
Similarly, Q, U, R lie on a straight line, and so $\angle QUR$ measures 180°.
Therefore, $\angle TUR = 180^\circ - \angle TUQ = 180^\circ - 95^\circ = 85^\circ$.
The sum of the angles in $\triangle TUR$ is 180°.
Thus, $\angle TRU = 180^\circ - \angle RTU - \angle TUR = 180^\circ - 60^\circ - 85^\circ = 35^\circ$.
Since $\triangle PQR$ is isosceles with $PQ = PR$, then $\angle PQR = \angle PRQ = 35^\circ$.
Finally, the sum of the angles in $\triangle PQR$ is 180°, and so $x^\circ = 180^\circ - \angle PQR - \angle PRQ$ or $x^\circ = 180^\circ - 35^\circ - 35^\circ$ and so $x = 110$.

ANSWER: (A)

21. The figure formed by combining a pair of adjacent small parallelograms, is also a parallelogram.
For example, each of the two figures shown is a parallelogram.
The reason for this is that opposite sides of these new figures are equal in length and they are parallel. We use the notation $a \times b$ to mean that the new figure has a rows of the small parallelograms and b columns of the small parallelograms.



Similarly, more than 2 small parallelograms can be combined to form new parallelograms.
In addition to the small parallelogram (1×1) and the 1×2 and 2×1 parallelograms shown above, the sizes of the remaining parallelograms that appear in the figure are shown below.



In the table below, the number of parallelograms of each of the different sizes is shown.

Size	1×1	1×2	2×1	1×3	1×4	2×2	2×3	2×4
Number of Parallelograms	8	6	4	4	2	3	2	1

The number of parallelograms appearing in the figure is $8 + 6 + 4 + 4 + 2 + 3 + 2 + 1 = 30$.

Answer: (B)

22. *Solution 1*
The number of dimes in the jar is one more than the number of nickels.
If we remove one dime from the jar, then the number of coins remaining in the jar is $50 - 1 = 49$, and the value of the coins remaining in the jar is $\$5.00 - \$0.10 = \$4.90$.
Also, the number of dimes remaining in the jar is now equal to the number of nickels remaining in the jar, and the number of nickels remaining in the jar is three times the number of quarters remaining in the jar.
That is, for every 1 quarter remaining in the jar, there are 3 nickels and 3 dimes.
Consider groups consisting of exactly 1 quarter, 3 nickels and 3 dimes.
In each of these groups, there are 7 coins whose total value is $\$0.25 + 3 \times \$0.05 + 3 \times \$0.10 = \$0.25 + \$0.15 + \$0.30 = \$0.70$.
Since there are 49 coins having a value of \$4.90 remaining in the jar, then there must be 7 such groups of 7 coins remaining in the jar (since $7 \times 7 = 49$).
(We may check that 7 such groups of coins, with each group having a value of \$0.70, has a total value of $7 \times \$0.70 = \$4.90$, as required.)
Therefore, there are 7 quarters in the jar.

Solution 2
To find the number of quarters in the jar, we need only focus on the total number of coins in the jar, 50, or on the total value of the coins in the jar, \$5.00.
In the solution that follows, we consider both the number of coins in the jar as well as the value of the coins in the jar, to demonstrate that each approach leads to the same answer.
We use a trial and error approach.
Suppose that the number of quarters in the jar is 5 (the smallest of the possible answers given).
The value of 5 quarters is $5 \times 25¢ = 125¢$.
Since the number of nickels in the jar is three times the number of quarters, there would be $3 \times 5 = 15$ nickels in the jar.
The value of 15 nickels is $15 \times 5¢ = 75¢$.

Since the number of dimes in the jar is one more than the number of nickels, there would be $15 + 1 = 16$ dimes in the jar.
The value of 16 dimes is $16 \times 10¢ = 160¢$.
If there were 5 quarters in the jar, then the total number of coins in the jar would be $5 + 15 + 16 = 36$, and so there must be more than 5 quarters in the jar.
Similarly, if there were 5 quarters in the jar, then the total value of the coins in the jar would be $125¢ + 75¢ + 160¢ = 360¢$.
Since the value of the coins in the jar is \$5.00 or 500¢ then the number of quarters in the jar is greater than 5.
We summarize our next two trials in the table below.

Number of Quarters	Value of Quarters	Number of Nickels	Value of Nickels	Number of Dimes	Value of Dimes	Total Value of Coins
6	150¢	18	90¢	19	190¢	430¢
7	175¢	21	105¢	22	220¢	500¢

When there are 7 quarters in the jar, there are $7 + 21 + 22 = 50$ coins in the jar, as required.
When there are 7 quarters in the jar, the value of the coins in the jar is $175¢ + 105¢ + 220¢ = 500¢$ or \$5.00, as required.
In either case, the number of quarters in the jar is 7.

Answer: (A)

23. In each block $1223334444 \cdots 999999999$, there is 1 digit 1, 2 digits 2, 3 digits 3, and so on.
The total number of digits written in each block is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$.
We note that $1953 \div 45$ gives a quotient of 43 and a remainder of 18 (that is, $1953 = 45 \times 43 + 18$).
Since each block contains 45 digits, then 43 blocks contain $43 \times 45 = 1935$ digits.
Since $1953 - 1935 = 18$, then the 18^{th} digit written in the next block (the 44^{th} block) will be the 1953^{rd} digit written.
Writing out the first 18 digits in a block, we get 122333444455555666, and so the 1953^{rd} digit written is a 6.

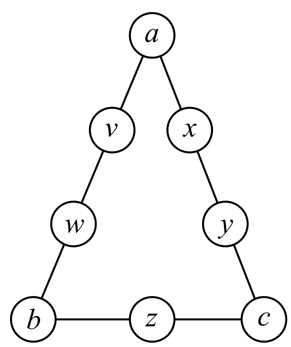
Answer: (C)

24. For a positive integer to be divisible by 9, the sum of its digits must be divisible by 9.
In this problem, we want to count the number of six-digit positive integers containing 2018 and divisible by 9.
Thus, we must find the remaining two digits, which together with 2018, form a six-digit positive integer that is divisible by 9.
The digits 2018 have a sum of $2 + 0 + 1 + 8 = 11$.
Let the remaining two digits be a and b so that the six-digit positive integer is $ab2018$ or $ba2018$ or $a2018b$ or $b2018a$ or $2018ab$ or $2018ba$.
When the digits a and b are added to 11, the sum must be divisible by 9.
That is, the sum $a + b + 11$ must be divisible by 9.
The smallest that each of a and b can be is 0, and so the smallest that the sum $a + b + 11$ can be is $0 + 0 + 11 = 11$.
The largest that each of a and b can be is 9, and so the largest that the sum $a + b + 11$ can be is $9 + 9 + 11 = 29$.
The only integers between 11 and 29 that are divisible by 9 are 18 and 27.
Therefore, either $a + b = 18 - 11 = 7$ or $a + b = 27 - 11 = 16$.
If $a + b = 7$, then the digits a and b are 0 and 7 or 1 and 6 or 2 and 5 or 3 and 4, in some order.
If a and b are 1 and 6, then the possible six-digit integers are $162\,018, 612\,018, 120\,186, 620\,181,$

201 816, and 201 861.
In this case, there are 6 possible six-digit positive integers.
Similarly, if a and b are 2 and 5, then there are 6 possible six-digit integers.
Likewise, if a and b are 3 and 4, then there are 6 possible six-digit integers.
If a and b are 0 and 7, then the possible six-digit integers are 702 018, 720 180, 201 870, and 201 807, since the integer cannot begin with the digit 0.
In this case, there are 4 possible six-digit positive integers.
Therefore, for the case in which the sum of a and b is 7, there are $6 + 6 + 6 + 4 = 22$ possible six-digit integers.
Finally, we consider the case for which the sum of the digits a and b is 16.
If $a + b = 16$, then the digits a and b are 7 and 9, or 8 and 8.
If a and b are 7 and 9, then there are again 6 possible six-digit integers (792 018, 972 018, 720 1869, 920 187, 201 879, and 201 897).
If a and b are 8 and 8, then there are 3 possible six-digit integers: 882 018, 820 188, and 201 888.
Therefore, for the case in which the sum of a and b is 16, there are $6 + 3 = 9$ possible six-digit integers, and so there are $22 + 9 = 31$ six-digit positive integers in total.
We note that all of these 31 six-digit positive integers are different from one another, and that they are the only six-digit positive integers satisfying the given conditions.
Therefore, there are 31 six-digit positive integers that are divisible by 9 and that contain the digits 2018 together and in this order.

ANSWER: (C)

25. We label the unknown numbers in the circles as shown:



Since the sum of the numbers along each side of the triangle is S then

$$S = a + v + w + b \qquad S = a + x + y + c \qquad S = b + z + c$$

When we add the numbers along each of the three sides of the triangles, we include each of a, b and c twice and obtain

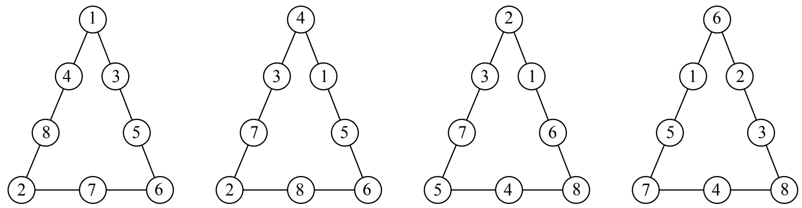
$$S+S+S = (a+v+w+b)+(a+x+y+c)+(b+z+c) = (a+v+w+b+z+c+y+x)+a+b+c$$

Now the numbers a, v, w, b, z, c, y, x are the numbers $1, 2, 3, 4, 5, 6, 7, 8$ in some order.
This means that $a + v + w + b + z + c + y + x = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$.
Therefore,

$$3S = 36 + a + b + c$$

Since $3S$ is a multiple of 3 and 36 is a multiple of 3, then $a+b+c$ (which equals $3S-36$) must also be a multiple of 3.
Looking at the possible numbers that can go in the circles, the smallest that $a+b+c$ can be is

$1 + 2 + 3$ or 6, which would make $3S = 36 + 6 = 42$ or $S = 14$. S cannot be any smaller than 14 because $a + b + c$ cannnot be any smaller than 6 and so $3S$ cannot be any smaller than 42. Looking at the possible numbers that can go in the circles, the largest that $a + b + c$ can be is $6 + 7 + 8$ or 21, which would make $3S = 36 + 21 = 57$ or $S = 19$. S cannot be any larger than 19 because $a + b + c$ cannot be any larger than 21 and so $3S$ cannot be any larger than 57.
So which of the values $S = 14, 15, 16, 17, 18, 19$ is actually possible?
The following diagrams show ways of completing the triangle with $S = 15, 16, 17, 19$:



Coming up with these examples requires a combination of reasoning and fiddling.
For example, consider the case when $S = 15$.
Since $3S = 36 + a + b + c$ and $S = 15$, then $a + b + c = 3 \times 15 - 36 = 9$.
In the given example, we have $a = 1$, $b = 2$ and $c = 6$.
Since the bottom row $(b + z + c)$ has the smallest number of circles, we put the largest of a, b, c here ($b = 2$ and $c = 6$) and then set $z = 15 - b - c = 7$. A bit of fiddling allows us to choose u, v, x, y appropriately to get the desired sums on the two other sides.
We note that there are other possible combinations of a, b, c with $a + b + c = 9$ (namely, $1, 3, 5$ and $2, 3, 4$). It turns out that neither of these possibilities can produce a triangle with $S = 15$.
In a similar way, we can determine examples like those shown with $S = 16, 17, 19$.

To complete the solution, we show that $S = 14$ and $S = 18$ are not possible.
Suppose that $S = 14$.
In this case, $a + b + c = 3S - 36 = 3 \times 14 - 36 = 6$.
The only integers from the list $1, 2, 3, 4, 5, 6, 7, 8$ which give this sum are $1, 2, 3$.
Consider the bottom row, which should have $b + z + c = 14$.
Since a, b, c are $1, 2, 3$ in some order, then $b + c$ is at most $2 + 3 = 5$.
Since the maximum number in the triangle is 8, then z is at most 8.
This makes $b + z + c$ at most $5 + 8 = 13$, which means that $b + z + c$ cannot equal 14.
This means that we cannot build a triangle with $S = 14$.
Suppose that $S = 18$.
In this case, $a + b + c = 3S - 36 = 3 \times 18 - 36 = 18$.
There are several possible sets of values for a, b, c: $3, 7, 8$ and $4, 6, 8$ and $5, 6, 7$.
Consider the bottom row again, which should have $b + z + c = 18$.
Here we have $a + b + c = 18$ and $b + z + c = 18$.
Since b and c are common to these sums and the total is the same in each case, then $a = z$, which is not allowed.
This means that we cannot build a triangle with $S = 18$.

In summary, the possible values of S are $15, 16, 17, 19$.
The sum of these values is 67.

Answer: (E)

Grade 8

1. Since the cost of 1 melon is \$3, then the cost of 6 melons is $6 \times \$3 = \$18$.

   Answer: (C)

2. The number line shown has length $1 - 0 = 1$.
   The number line is divided into 10 equal parts, and so each part has length $1 \div 10 = 0.1$.
   The P is positioned 2 of these equal parts before 1, and so the value of P is $1 - (2 \times 0.1) = 1 - 0.2 = 0.8$.
   (Similarly, we could note that P is positioned 8 equal parts after 0, and so the value of P is $8 \times 0.1 = 0.8$.)

   Answer: (D)

3. Following the correct order of operations, we get $(2+3)^2 - (2^2+3^2) = 5^2 - (4+9) = 25 - 13 = 12$.

   Answer: (B)

4. Since Lakshmi is travelling at 50 km each hour, then in one half hour (30 minutes) she will travel $50 \div 2 = 25$ km.

   Answer: (C)

5. Exactly 2 of the $3 + 2 + 4 + 6 = 15$ flowers are tulips.
   Therefore, the probability that Evgeny randomly chooses a tulip is $\frac{2}{15}$.

   Answer: (E)

6. The range of the students' heights is equal to the difference between the height of the tallest student and the height of the shortest student.
   Reading from the graph, Emma is the tallest student and her height is approximately 175 cm.
   Kinley is the shortest student and her height is approximately 100 cm.
   Therefore, the range of heights is closest to $175 - 100 = 75$ cm.

   Answer: (A)

7. *Solution 1*
   The circumference of a circle, C, is given by the formula $C = \pi \times d$, where d is the diameter of the circle.
   Since the circle has a diameter of 1 cm, then its circumference is $C = \pi \times 1 = \pi$ cm.
   Since π is approximately 3.14, then the circumference of the circle is between 3 cm and 4 cm.

   Solution 2
   The circumference of a circle, C, is given by the formula $C = 2 \times \pi \times r$, where r is the radius of the circle.
   Since the circle has a diameter of 1 cm, then its radius is $r = \frac{1}{2}$ cm, and so the circumference is $C = 2 \times \pi \times \frac{1}{2} = \pi$ cm.
   Since π is approximately 3.14, then the circumference of the circle is between 3 cm and 4 cm.

   Answer: (B)

8. The ratio of the amount of cake eaten by Rich to the amount of cake eaten by Ben is $3 : 1$.
   Thus, if the cake was divided into 4 pieces of equal size, then Rich ate 3 pieces and Ben ate 1 piece or Ben ate $\frac{1}{4}$ of the cake.
   Converting to a percent, Ben ate $\frac{1}{4} \times 100\% = 0.25 \times 100\% = 25\%$ of the cake.

   Answer: (D)

9. Moving 3 letters clockwise from W, we arrive at the letter Z.
Moving 1 letter clockwise from the letter Z, the alphabet begins again at A.
Therefore, the letter that is 4 letters clockwise from W is A.
Moving 4 letters clockwise from I, we arrive at the letter M.
Moving 4 letters clockwise from N, we arrive at the letter R.
The ciphertext of the message WIN is AMR.

Answer: (C)

10. The smallest of 3 consecutive even numbers is 2 less than the middle number.
The largest of 3 consecutive even numbers is 2 more than the middle number.
Therefore, the sum of 3 consecutive even numbers is three times the middle number.
To see this, consider subtracting 2 from the largest of the 3 numbers, and adding 2 to the smallest of the 3 numbers.
Since we have subtracted 2 and also added 2, then the sum of these 3 numbers is equal to the sum of the original 3 numbers.
However, if we subtract 2 from the largest number, the result is equal to the middle number, and if we add 2 to the smallest number, the result is equal to the middle number.
Therefore, the sum of any 3 consecutive even numbers is equal to three times the middle number.
Since the sum of the 3 consecutive even numbers is 312, then the middle number is equal to $312 \div 3 = 104$.
If the middle number is 104, then the largest of the 3 consecutive even numbers is $104+2 = 106$.
(We may check that $102 + 104 + 106$ is indeed equal to 312.)

Answer: (B)

11. If $4x + 12 = 48$, then $4x = 48 - 12$ or $4x = 36$, and so $x = \frac{36}{4} = 9$.

Answer: (E)

12. The time in Vancouver is 3 hours earlier than the time in Toronto.
Therefore, when it is 6:30 p.m. in Toronto, the time in Vancouver is 3:30 p.m..

Answer: (C)

13. *Solution 1*
Mateo receives \$20 every hour for one week.
Since there are 24 hours in each day, and 7 days in each week, then Mateo receives $\$20 \times 24 \times 7 = \$3360$ over the one week period.
Sydney receives \$400 every day for one week.
Since there are 7 days in each week, then Sydney receives $\$400 \times 7 = \$2800$ over the one week period.
The difference in the total amounts of money that they receive over the one week period is $\$3360 - \$2800 = \$560$.

Solution 2
Mateo receives \$20 every hour for one week.
Since there are 24 hours in each day, then Mateo receives $\$20 \times 24 = \$480$ each day.
Sydney receives \$400 each day, and so Mateo receives $\$480 - \$400 = \$80$ more than Sydney each day.
Since there are 7 days in one week, then the difference in the total amounts of money that they receive over the one week period is $\$80 \times 7 = \$560$.

Answer: (A)

14. Since $2018 = 2 \times 1009$, and both 2 and 1009 are prime numbers, then the required sum is $2 + 1009 = 1011$.
Note: In the question, we are given that 2018 has exactly two divisors that are prime numbers, and since 2 is a prime divisor of 2018, then 1009 must be the other prime divisor.

ANSWER: (B)

15. The first place award can be given out to any one of the 5 classmates.
Once the first place award has been given, there are 4 classmates remaining who could be awarded second place (since the classmate who was awarded first place cannot also be awarded second place).
For each of the 5 possible first place winners, there are 4 classmates who could be awarded second place, and so there are 5×4 ways that the first and second place awards can be given out.
Once the first and second place awards have been given, there are 3 classmates remaining who could be awarded the third place award (since the classmates who were awarded first place and second place cannot also be awarded third place).
For each of the 5 possible first place winners, there are 4 classmates who could be awarded second place, and there are 3 classmates who could be awarded third place.
That is, there are $5 \times 4 \times 3 = 60$ ways that the first, second and third place awards can be given out.

ANSWER: (B)

16. For the product of two integers to equal 1, the two integers must both equal 1 or must both equal -1.
Similarly, if the product of six integers is equal to 1, then each of the six integers must equal 1 or -1.
For the product of six integers, each of which is equal to 1 or -1, to equal 1, the number of -1s must be even, because an odd number of -1s would give a product that is negative.
That is, there must be zero, two, four or six -1s among the six integers.
We summarize these four possibilities in the table below.

Number of -1s	Product of the six integers	Sum of the six integers
0	$(1)(1)(1)(1)(1)(1) = 1$	$1+1+1+1+1+1 = 6$
2	$(-1)(-1)(1)(1)(1)(1) = 1$	$(-1)+(-1)+1+1+1+1 = 2$
4	$(-1)(-1)(-1)(-1)(1)(1) = 1$	$(-1)+(-1)+(-1)+(-1)+1+1 = -2$
6	$(-1)(-1)(-1)(-1)(-1)(-1) = 1$	$(-1)+(-1)+(-1)+(-1)+(-1)+(-1) = -6$

Of the answers given, the sum of such a group of six integers cannot equal 0.

ANSWER: (C)

17. *Solution 1*
Each translation to the right 5 units increases the x-coordinate of point A by 5.
Similarly, each translation up 3 units increases the y-coordinate of point A by 3.
After 1 translation, the point $A(-3,2)$ would be at $B(-3+5, 2+3)$ or $B(2,5)$.
After 2 translations, the point $A(-3,2)$ would be at $C(2+5, 5+3)$ or $C(7,8)$.
After 3 translations, the point $A(-3,2)$ would be at $D(7+5, 8+3)$ or $D(12,11)$.
After 4 translations, the point $A(-3,2)$ would be at $E(12+5, 11+3)$ or $E(17,14)$.
After 5 translations, the point $A(-3,2)$ would be at $F(17+5, 14+3)$ or $F(22,17)$.
After 6 translations, the point $A(-3,2)$ would be at $G(22+5, 17+3)$ or $G(27,20)$.
After these 6 translations, the point is at $(27,20)$ and so $x+y = 27+20 = 47$.

Solution 2
Each translation to the right 5 units increases the x-coordinate of point A by 5.
Similarly, each translation up 3 units increases the y-coordinate of point A by 3.
Therefore, each translation of point $A(-3, 2)$ to the right 5 units and up 3 units increases the sum of the x- and y-coordinates, $x + y$, by $5 + 3 = 8$.
After 6 of these translations, the sum $x + y$ will increase by $6 \times 8 = 48$.
The sum of the x- and y-coordinates of point $A(-3, 2)$ is $-3 + 2 = -1$.
After these 6 translations, the value of $x + y$ is $-1 + 48 = 47$.

ANSWER: (D)

18. *Solution 1*
The volume of any rectangular prism is given by the product of the length, the width, and the height of the prism.
When the length of the prism is doubled, the product of the new length, the width, and the height of the prism doubles, and so the volume of the prism doubles.
Since the original prism has a volume of 30 cm^3, then doubling the length creates a new prism with volume $30 \times 2 = 60$ cm^3.
When the width of this new prism is tripled, the product of the length, the new width, and the height of the prism is tripled, and so the volume of the prism triples.
Since the prism has a volume of 60 cm^3, then tripling the width creates a new prism with volume $60 \times 3 = 180$ cm^3.
When the height of the prism is divided by four, the product of the length, the width, and the new height of the prism is divided by four, and so the volume of the prism is divided by four.
Since the prism has a volume of 180 cm^3, then dividing the height by four creates a new prism with volume $180 \div 4 = 45$ cm^3.

Solution 2
The volume of any rectangular prism is given by the product of its length, l, its width, w, and its height, h, which equals lwh.
When the length of the prism is doubled, the length of the new prism is $2l$.
Similarly, when the width is tripled, the new width is $3w$, and when the height is divided by four, the new height is $\frac{1}{4}h$.
Therefore, the volume of the new prism is the product of its length, $2l$, its width, $3w$, and its height, $\frac{1}{4}h$, which equals $(2l)(3w)(\frac{1}{4}h)$ or $\frac{3}{2}lwh$.
That is, the volume of the new prism is $\frac{3}{2}$ times larger than the volume of the original prism.
Since the original prism has a volume of 30 cm^3, then doubling the length, tripling the width, and dividing the height by four, creates a new prism with volume $30 \times \frac{3}{2} = \frac{90}{2} = 45$ cm^3.

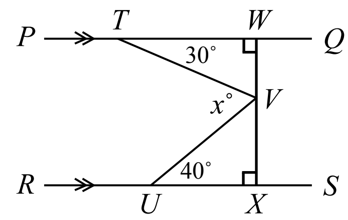
ANSWER: (E)

19. The mean height of the group of children is equal to the sum of the heights of the children divided by the number of children in the group.
Therefore, the mean height of the group of children increases by 6 cm if the sum of the increases in the heights of the children, divided by the number of children in the group, is equal to 6.
If 12 of the children were each 8 cm taller, then the sum of the increases in the heights of the children would be $12 \times 8 = 96$ cm.
Thus, 96 divided by the number of children in the group is equal to 6.
Since $96 \div 16 = 6$, then the number of children in the group is 16.

ANSWER: (A)

20. *Solution 1*

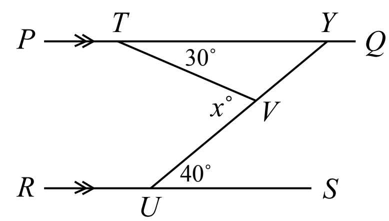
We begin by constructing a line segment WX perpendicular to PQ and passing through V, as shown.



Since RS is parallel to PQ, then WX is also perpendicular to RS.
In $\triangle TWV$, $\angle TWV = 90°$ and $\angle WTV = 30°$.
Since the sum of the angles in a triangle is $180°$, then $\angle TVW = 180° - 30° - 90° = 60°$.
In $\triangle UXV$, $\angle UXV = 90°$ and $\angle VUX = 40°$.
Similarly, $\angle UVX = 180° - 40° - 90° = 50°$.
Since WX is a straight line segment, then $\angle TVW + \angle TVU + \angle UVX = 180°$.
That is, $60° + \angle TVU + 50° = 180°$ or $\angle TVU = 180° - 60° - 50°$ or $\angle TVU = 70°$, and so $x = 70$.

Solution 2

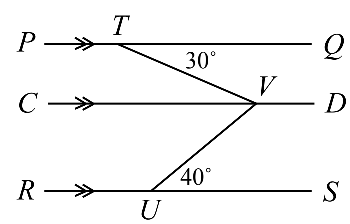
We begin by extending line segment UV to meet PQ at Y, as shown.



Since RS is parallel to PQ, then $\angle TYV$ and $\angle VUS$ are alternate angles, and so $\angle TYV = \angle VUS = 40°$.
In $\triangle TYV$, $\angle TYV = 40°$ and $\angle YTV = 30°$.
Since the sum of the angles in a triangle is $180°$, then $\angle TVY = 180° - 40° - 30° = 110°$.
Since UY is a straight line segment, then $\angle TVU + \angle TVY = 180°$.
That is, $\angle TVU + 110° = 180°$ or $\angle TVU = 180° - 110°$ or $\angle TVU = 70°$, and so $x = 70$.

Solution 3

We begin by constructing a line segment CD parallel to both PQ and RS, and passing through V, as shown.



Since CD is parallel to PQ, then $\angle QTV$ and $\angle TVC$ are alternate angles, and so $\angle TVC = \angle QTV = 30°$.
Similarly, since CD is parallel to RS, then $\angle CVU$ and $\angle VUS$ are alternate angles, and so $\angle CVU = \angle VUS = 40°$.
Since $\angle TVU = \angle TVC + \angle CVU$, then $\angle TVU = 30° + 40° = 70°$, and so $x = 70$.

ANSWER: (D)

21. *Solution 1*

We begin by assuming that there are 100 marbles in the bag.
The probability of choosing a brown marble is 0.3, and so the number of brown marbles in the bag must be 30 since $\frac{30}{100} = 0.3$.
Choosing a brown marble is three times as likely as choosing a purple marble, and so the number of purple marbles in the bag must be $30 \div 3 = 10$.
Choosing a green marble is equally likely as choosing a purple marble, and so there must also be 10 green marbles in the bag.
Since there are 30 brown marbles, 10 purple marbles, and 10 green marbles in the bag, then there are $100 - 30 - 10 - 10 = 50$ marbles in the bag that are either red or yellow.
Choosing a red marble is equally likely as choosing a yellow marble, and so the number of red marbles in the bag must equal the number of yellow marbles in the bag.
Therefore, the number of red marbles in the bag is $50 \div 2 = 25$.
Of the 100 marbles in the bag, there are $25 + 10 = 35$ marbles that are either red or green.
The probability of choosing a marble that is either red or green is $\frac{35}{100} = 0.35$.

Solution 2

The probability of choosing a brown marble is 0.3.

The probability of choosing a brown marble is three times that of choosing a purple marble, and so the probability of choosing a purple marble is $0.3 \div 3 = 0.1$.

The probability of choosing a green marble is equal to that of choosing a purple marble, and so the probability of choosing a green marble is also 0.1.

Let the probability of choosing a red marble be p.

The probability of choosing a red marble is equal to that of choosing a yellow marble, and so the probability of choosing a yellow marble is also p.

The total of the probabilities of choosing a marble must be 1.

Therefore, $0.3 + 0.1 + 0.1 + p + p = 1$ or $0.5 + 2p = 1$ or $2p = 0.5$, and so $p = 0.5 \div 2 = 0.25$.

The probability of choosing a red marble is 0.25 and the probability of choosing a green marble is 0.1, and so the probability of choosing a marble that is either red or green is $0.25 + 0.1 = 0.35$.

ANSWER: (C)

22. The area of square $PQRS$ is $(30)(30) = 900$.

Each of the 5 regions has equal area, and so the area of each region is $900 \div 5 = 180$.

The area of $\triangle SPT$ is equal to $\frac{1}{2}(PS)(PT) = \frac{1}{2}(30)(PT) = 15(PT)$.

The area of $\triangle SPT$ is 180, and so $15(PT) = 180$ or $PT = 180 \div 15 = 12$.

The area of $\triangle STU$ is 180.

Let the base of $\triangle STU$ be UT.

The height of $\triangle STU$ is equal to PS since PS is the perpendicular distance between base UT (extended) and the vertex S.

The area of $\triangle STU$ is equal to $\frac{1}{2}(PS)(UT) = \frac{1}{2}(30)(UT) = 15(UT)$.

The area of $\triangle STU$ is 180, and so $15(UT) = 180$ or $UT = 180 \div 15 = 12$.

In $\triangle SPT$, $\angle SPT = 90^\circ$. By the Pythagorean Theorem, $ST^2 = PS^2 + PT^2$ or $ST^2 = 30^2 + 12^2$ or $ST^2 = 900 + 144 = 1044$, and so $ST = \sqrt{1044}$ (since $ST > 0$).

In $\triangle SPU$, $\angle SPU = 90^\circ$ and $PU = PT + UT = 12 + 12 = 24$.

By the Pythagorean Theorem, $SU^2 = PS^2 + PU^2$ or $ST^2 = 30^2 + 24^2$ or $SU^2 = 900 + 576 = 1476$, and so $SU = \sqrt{1476}$ (since $SU > 0$).

Therefore, $\dfrac{SU}{ST} = \dfrac{\sqrt{1476}}{\sqrt{1044}}$ which is approximately equal to 1.189.

Of the answers given, $\dfrac{SU}{ST}$ is closest to 1.19.

ANSWER: (B)

23. *Solution 1*

In the table, we determine the value of the product $n(n + 1)(n + 2)$ for the first 10 positive integers:

n	$n(n+1)(n+2)$
1	$1 \times 2 \times 3 = 6$
2	$2 \times 3 \times 4 = 24$
3	$3 \times 4 \times 5 = 60$
4	$4 \times 5 \times 6 = 120$
5	$5 \times 6 \times 7 = 210$
6	$6 \times 7 \times 8 = 336$
7	$7 \times 8 \times 9 = 504$
8	$8 \times 9 \times 10 = 720$
9	$9 \times 10 \times 11 = 990$
10	$10 \times 11 \times 12 = 1320$

From the table, we see that $n(n+1)(n+2)$ is a multiple of 5 when $n = 3, 4, 5, 8, 9, 10$.
In general, because 5 is a prime number, the product $n(n+1)(n+2)$ is a multiple of 5 exactly when at least one of its factors $n, n+1, n+2$ is a multiple of 5.
A positive integer is a multiple of 5 when its units (ones) digit is either 0 or 5.
Next, we make a table that lists the units digits of $n+1$ and $n+2$ depending on the units digit of n:

Units digit of n	Units digit of $n+1$	Units digit of $n+2$
1	2	3
2	3	4
3	4	5
4	5	6
5	6	7
6	7	8
7	8	9
8	9	0
9	0	1
0	1	2

From the table, one of the three factors has a units digit of 0 or 5 exactly when the units digit of n is one of $3, 4, 5, 8, 9, 0$. (Notice that this agrees with the first table above.)
This means that 6 out of each block of 10 values of n ending at a multiple of 10 give a value for $n(n+1)(n+2)$ that is a multiple of 5.
We are asked for the 2018^{th} positive integer n for which $n(n+1)(n+2)$ is a multiple of 5.
Note that $2018 = 336 \times 6 + 2$.
This means that, in the first $336 \times 10 = 3360$ positive integers, there are $336 \times 6 = 2016$ integers n for which $n(n+1)(n+2)$ is a multiple of 5. (Six out of every ten integers have this property.)
We need to count two more integers along the list.
The next two integers n for which $n(n+1)(n+2)$ is a multiple of 5 will have units digits 3 and 4, and so are 3363 and 3364.
This means that 3364 is the 2018^{th} integer with this property.

Solution 2
In the table below, we determine the value of the product $n(n+1)(n+2)$ for the first 10 positive integers n.

n	$n(n+1)(n+2)$
1	$1 \times 2 \times 3 = 6$
2	$2 \times 3 \times 4 = 24$
3	$3 \times 4 \times 5 = 60$
4	$4 \times 5 \times 6 = 120$
5	$5 \times 6 \times 7 = 210$
6	$6 \times 7 \times 8 = 336$
7	$7 \times 8 \times 9 = 504$
8	$8 \times 9 \times 10 = 720$
9	$9 \times 10 \times 11 = 990$
10	$10 \times 11 \times 12 = 1320$

From the table, we see that the value of $n(n+1)(n+2)$ is not a multiple of 5 when $n = 1$ or when $n = 2$, but that $n(n+1)(n+2)$ is a multiple of 5 when $n = 3, 4, 5$.

Similarly, we see that the value of $n(n+1)(n+2)$ is a not multiple of 5 when $n = 6, 7$, but that $n(n+1)(n+2)$ is a multiple of 5 when $n = 8, 9, 10$.
That is, if we consider groups of 5 consecutive integers beginning at $n = 1$, it appears that for the first 2 integers in the group, the value of $n(n+1)(n+2)$ is not a multiple of 5, and for the last 3 integers in the group, the value of $n(n+1)(n+2)$ is a multiple of 5.
Will this pattern continue?
Since 5 is a prime number, then for each value of $n(n+1)(n+2)$ that is a multiple of 5, at least one of the factors $n, n+1$ or $n+2$ must be divisible by 5.
(We also note that for each value of $n(n+1)(n+2)$ that is not a multiple of 5, each of n, $n+1$ and $n+2$ is not divisible by 5.)
For what values of n is at least one of n, $n+1$ or $n+2$ divisible by 5, and thus $n(n+1)(n+2)$ divisible by 5?
When n is a multiple of 5, then the value of $n(n+1)(n+2)$ is divisible by 5.
When n is one 1 less than a multiple of 5, then $n+1$ is a multiple of 5 and and so $n(n+1)(n+2)$ is divisible by 5.
Finally, when n is 2 less than a multiple of 5, then $n+2$ is a multiple of 5 and and so $n(n+1)(n+2)$ is divisible by 5.
We also note that when n is 3 less than a multiple of 5, each of n, $n+1$ (which is 2 less than a multiple of 5), and $n+2$ (which is 1 less than a multiple of 5) is not divisible by 5, and so $n(n+1)(n+2)$ is not divisible by 5.
Similarly, when n is 4 less than a multiple of 5, then each of n, $n+1$ (which is 3 less than a multiple of 5), and $n+2$ (which is 2 less than a multiple of 5) is not divisible by 5, and so $n(n+1)(n+2)$ is not divisible by 5.
We have shown that the value of $n(n+1)(n+2)$ is a multiple of 5 when n is: a multiple of 5, or 1 less than a multiple of 5, or 2 less than a multiple of 5.
We have also shown that the value of $n(n+1)(n+2)$ is not a multiple of 5 when n is: 3 less than a multiple of 5, or 4 less than a multiple of 5.
Since every positive integer is either a multiple of 5, or 1, 2, 3, or 4 less than a multiple of 5, we have considered the value of $n(n+1)(n+2)$ for all positive integers n.
In the first group of 5 positive integers from 1 to 5, there are exactly 3 integers n ($n = 3, 4, 5$) for which $n(n+1)(n+2)$ is a multiple of 5.
Similarly, in the second group of 5 positive integers from 6 to 10, there are exactly 3 integers n ($n = 8, 9, 10$) for which $n(n+1)(n+2)$ is a multiple of 5.
As was shown above, this pattern continues giving 3 values for n for which $n(n+1)(n+2)$ is a multiple of 5 in each successive group of 5 consecutive integers.
When these positive integers, n, are listed in increasing order, we are required to find the 2018^{th} integer in the list.
Since $2018 = 3 \times 672 + 2$, then among the first 672 successive groups of 5 consecutive integers (which is the first $5 \times 672 = 3360$ positive integers), there are exactly $3 \times 672 = 2016$ integers n for which $n(n+1)(n+2)$ is a multiple of 5.
The next two integers, 3361 and 3362, do not give values for n for which $n(n+1)(n+2)$ is a multiple of 5 (since 3361 is 4 less than a multiple of 5 and 3362 is 3 less than a multiple of 5).
The next two integers, 3363 and 3364, do give values for n for which $n(n+1)(n+2)$ is a multiple of 5.
Therefore, the 2018^{th} integer in the list is 3364.

Answer: (E)

24. Let a, b, c, and d be the four distinct digits that are chosen from the digits 1 to 9. These four digits can be arranged in 24 different ways to form 24 distinct four-digit numbers. Consider breaking the solution up into the 3 steps that follow.

Step 1: Determine how many times each of the digits a, b, c, d appears as thousands, hundreds, tens, and units (ones) digits among the 24 four-digit numbers

If one of the 24 four-digit numbers has thousands digit equal to a, then the remaining three digits can be arranged in 6 ways: bcd, bdc, cbd, cdb, dbc, and dcb.
That is, there are exactly 6 four-digit numbers whose thousands digit is a.
If the thousands digit of the four-digit number is b, then the remaining three digits can again be arranged in 6 ways to give 6 different four-digit numbers whose thousands digit is b.
Similarly, there are 6 four-digit numbers whose thousands digit is c and 6 four-digit numbers whose thousands digit is d.
The above reasoning can be used to explain why there are also 6 four-digit numbers whose hundreds digit is a, 6 four-digit numbers whose hundreds digit is b, 6 four-digit numbers whose hundreds digit is c, and 6 four-digit numbers whose hundreds digit is d.
In fact, we can extend this reasoning to conclude that among the 24 four-digit numbers, each of the digits a, b, c, d, appears exactly 6 times as the thousands digit, 6 times as the hundreds digit, 6 times as the tens digit, and 6 times as the units digit.

Step 2: Determine N, the sum of the 24 four-digit numbers

Since each of the digits a, b, c, d appear 6 times as the units digit of the 24 four-digit numbers, then the sum of the units digits of the 24 four-digit numbers is $6a+6b+6c+6d$ or $6\times(a+b+c+d)$.
Similarly, since each of the digits a, b, c, d appear 6 times as the tens digit of the 24 four-digit numbers, then the sum of the tens digits of the 24 four-digit numbers is $10\times 6\times(a+b+c+d)$.
Continuing in this way for the hundreds digits and the thousands digits, we get

$$\begin{aligned} N &= 1000\times 6\times(a+b+c+d)+100\times 6\times(a+b+c+d)+10\times 6\times(a+b+c+d) \\ &\quad + 6\times(a+b+c+d) \\ &= 6000(a+b+c+d)+600(a+b+c+d)+60(a+b+c+d)+6(a+b+c+d) \end{aligned}$$

If we let $s=a+b+c+d$, then $N=6000s+600s+60s+6s=6666s$.

Step 3: Determine the largest sum of the distinct prime factors of $N=6666s$

Writing 6666 as a product of prime numbers, we get $6666=6\times 1111=2\times 3\times 11\times 101$.
So then $N=2\times 3\times 11\times 101\times s$ and thus the sum of the distinct prime factors of N is $2+3+11+101$ added to the prime factors of s which are distinct from $2, 3, 11$, and 101.
That is, to determine the largest sum of the distinct prime factors of N, we need to find the largest possible sum of the prime factors of s which are not equal to $2, 3, 11$, and 101.
Since $s=a+b+c+d$ for distinct digits a, b, c, d chosen from the digits 1 to 9, then the largest possible value of s is $9+8+7+6=30$ and the smallest possible value is $1+2+3+4=10$.
If $s=29$ (which occurs when a, b, c, d are equal to $9, 8, 7, 5$ in some order), then $N=2\times 3\times 11\times 101\times 29$ and the sum of the prime factors of N is $2+3+11+101+29=146$ (since 29 is a prime number).
If s is any other integer between 10 and 30 inclusive, the sum of its prime factors is less than 29. (See if you can convince yourself that all other possible values of s have prime factors whose sum is less than 29. Alternately, you could list the prime factors of each of the integers from 10 to 30 to see that 29 is indeed the largest sum.)
Therefore, the largest sum of the distinct prime factors of N is 146.

Answer: (D)

25. Since the grid has height 2, then there are only two possible lengths for vertical arrows: 1 or 2. Since all arrows in any path have different lengths, then there can be at most 2 vertical arrows in any path.
This means that there cannot be more than 3 horizontal arrows in any path. (If there were 4 or more horizontal arrows then there would have to be 2 consecutive horizontal arrows in the path, which is forbidden by the requirement that two consecutive arrows must be perpendicular.)
This means that any path consists of at most 5 arrows.

Using the restriction that all arrows in any path must have different lengths, we now determine the possible combinations of lengths of vertical arrows and of horizontal arrows to get from A to F.
Once we have determined the possible combinations of vertical and horizontal arrows independently, we then try to combine and arrange them.
First, we look at vertical arrows.
The grid has height 2, and A is 1 unit below F so any combination of vertical arrows in a path must have a net results of 1 unit up.
We use "U" for up and "D" for down.
The possible combinations are:

U1 (up arrow with length 1)
D1, U2 (down arrow with length 1, up arrow with length 2)

Next, we look at horizontal arrows.
The grid has width 12, and A is 9 units to the left of F so any combination of horizontal arrows in a path must have a net result of 9 units right.
We use "R" for right and "L" for left.
Many of these combinations of arrows can be re-arranged in different orders. We will deal with this later.
We proceed by looking at combinations of 1 arrow, then 2 arrows, then 3 arrows.
We note that every combination of vertical arrows includes an arrow with length 1 so we can ignore any horizontal combination that uses an arrow of length 1.
Also, any combination of 3 horizontal arrows must be combined with a combination of 2 horizontal arrows, which have lengths 1 and 2.
Thus, we can ignore any combination of 3 horizontal arrows that includes either or both of an arrow of length 1 and length 2.

a) R9
b) R2, R7
c) R3, R6
d) R4, R5
e) L2, R11
f) L3, R12
g) L3, R4, R8
h) L3, R5, R7
i) L4, R3, R10
j) L4, R5, R8
k) L4, R6, R7
l) L5, R3, R11
m) L5, R4, R10
n) L5, R6, R8
o) L6, R3, R12
p) L6, R4, R11
q) L6, R5, R10
r) L6, R7, R8
s) L7, R4, R12
t) L7, R5, R11
u) L7, R6, R10
v) L8, R5, R12
w) L8, R6, R11
x) L8, R7, R10
y) L9, R6, R12
z) L9, R7, R11
aa) L9, R8, R10
ab) L10, R7, R12
ac) L10, R8, R11
ad) L11, R8, R12

There is only one combination of 1 horizontal arrow.
The combinations of 2 horizontal arrows are listed by including those with two right arrows first (in increasing order of length) and then those with left and right arrows (in increasing order of length).
The combinations of 3 arrows are harder to list completely.
There are no useful combinations that include either 3 right arrows or 2 left arrows, since in either case an arrow of length 1 or 2 would be required.
Here, we have listed combinations of 2 right arrows, then those with "L3" (left arrow of length 3), then those with "L4", and so on.

Now we combine the vertical and horizontal combinations to get the paths.
Each combination of arrow directions and lengths can be drawn to form a path.
Vertical combination U1 can only be combined with horizontal paths a through f, since it cannot be combined with 3 horizontal arrows.

a) There are 2 paths: U1/R9 or R9/U1.

b) There are 2 paths: R2/U1/R7 or R7/U1/R2.

c) Again, there are 2 paths.

d) Again, there are 2 paths.

e) There is 1 path: L2/U1/R11. This is because the arrows must alternate horizontal, vertical, horizontal and we cannot end with a left arrow.

f) Again, there is 1 path.

This is 10 paths so far.
Vertical combination D1, U2 can be combined with horizontal paths of lengths 1, 2 or 3.

a) There is 1 path: D1/R9/U2. This is because we cannot end with a down arrow.

b) Not possible because this would include two arrows of length 2.

c) There are 4 paths: R3/D1/R6/U2, R6/D1/R3/U2, D1/R3/U2/R6, D1/R6/U2/R3. We can interchange R3 and R6 as well as picking whether to start with a vertical or horizontal arrow.

d) Again, there are 4 paths.

e) Not possible because this would include two arrows of length 2.

f) There are 2 paths: L3/D1/R12/U2 and D1/L3/U2/R12.

g) There are 4 paths: L3/D1/R4/U2/R8, R4/D1/L3/U2/R8, L3/D1/R8/U2/R4, R8/D1/L3/U2/R4. Each such combination must start with a horizontal arrow, must end with a right arrow, and must have the down arrow before the up arrow.

h) Again, there are 4 paths.

i) There is 1 path: R3/D1/L4/U2/R10. We cannot begin with R10 or L4 since either would take the path off of the grid, and we must end with an arrow to the right.

j,k) There are 2 paths in each case. For example, with j we have R5/D1/L4/U2/R8 and R8/D1/L4/U2/R5.

l,m) As with i, there is 1 path in each case.

n) As with j, there are 2 paths.

o,p,q) As with i, there is 1 path in each case.

r) As with j, there are 2 paths.

s) to ad) In each of these 12 cases, there is 1 path as with i.

Including the previously counted 10 paths that use U1 only, we have

$$10 + 1 + 4 + 4 + 2 + 4 + 4 + 1 + 2(2) + 2(1) + 2 + 3(1) + 2 + 12(1) = 55$$

paths in total.

Answer: (B)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2017 Gauss Contests

(Grades 7 and 8)

Wednesday, May 10, 2017
(in North America and South America)

Thursday, May 11, 2017
(outside of North America and South America)

Solutions

Centre for Education in Mathematics and Computing Faculty and Staff

Gauss Contest Committee

Grade 7

1. Evaluating, $(2 + 4 + 6) - (1 + 3 + 5) = 12 - 9 = 3$.

   Answer: (B)

2. Reading from the tallest bar on the graph, approximately 50 students play soccer.
   Since this is larger than the number of students who play any of the other sports, then soccer is played by the most students.

   Answer: (C)

3. Michael has \$280 in \$20 bills and so the number of \$20 bills that he has is $280 \div 20 = 14$.

   Answer: (C)

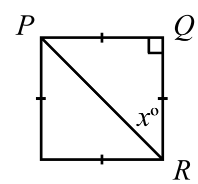
4. There are only two different products of two positive integers whose result is 14.
   These are 2×7 and 1×14.
   Since the two integers must be between 1 and 10, then the product must be 2×7.
   The sum of these two integers is $2 + 7 = 9$.

   Answer: (D)

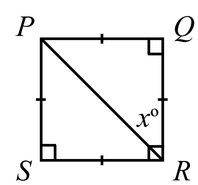
5. Written as a fraction, three thousandths is equal to $\frac{3}{1000}$.
   As a decimal, three thousandths is equal to $3 \div 1000 = 0.003$.

   Answer: (E)

6. *Solution 1*
   Since the given figure is a square, then $PQ = QR$ and $\angle PQR = 90^\circ$.
   Since $PQ = QR$, $\triangle PQR$ is isosceles and so $\angle QPR = \angle QRP = x^\circ$.
   The three angles in any triangle add to 180° and since $\angle PQR = 90^\circ$, then $\angle QPR + \angle QRP = 180^\circ - 90^\circ = 90^\circ$.
   Since $\angle QPR = \angle QRP$, then $\angle QRP = 90^\circ \div 2 = 45^\circ$, and so $x = 45$.



   Solution 2
   Diagonal PR divides square $PQRS$ into two identical triangles: $\triangle PQR$ and $\triangle PSR$.
   Since these triangles are identical, $\angle PRS = \angle PRQ = x^\circ$.
   Since $PQRS$ is a square, then $\angle QRS = 90^\circ$.
   That is, $\angle PRS + \angle PRQ = 90^\circ$ or $x^\circ + x^\circ = 90^\circ$ or $2x = 90$ and so $x = 45$.



   Answer: (B)

7. *Solution 1*
   Written as a mixed fraction, $\frac{35}{4} = 8\frac{3}{4}$.
   Since $\frac{3}{4}$ is closer to 1 than it is to 0, then $8\frac{3}{4}$ is closer to 9 than it is to 8.
   The integer closest in value to $\frac{35}{4}$ is 9.

   Solution 2
   Written as a decimal $\frac{35}{4} = 35 \div 4 = 8.75$.
   Since 0.75 is closer to 1 than it is to 0, then 8.75 is closer to 9 than it is to 8.
   The integer closest in value to $\frac{35}{4}$ is 9.

   Answer: (C)

8. When $n = 101$:

$$\begin{aligned} 3n &= 3 \times 101 = 303 \\ n + 2 &= 101 + 2 = 103 \\ n - 12 &= 101 - 12 = 89 \\ 2n - 2 &= 2 \times 101 - 2 = 202 - 2 = 200 \\ 3n + 2 &= 3 \times 101 + 2 = 303 + 2 = 305. \end{aligned}$$

Of the expressions given, $2n - 2$ is the only expression which has an even value when $n = 101$. (In fact, the value of $2n - 2$ is an even integer for every integer n. Can you see why?)

Answer: (D)

9. The mean (average) of three integers whose sum is 153 is $\frac{153}{3} = 51$.
The mean of three consecutive integers equals the middle of the three integers.
That is, 51 is the middle integer of three consecutive integers and so the largest of these integers is 52.
(We may check that $50 + 51 + 52 = 153$.)

Answer: (A)

10. Each of the 4 smaller triangles is equilateral and thus has sides of equal length.
Each of these smaller triangles has a perimeter of 9 cm and so has sides of length $\frac{9}{3} = 3$ cm.
In $\triangle PQR$, side PQ is made up of two such sides of length 3 cm and thus $PQ = 2 \times 3 = 6$ cm.
Since $\triangle PQR$ is equilateral, then $PR = QR = PQ = 6$ cm.
Therefore, the perimeter of $\triangle PQR$ is $3 \times 6 = 18$ cm.

Answer: (E)

11. The denominators of the two fractions are 7 and 63.
Since $7 \times 9 = 63$, then we must also multiply the numerator 3 by 9 so that the fractions are equivalent.
That is, $\dfrac{3}{7} = \dfrac{3 \times 9}{7 \times 9} = \dfrac{27}{63}$.
Therefore, the number that goes into the $\square$ so that the statement true is 27.

Answer: (A)

12. If puzzles are bought individually for \$10 each, then 6 puzzles will cost $\$10 \times 6 = \$60$.
Since the cost for a box of 6 puzzles is \$50, it is less expensive to buy puzzles by the box than it is to buy them individually.
Buying 4 boxes of 6 puzzles gives the customer $4 \times 6 = 24$ puzzles and the cost is $4 \times \$50 = \$200$.
Buying one additional puzzle for \$10 gives the customer 25 puzzles at a minimum cost of \$210.

Answer: (A)

13. A translation moves (slides) an object some distance without altering it in any other way.
That is, the object is not rotated, reflected, and its exact size and shape are maintained.
Of the triangles given, the triangle labelled D is the only triangle whose orientation is identical to that of the shaded triangle.
Thus, D is the triangle which can be obtained when the shaded triangle is translated.

Answer: (D)

14. Since the time in Toronto, ON is 1:00 p.m. when the time in Gander, NL is 2:30 p.m., then the time in Gander is 1 hour and 30 minutes ahead of the time in Toronto.
A flight that departs from Toronto at 3:00 p.m. and takes 2 hours and 50 minutes will land in Gander at 5:50 p.m. Toronto time.
When the time in Toronto is 5:50 p.m., the time in Gander is 1 hour and 30 minutes ahead which is 7:20 p.m.

Answer: (A)

15. Henry was slower than Faiz and thus finished the race behind Faiz.
Ryan was faster than Henry and Faiz and thus finished the race ahead of both of them.
From fastest to slowest, these three runners finished in the order Ryan, Faiz and then Henry.
Toma was faster than Ryan but slower than Omar.
Therefore, from fastest to slowest, the runners finished in the order Omar, Toma, Ryan, Faiz, and Henry.
The student who finished fourth was Faiz.

Answer: (A)

16. The positive divisors of 20 are: $1, 2, 4, 5, 10, 20$.
Of the 20 numbers on the spinner, 6 of the numbers are divisors of 20.
It is equally likely that the spinner lands on any of the 20 numbers.
Therefore, the probability that the spinner lands on a number that is a divisor of 20 is $\frac{6}{20}$.

Answer: (E)

17. *Solution 1*
Since 78 is 2 less than 80 and 82 is 2 greater than 80, the mean of 78 and 82 is 80.
Since the mean of all four integers is 80, then the mean of 83 and x must also equal 80.
The integer 83 is 3 greater than 80, and so x must be 3 less than 80.
That is, $x = 80 - 3 = 77$.
(We may check that the mean of $78, 83, 82$, and 77 is indeed 80.)

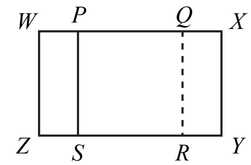
Solution 2
Since the mean of the four integers is 80, then the sum of the four integers is $4 \times 80 = 320$.
Since the sum of 78, 83 and 82 is 243, then $x = 320 - 243 = 77$.
Therefore, x is 77 which is 3 less than the mean 80.

Answer: (D)

18. A discount of 20% on a book priced at \$100 is a $0.20 \times \$100 = \$20$ discount.
Thus for option (A), Sara's discounted price is $\$100 - \$20 = \$80$.
A discount of 10% on a book priced at \$100 is a $0.10 \times \$100 = \$10$ discount, giving a discounted price of $\$100 - \$10 = \$90$.
A second discount of 10% on the new price of \$90 is a $0.10 \times \$90 = \$9$ discount.
Thus for option (B), Sara's discounted price is $\$90 - \$9 = \$81$.
A discount of 15% on a book priced at \$100 is a $0.15 \times \$100 = \$15$ discount, giving a discounted price of $\$100 - \$15 = \$85$.
A further discount of 5% on the new price of \$85 is a $0.05 \times \$85 = \$4.25$ discount.
Thus for option (C), Sara's discounted price is $\$85 - \$4.25 = \$80.75$.
A discount of 5% on a book priced at \$100 is a $0.05 \times \$100 = \$5$ discount, giving a discounted price of $\$100 - \$5 = \$95$.
A further discount of 15% on the new price of \$95 is a $0.15 \times \$95 = \$14.25$ discount.
Thus for option (D), Sara's discounted price is $\$95 - \$14.25 = \$80.75$.
Therefore, the four options do not give the same price and option (A) gives Sara the best discounted price.

Answer: (A)

19. In the diagram, rectangles $WQRZ$ and $PXYS$ are the two sheets of 11 cm$\times$ 8 cm paper.



The overlapping square $PQRS$ has sides of length 8 cm.
That is, $WQ = ZR = PX = SY = 11$ cm and
$WZ = QR = PS = XY = PQ = SR = 8$ cm.
In rectangle $WQRZ$, $WQ = WP + PQ = 11$ cm and so
$WP = 11 - PQ = 11 - 8 = 3$ cm.
In rectangle $WXYZ$, $WX = WP + PX = 3 + 11 = 14$ cm.
Since $WX = 14$ cm and $XY = 8$ cm, the area of $WXYZ$ is
$14 \times 8 = 112$ cm^2.

Answer: (B)

20. Points P, Q, R, S, T divide the bottom edge of the park into six segments of equal length, each of which has length $600 \div 6 = 100$ m.
If Betty and Ann had met for the first time at point Q, then Betty would have walked a total distance of $600 + 400 + 4 \times 100 = 1400$ m and Ann would have walked a total distance of $400 + 2 \times 100 = 600$ m.
When they meet, the time that Betty has been walking is equal to the time that Ann has been walking and so the ratio of Betty's speed to Ann's speed is equal to the ratio of the distance that Betty has walked to the distance that Ann has walked.
That is, if they had met for the first time at point Q, the ratio of their speed's would be $1400 : 600$ or $14 : 6$ or $7 : 3$.
Similarly, if Betty and Ann had met for the first time at point R, then Betty would have walked a total distance of $600 + 400 + 3 \times 100 = 1300$ m and Ann would have walked a total distance of $400 + 3 \times 100 = 700$ m.
In this case, the ratio of their speed's would be $1300 : 700$ or $13 : 7$.
When Betty and Ann actually meet for the first time, they are between Q and R.
Thus Betty has walked less distance than she would have had they met at Q and more distance than she would have had they met at R.
That is, the ratio of Betty's speed to Ann's speed must be less than $7 : 3$ and greater than $13 : 7$.
We must determine which of the five given answers is a ratio that is less than $7 : 3$ and greater than $13 : 7$.
One way to do this is to convert each ratio into a mixed fraction.
That is, we must determine which of the five answers is less than $7 : 3 = \frac{7}{3} = 2\frac{1}{3}$ and greater than $13 : 7 = \frac{13}{7} = 1\frac{6}{7}$.
Converting the answers, we get $\frac{5}{3} = 1\frac{2}{3}$, $\frac{9}{4} = 2\frac{1}{4}$, $\frac{11}{6} = 1\frac{5}{6}$, $\frac{12}{5} = 2\frac{2}{5}$, and $\frac{17}{7} = 2\frac{3}{7}$.
Of the five given answers, the only fraction that is less than $2\frac{1}{3}$ and greater than $1\frac{6}{7}$ is $2\frac{1}{4}$.
If Betty and Ann meet for the first time between Q and R, then the ratio of Betty's speed to Ann's speed could be $9 : 4$.

Answer: (B)

21. *Solution 1*
The first and tenth rectangles each contribute an equal amount to the perimeter.
They each contribute two vertical sides (each of length 2), one full side of length 4 (the top side for the first rectangle and the bottom side for the tenth rectangle), and one half of the length of the opposite side.
That is, the first and tenth rectangles each contribute $2 + 2 + 4 + 2 = 10$ to the perimeter.
Rectangles two through nine each contribute an equal amount to the perimeter.

They each contribute two vertical sides (each of length 2), one half of a side of length 4, and one half of the length of the opposite side (which also has length 4).
That is, rectangles two through nine each contribute $2+2+2+2=8$ to the perimeter.
Therefore, the total perimeter of the given figure is $(2\times 10)+(8\times 8)=20+64=84$.

Solution 2
One method for determining the perimeter of the given figure is to consider vertical lengths and horizontal lengths.
Each of the ten rectangles has two vertical sides (a left side and a right side) which contribute to the perimeter.
These 20 sides each have length 2, and thus contribute $20\times 2=40$ to the perimeter of the figure.
Since these are the only vertical lengths contributing to the perimeter, we now determine the sum of the horizontal lengths.

There are two types of horizontal lengths which contribute to the perimeter: the bottom side of a rectangle, and the top side of a rectangle.
The bottom side of each of the first nine rectangles contributes one half of its length to the perimeter.
That is, the bottom sides of the first nine rectangles contribute $\frac{1}{2}\times 4\times 9=18$ to the perimeter.
The entire bottom side of the tenth rectangle is included in the perimeter and thus contributes a length of 4.
Similarly, the top sides of the second rectangle through to the tenth rectangle contribute one half of their length to the perimeter.
That is, the top sides of rectangles two through ten contribute $\frac{1}{2}\times 4\times 9=18$ to the perimeter.
The entire top side of the first rectangle is included in the perimeter and thus contributes a length of 4.
In total, the horizontal lengths included in the perimeter sum to $18+4+18+4=44$.
Since there are no additional lengths which contribute to the perimeter of the given figure, the total perimeter is $40+44=84$.

Solution 3
Before they were positioned to form the given figure, each of the ten rectangles had a perimeter of $2\times(2+4)=12$.
When the figure was formed, some length of each of the ten rectangles' perimeter was "lost" (and thus is not included) in the perimeter of the given figure.
These lengths that were lost occur where the rectangles touch one another.
There are nine such locations where two rectangles touch one another (between the first and second rectangle, between the second and third rectangle, and so on).
In these locations, each of the two rectangles has one half of a side of length 4 which is not included in the perimeter of the given figure.
That is, the portion of the total perimeter of the ten rectangles that is not included in the perimeter of the figure is $9\times(2+2)=36$.
Since the total perimeter of the ten rectangles before they were positioned into the given figure is $10\times 12=120$, then the perimeter of the given figure is $120-36=84$.

ANSWER: (D)

22. The units digit of the product $1ABCDE\times 3$ is 1, and so the units digit of $E\times 3$ must equal 1.
Therefore, the only possible value of E is 7.
Substituting $E=7$, we get

$$\begin{array}{r} 1\,A\,B\,C\,D\,7 \\ \times \qquad\quad 3 \\ \hline A\,B\,C\,D\,7\,1 \end{array}$$

Since $7 \times 3 = 21$, 2 is carried to the tens column.
Thus, the units digit of $D \times 3 + 2$ is 7, and so the units digit of $D \times 3$ is 5.
Therefore, the only possible value of D is 5.
Substituting $D = 5$, we get

$$\begin{array}{r} 1\,A\,B\,C\,5\,7 \\ \times \qquad\quad 3 \\ \hline A\,B\,C\,5\,7\,1 \end{array}$$

Since $5 \times 3 = 15$, 1 is carried to the hundreds column.
Thus, the units digit of $C \times 3 + 1$ is 5, and so the units digit of $C \times 3$ is 4.
Therefore, the only possible value of C is 8.
Substituting $C = 8$, we get

$$\begin{array}{r} 1\,A\,B\,8\,5\,7 \\ \times \qquad\quad 3 \\ \hline A\,B\,8\,5\,7\,1 \end{array}$$

Since $8 \times 3 = 24$, 2 is carried to the thousands column.
Thus, the units digit of $B \times 3 + 2$ is 8, and so the units digit of $B \times 3$ is 6.
Therefore, the only possible value of B is 2.
Substituting $B = 2$, we get

$$\begin{array}{r} 1\,A\,2\,8\,5\,7 \\ \times \qquad\quad 3 \\ \hline A\,2\,8\,5\,7\,1 \end{array}$$

Since $2 \times 3 = 6$, there is no carry to the ten thousands column.
Thus, the units digit of $A \times 3$ is 2.
Therefore, the only possible value of A is 4.
Substituting $A = 4$, we get

$$\begin{array}{r} 1\,4\,2\,8\,5\,7 \\ \times \qquad\quad 3 \\ \hline 4\,2\,8\,5\,7\,1 \end{array}$$

Checking, we see that the product is correct and so $A+B+C+D+E = 4+2+8+5+7 = 26$.

Answer: (B)

23. Given 8 dimes (10 ¢ coins) and 3 quarters (25 ¢ coins), we list the different amounts of money (in cents) that can be created in the table below.
When an amount of money already appears in the table, it has been stroked out.

Number of Dimes

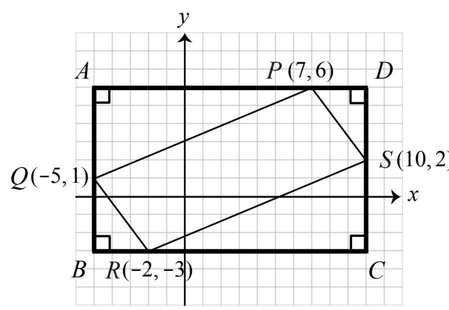
Number of Quarters

25 ¢ \ 10 ¢	0	1	2	3	4	5	6	7	8
0	0	10	20	30	40	50	60	70	80
1	25	35	45	55	65	75	85	95	105
2	~~50~~	~~60~~	~~70~~	~~80~~	90	100	110	120	130
3	~~75~~	~~85~~	~~95~~	~~105~~	115	125	135	145	155

We may ignore the first entry in the table, 0, since we are required to use at least one of the 11 coins.
We are left with 27 different amounts of money that can be created using one or more of the 8 dimes and 3 quarters.

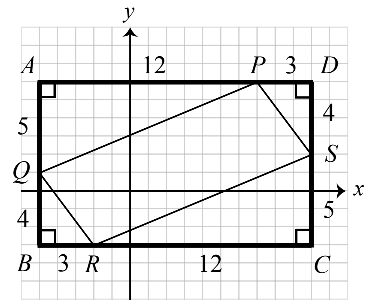
Answer: (A)

24. We begin by constructing rectangle $ABCD$ around the given quadrilateral $PQRS$, as shown.
The vertical sides AB and DC pass through points Q and S, respectively.
The horizontal sides AD and BC pass through points P and R, respectively.
We determine the area of $PQRS$ by subtracting the areas of the four right-angled triangles, AQP, QBR, CSR, and SDP, from the area of $ABCD$.



To determine the horizontal side lengths of the right-angled triangles we count units along the x-axis, or we subtract the x-coordinates of two vertices.
For example, since AB is vertical and passes through $Q(-5, 1)$, the x-coordinates of A and B are equal to that of Q, which is -5.
Thus, the length of AP is determined by subtracting the x-coodinate of A from the x-coordinate of P, which is 7.
Therefore the length of AP is $7 - (-5) = 12$.
Similarly, the length of BR is $-2 - (-5) = 3$.
Since DC is vertical and passes through $S(10, 2)$, the x-coordinates of D and C are equal to that of S, which is 10.
Thus, the length of PD is $10 - 7 = 3$, and the length of RC is $10 - (-2) = 12$.
To determine the vertical side lengths of the right-angled triangles we may count units along the y-axis, or we may subtract the y-coordinates of two vertices.
For example, since AD is horizontal and passes through $P(7, 6)$, the y-coordinates of A and D are equal to that of P, which is 6.
Thus, the length of AQ is determined by subtracting the y-coodinate of Q (which is 1) from the y-coordinate of A.
Therefore the length of AQ is $6 - 1 = 5$.
Similarly, the length of DS is $6 - 2 = 4$.

Since BC is horizontal and passes through $R(-2,-3)$, the y-coordinates of B and C are equal to that of R, which is -3. Thus, the length of QB is $1-(-3)=4$, and the length of SC is $2-(-3)=5$.



The area of $\triangle AQP$ is $\frac{1}{2}\times AQ\times AP=\frac{1}{2}\times 5\times 12=30$.
The area of $\triangle CSR$ is also 30.
The area of $\triangle QBR$ is $\frac{1}{2}\times QB\times BR=\frac{1}{2}\times 4\times 3=6$.
The area of $\triangle SDP$ is also 6.
Since $AB=AQ+QB=5+4=9$ and $BC=BR+RC=3+12=15$, the area of $ABCD$ is $9\times 15=135$.
Finally, the area of $PQRS$ is $135-30\times 2-6\times 2=135-60-12=63$.

ANSWER: (B)

25. *Solution 1*

The sum of the positive integers from 1 to n is given by the expression $\dfrac{n(n+1)}{2}$.
For example when $n=6$, the sum $1+2+3+4+5+6$ can be determined by adding these integers to get 21, or by using the expression $\dfrac{6(6+1)}{2}=\dfrac{42}{2}=21$.
Using this expression, the sum of the positive integers from 1 to 2017, or $1+2+3+4+\cdots+2016+2017$ is $\dfrac{2017(2018)}{2}=\dfrac{4\,070\,306}{2}=2\,035\,153$.
To determine the sum of the integers which Ashley has not underlined, we must subtract from 2 035 153 any of the 2017 integers which is a multiple of 2, or a multiple of 3, or a multiple of 5, while taking care not to subtract any number more than once.
First, we find the sum of all of the 2017 numbers which are a multiple of 2.
This sum contains 1008 integers and is equal to $2+4+6+8+\cdots+2014+2016$.
Since each number in this sum is a multiple of 2, then this sum is equal to twice the sum $1+2+3+4+\cdots+1007+1008$, since $2\times 1=2$, $2\times 2=4$, $2\times 3=6$, and so on.
That is, $2+4+6+8+\cdots+2014+2016=2(1+2+3+4+\cdots+1007+1008)$.
Using the formula above, the sum of the first 1008 positive integers is equal to $\dfrac{1008(1009)}{2}=\dfrac{1\,017\,072}{2}=508\,536$, and so
$2+4+6+8+\cdots+2014+2016=2\times 508\,536=1\,017\,072$.
We may similarly determine the sum of all of the 2017 numbers which are a multiple of 3.
This sum is equal to $3+6+9+12+\cdots+2013+2016$ and contains 672 integers (since $3\times 672=2016$).
Since each of these numbers is a multiple of 3, $3+6+9+12+\cdots+2013+2016$ is equal to $3(1+2+3+4+\cdots+671+672)=3\times\dfrac{672(673)}{2}=3\times\dfrac{452\,256}{2}=3\times 226\,128=678\,384$.
The sum of all of the 2017 numbers which are a multiple of 5 is equal to
$5+10+15+20+\cdots+2010+2015=5(1+2+3+4+\cdots+402+403)=5\times\dfrac{403(404)}{2}$ or $5\times 81\,406$ which is equal to $407\,030$.
We summarize this work in the table below.

Description	Sum	Result
All integers from 1 to 2017	$1+2+3+4+\cdots+2016+2017$	2 035 153
Integers that are a multiple of 2	$2+4+6+8+\cdots+2014+2016$	1 017 072
Integers that are a multiple of 3	$3+6+9+12+\cdots+2013+2016$	678 384
Integers that are a multiple of 5	$5+10+15+20+\cdots+2010+2015$	407 030

If we now subtract the sum of any of the 2017 integers which is a multiple of 2, or a multiple of 3, or a multiple of 5 from the sum of all 2017 integers, is the result our required sum?
The answer is no. Why?
There is overlap between the list of numbers that are a multiple of 2 and those that are a multiple of 3, and those that are a multiple of 5.
For example, any number that is a multiple of both 2 and 3 (and thus a multiple of 6) has been included in both lists and therefore has been counted twice in our work above.
We must add back into our sum those numbers that are a multiple of 6 (multiple of both 2 and 3), those that are a multiple of 10 (multiple of both 2 and 5), and those that are a multiple of 15 (multiple of both 3 and 5).
The sum of all of the 2017 numbers which are a multiple of 6 is equal to
$6+12+18+24+\cdots+2010+2016 = 6(1+2+3+4+\cdots+335+336)$, which is equal to
$6\left(\dfrac{336(337)}{2}\right) = 6 \times 56\,616 = 339\,696$.
The sum of all of the 2017 numbers which are a multiple of 10 is equal to
$10+20+30+40+\cdots+2000+2010 = 10(1+2+3+4+\cdots+200+201)$, which is equal to
$10\left(\dfrac{201(202)}{2}\right) = 10 \times 20\,301 = 203\,010$.
The sum of all of the 2017 numbers which are a multiple of 15 is equal to
$15+30+45+60+\cdots+1995+2010 = 15(1+2+3+4+\cdots+133+134)$, which is equal to
$15\left(\dfrac{134(135)}{2}\right) = 15 \times 9045 = 135\,675$.
We again summarize this work in the table below.

Description	Sum	Result
All integers from 1 to 2017	$1+2+3+4+\cdots+2016+2017$	2 035 153
Integers that are a multiple of 2	$2+4+6+8+\cdots+2014+2016$	1 017 072
Integers that are a multiple of 3	$3+6+9+12+\cdots+2013+2016$	678 384
Integers that are a multiple of 5	$5+10+15+20+\cdots+2010+2015$	407 030
Integers that are a multiple of 6	$6+12+18+24+\cdots+2010+2016$	339 696
Integers that are a multiple of 10	$10+20+30+40+\cdots+2000+2010$	203 010
Integers that are a multiple of 15	$15+30+45+60+\cdots+1995+2010$	135 675

If we take the sum of all 2017 integers, subtract those that are a multiple of 2, and those that are a multiple of 3, and those that are a multiple of 5, and then add those numbers that were subtracted twice (the multiples of 6, the multiples of 10, and the multiples of 15), then we get:

$$2\,035\,153 - 1\,017\,072 - 678\,384 - 407\,030 + 339\,696 + 203\,010 + 135\,675 = 611\,048$$

Is this the required sum?
The answer is still no, but we are close!
Consider any of the 2017 integers that is a multiple of 2, 3 and 5 (that is, a multiple of $2 \times 3 \times 5 = 30$).
Each number that is a multiple of 30 would have been underlined by Ashley, and therefore should not be included in our sum.
Each multiple of 30 was subtracted from the sum three times (once for each of the multiples of 2, 3 and 5), but then added back into our sum three times (once for each of the mutiples of 6, 10 and 15).
Thus, any of the 2017 integers that is a multiple of 30 must still be subtracted from 611 048 to achieve our required sum.

The sum of all of the 2017 numbers which are a multiple of 30 is equal to $30 + 60 + 90 + 120 + \cdots + 1980 + 2010 = 30(1 + 2 + 3 + 4 + \cdots + 66 + 67)$, which is equal to $30\left(\frac{67(68)}{2}\right) = 30 \times 2278 = 68\,340$.
Finally, the sum of the 2017 integers which Ashley has not underlined is $611\,048 - 68\,340 = 542\,708$.

Solution 2
We begin by considering the integers from 1 to 60.
When Ashley underlines the integers divisible by 2 and by 5, this will eliminate all of the integers ending in 0, 2, 4, 5, 6, and 8.
This leaves $1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, 31, 33, 37, 39, 41, 43, 47, 49, 51, 53, 57, 59$.
Of these, the integers $3, 9, 21, 27, 33, 39, 51, 57$ are divisible by 3.
Therefore, of the first 60 integers, only the integers

$$1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59$$

will not be underlined.
Among these 16 integers, we notice that the second set of 8 integers consists of the first 8 integers with 30 added to each.
This pattern continues, so that a corresponding set of 8 out of each block of 30 integers will not be underlined.
Noting that 2010 is the largest multiple of 30 less than 2017, this means that Ashley needs to add the integers

$$\begin{array}{rrrrrrrr}
1 & 7 & 11 & 13 & 17 & 19 & 23 & 29 \\
31 & 37 & 41 & 43 & 47 & 49 & 53 & 59 \\
61 & 67 & 71 & 73 & 77 & 79 & 83 & 89 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1981 & 1987 & 1991 & 1993 & 1997 & 1999 & 2003 & 2009 \\
2011 & 2017 & & & & & &
\end{array}$$

Let S equal the sum of these integers.
Before proceeding, we justify briefly why the pattern continues:

> Every positive integer is a multiple of 30, or 1 more than a multiple of 30, or 2 more than a multiple of 30, and so on, up to 29 more than a multiple of 30. Algebraically, this is saying that every positive integer can be written in one of the forms
>
> $$30k, 30k + 1, 30k + 2, 30k + 3, \ldots, 30k + 27, 30k + 28, 30k + 29$$
>
> depending on its remainder when divided by 30.
> Every integer with an even remainder when divided by 30 is even, since 30 is also even.
> Similarly, every integer with a remainder divisible by 3 or 5 when divided by 30 is divisible by 3 or 5, respectively.
> This leaves us with the forms
>
> $$30k + 1, 30k + 7, 30k + 11, 30k + 13, 30k + 17, 30k + 19, 30k + 23, 30k + 29.$$
>
> No integer having one of these forms will be underlined, since, for example, $30k + 11$ is one more than a multiple of 2 and 5 (namely, $30k + 10$) and is 2 more than a multiple of 3 (namely, $30k + 9$) so is not divisible by 2, 3 or 5.

The sum of the 8 integers in the first row of the table above is 120.
Since each of the integers in the second row of the table is 30 greater than the corresponding integer in the first row, then the sum of the numbers in the second row of the table is $120+8\times30$.
Similarly, the sum of the integers in the third row is $120 + 8 \times 60$, and so on.
We note that $2010 = 67 \times 30$ and $1980 = 66 \times 30$, so there are 67 complete rows in the table.
Therefore,

$$\begin{aligned} S &= 120 + (120 + 8 \times 30) + (120 + 8 \times 60) + \cdots + (120 + 8 \times 1980) + (2011 + 2017) \\ &= 120 \times 67 + 8 \times (30 + 60 + \cdots + 1980) + 4028 \\ &= 8040 + 8 \times 30 \times (1 + 2 + \cdots + 65 + 66) + 4028 \\ &= 12\,068 + 240 \times (33 \times 67) \\ &= 12\,068 + 530\,640 \\ &= 542\,708 \end{aligned}$$

Here, we have used the fact that the integers from 1 to 66 can be grouped into 33 pairs each of which adds to 67, as shown here:

$$1 + 2 + \cdots + 65 + 66 = (1 + 66) + (2 + 65) + \cdots + (33 + 34) = 67 + 67 + \cdots + 67 = 33 \times 67$$

Answer: (A)

Grade 8

1. Michael has \$280 in \$20 bills and so the number of \$20 bills that he has is $280 \div 20 = 14$.
   Answer: (C)

2. Evaluating, we get $4^2 - 2^3 = 16 - 8 = 8$.
   Answer: (A)

3. Exactly 1 of the 5 equal sections contains the number 4.
   Therefore, the probability that the spinner lands on 4 is $\frac{1}{5}$.
   Answer: (E)

4. The number of grade 8 students on the chess team is $160 \times 10\% = 160 \times \frac{10}{100} = 160 \times 0.10 = 16$.
   Answer: (B)

5. Since $22 = 2 \times 11$, then $44 \times 22 = 44 \times 2 \times 11$.
   Since $44 \times 2 = 88$, then $44 \times 2 \times 11 = 88 \times 11$.
   Therefore, $44 \times 22 = 88 \times 11$.
   Answer: (B)

6. In terms of x, the sum of the three side lengths of the triangle is $x + x + 1 + x - 1 = 3x$.
   Since the perimeter is 21, then $3x = 21$ and so $x = 7$.
   Answer: (B)

7. Reading from the graph, 20 students chose pink and 25 students chose blue.
   The ratio of the number of students who chose pink to the number of students who chose blue is $20 : 25$.
   After simplifying this ratio (dividing each number by 5), $20 : 25$ is equal to $4 : 5$.
   Answer: (A)

8. *Solution 1*
   To get the original number, we reverse the steps.
   That is, we add 6 and then divide by 3.
   Therefore, the original number is $(15 + 6) \div 3 = 21 \div 3 = 7$.

   Solution 2
   If the original number is x, then when it is tripled the result is $3x$.
   When this result is decreased by 6, we get $3x - 6$.
   Solving $3x - 6 = 15$, we get $3x = 15 + 6$ or $3x = 21$ and so $x = 7$.
   Answer: (D)

9. Tian travels 500 m every 625 steps and so she travels $500 \div 625 = 0.8$ m with each step.
   If Tian walks 10 000 steps at this same rate, she will walk a distance of $0.8 \times 10\,000 = 8000$ m.
   Since there are 1000 m in each kilometre, Tian will walk $8000 \div 1000 = 8$ km.
   Answer: (D)

10. Line segments PQ and RS intersect at T and so $\angle PTR$ and $\angle STQ$ are opposite angles and therefore $\angle STQ = \angle PTR = 88^\circ$.
    Since $TS = TQ$, then $\triangle STQ$ is an isosceles triangle and so $\angle TSQ = \angle TQS = x^\circ$.
    The three interior angles of any triangle add to 180°.
    Thus, $88^\circ + x^\circ + x^\circ = 180^\circ$, and so $2x = 180 - 88$ or $2x = 92$ which gives $x = 46$.
    Answer: (B)

11. The volume of a rectangular prism is determined by multiplying the area of its base by its height.
The area of the base for the given prism is $4 \times 5 = 20$ cm^2 and its height is x cm.
Since the prism's volume is 60 cm^3, then $20x = 60$ and so $x = 3$.

Answer: (D)

12. Since $\angle ACB = 90^\circ$, then $\triangle ACB$ is a right-angled triangle.
By the Pythagorean Theorem, $AB^2 = AC^2 + CB^2 = 8^2 + 15^2 = 64 + 225 = 289$ m^2.
Since $AB > 0$, then $AB = \sqrt{289} = 17$ m and so Cindy walks a distance of 17 m.
Walking from A to C to B, David walks a total distance of $8 + 15 = 23$ m.
Thus, David walks $23 - 17 = 6$ m farther than Cindy.

Answer: (D)

13. Each term of the sum $10 + 20 + 30 + \cdots + 990 + 1000$ is 10 times larger than its corresponding term in the sum $1 + 2 + 3 + \cdots + 99 + 100$, and so the required sum is 10 times larger than the given sum.
Since $1+2+3+\cdots+99+100 = 5050$, then $10+20+30+\cdots+990+1000 = 5050 \times 10 = 50\,500$.

Answer: (C)

14. If three of the students receive the smallest total number of pens possible, then the remaining student will receive the largest number of pens possible.
The smallest number of pens that a student can receive is 1, since each student receives at least 1 pen.
Since each student receives a different number of pens, the second smallest number of pens that a student can receive is 2 and the third smallest number of pens that a student can receive is 3.
The smallest total number of pens that three students can receive is $1 + 2 + 3 = 6$.
Therefore, the largest number of pens that a student can receive is $20 - 6 = 14$.

Answer: (C)

15. The even integers between 1 and 103 are $2 = 2 \times 1, 4 = 2 \times 2, 6 = 2 \times 3, 8 = 2 \times 4$, and so on up to and including $102 = 2 \times 51$.
Since there are 51 even integers in the list $2, 4, 6, \ldots, 100, 102$, then there are 51 even integers between 1 and 103.
Next, we want to find a number N such that there are 51 odd integers between 4 and N.
We notice that our lower bound, 4, is 3 greater than our original lower bound of 1.
By increasing each of the 51 even integers from above by 3, we create the first 51 odd integers which are greater than 4.
These odd integers are $2 \times 1 + 3 = 5, 2 \times 2 + 3 = 7, 2 \times 3 + 3 = 9, 2 \times 4 + 3 = 11$, and so on up to and including $2 \times 51 + 3 = 105$.
Since there are 51 odd integers in the list $5, 7, 9, \ldots, 103, 105$, then there are 51 odd integers between 4 and 106.
That is, the number of even integers between 1 and 103 is the same as the number of odd integers between 4 and 106.

Answer: (E)

16. Label points S, T, U, V, W, as shown in Figure 1.
Each shaded triangle is equilateral, $\triangle PQR$ is equilateral, and so $\angle VSU = \angle VTW = \angle SPT = 60^\circ$.
Therefore, $\angle PSV = 180^\circ - \angle VSU = 120^\circ$ (since PSU is a straight angle), and similarly, $\angle PTV = 120^\circ$.
In quadrilateral $PSVT$, $\angle SVT = 360^\circ - \angle PSV - \angle SPT - \angle PTV$ or $\angle SVT = 360^\circ - 120^\circ - 60^\circ - 120^\circ = 60^\circ$.
Therefore, $PSVT$ is a parallelogram, and since $SV = TV = 2$, then $PS = PT = 2$ (opposite sides of a parallelogram are equal in length).
Join S to T, as shown in Figure 2.
Since $SV = TV$, then $\angle VST = \angle VTS = \frac{1}{2}(180^\circ - 60^\circ) = 60^\circ$.
That is, $\triangle SVT$ is equilateral with side length 2, and is therefore congruent to each of the shaded triangles.
Similarly, $\angle SPT = 60^\circ$, $PS = PT = 2$, and so $\triangle PST$ is congruent to each of the shaded triangles.
We may also join U to X and W to Y (as in Figure 3), and similarly show that $\triangle UQX, \triangle UXV, \triangle WRY$, and $\triangle WYV$ are also congruent to the shaded triangles.
Thus, $\triangle PQR$ can be divided into 9 congruent triangles.
Since 3 of these 9 triangles are shaded, the fraction of the area of $\triangle PQR$ that is shaded is $\frac{3}{9}$ or $\frac{1}{3}$.

Figure 1

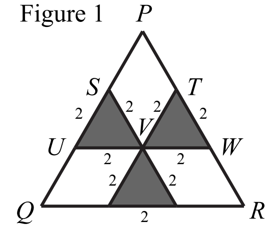


Figure 2

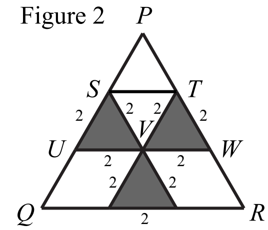
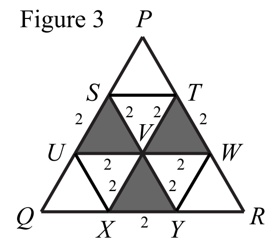


Figure 3



ANSWER: (B)

17. The range of the players' heights is equal to the difference between the height of the tallest player and the height of the shortest player.
Since the tallest player, Meghan, has a height of 188 cm, and the range of the players' heights is 33 cm, then the shortest player, Avery, has a height of $188 - 33 = 155$ cm.
Thus, answer (D) is a statement which provides enough information to determine Avery's height, and so must be the only one of the five statements which is enough to determine Avery's height.
(Can you give a reason why each of the other four answers does not provide enough information to determine Avery's height?)

ANSWER: (D)

18. When Brodie and Ryan are driving directly towards each other at constant speeds of 50 km/h and 40 km/h respectively, then the distance between them is decreasing at a rate of $50 + 40 = 90$ km/h.
If Brodie and Ryan are 120 km apart and the distance between them is decreasing at 90 km/h, then they will meet after $\dfrac{120}{90}$ h or $\dfrac{4}{3}$ h or $1\frac{1}{3}$ h.
Since $\frac{1}{3}$ of an hour is $\frac{1}{3} \times 60 = 20$ minutes, then it will take Brodie and Ryan 1 h 20 min to meet.

ANSWER: (E)

19. The mean age of three of the friends is 12 years and 3 months which is equal to $12 \times 12 + 3 = 144 + 3 = 147$ months.
Since the mean equals the sum of the ages divided by 3, then the sum of the ages of these three friends is $3 \times 147 = 441$ months.
The mean age of the remaining four friends is 13 years and 5 months or $12 \times 13 + 5 = 156 + 5 = 161$ months.

Thus, the sum of the ages of these four friends is $4 \times 161 = 644$ months.
The sum of the ages of all seven friends is $441 + 644 = 1085$ months, and so the mean age of all seven friends is $\frac{1085}{7} = 155$ months.

Answer: (E)

20. The units digit of the product $1ABCDE \times 3$ is 1, and so the units digit of $E \times 3$ must equal 1.
Therefore, the only possible value of E is 7.
Substituting $E = 7$, we get

$$\begin{array}{r} 1\,A\,B\,C\,D\,7 \\ \times \qquad\quad 3 \\ \hline A\,B\,C\,D\,7\,1 \end{array}$$

Since $7 \times 3 = 21$, 2 is carried to the tens column.
Thus, the units digit of $D \times 3 + 2$ is 7, and so the units digit of $D \times 3$ is 5.
Therefore, the only possible value of D is 5.
Substituting $D = 5$, we get

$$\begin{array}{r} 1\,A\,B\,C\,5\,7 \\ \times \qquad\quad 3 \\ \hline A\,B\,C\,5\,7\,1 \end{array}$$

Since $5 \times 3 = 15$, 1 is carried to the hundreds column.
Thus, the units digit of $C \times 3 + 1$ is 5, and so the units digit of $C \times 3$ is 4.
Therefore, the only possible value of C is 8.
Substituting $C = 8$, we get

$$\begin{array}{r} 1\,A\,B\,8\,5\,7 \\ \times \qquad\quad 3 \\ \hline A\,B\,8\,5\,7\,1 \end{array}$$

Since $8 \times 3 = 24$, 2 is carried to the thousands column.
Thus, the units digit of $B \times 3 + 2$ is 8, and so the units digit of $B \times 3$ is 6.
Therefore, the only possible value of B is 2.
Substituting $B = 2$, we get

$$\begin{array}{r} 1\,A\,2\,8\,5\,7 \\ \times \qquad\quad 3 \\ \hline A\,2\,8\,5\,7\,1 \end{array}$$

Since $2 \times 3 = 6$, there is no carry to the ten thousands column.
Thus, the units digit of $A \times 3$ is 2.
Therefore, the only possible value of A is 4.
Substituting $A = 4$, we get

$$\begin{array}{r} 1\,4\,2\,8\,5\,7 \\ \times \qquad\quad 3 \\ \hline 4\,2\,8\,5\,7\,1 \end{array}$$

Checking, we see that the product is correct and so $A+B+C+D+E = 4+2+8+5+7 = 26$.

Answer: (B)

21. On the bottom die, the two visible faces are showing 2 dots and 4 dots.
Since the number of dots on opposite faces of this die add to 7, then there are 5 dots on the face opposite the face having 2 dots, and 3 dots on the face opposite the face having 4 dots.

Therefore, the top face of this bottom die (which is a face that is hidden between the dice) has either 1 dot on it or it has 6 dots on it.

On the second die from the bottom, the sum of the number of dots on the top and bottom faces (the faces hidden between the dice) is 7 since the number of dots on opposite faces add to 7. (We do not need to know which faces these are, though we could determine that they must have 1 and 6 dots on them.)

Similarly, on the third die from the bottom, the sum of the number of dots on the top and bottom faces (the faces hidden between the dice) is also 7.

Finally, the top face of the top die shows 3 dots, and so the bottom face of this die (which is a face that is hidden between the dice) contains 4 dots.

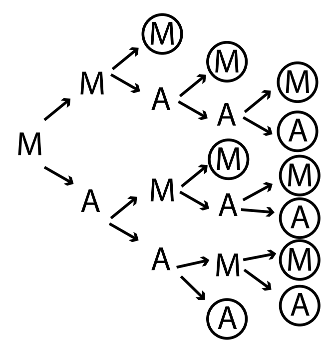
Therefore, the sum of the number of dots hidden between the dice is either $4 + 7 + 7 + 1 = 19$ or $4 + 7 + 7 + 6 = 24$.
Of these two possible answers, 24 is the only answer which appears among the five given answers.

Answer: (C)

22. To give $Y^X - W^V$ the greatest possible value, we make Y^X as large as possible while making W^V as small as possible.
To make Y^X as large as possible, we make Y and X as large as possible.
Thus, we must assign Y and X the two largest values, and assign W and V the two smallest values.
Let Y and X equal 4 and 5 in some order.
Since $4^5 = 1024$ and $5^4 = 625$, we let $Y = 4$ and $X = 5$ so that Y^X is as large as possible.
Similarly, we assign W and V the smallest possible numbers, 2 and 3.
Since $2^3 = 8$ and $3^2 = 9$, we let $W = 2$ and $V = 3$ so that W^V is as small as possible.
Thus, the greatest possible value of $Y^X - W^V$ is equal to $4^5 - 2^3 = 1024 - 8 = 1016$, which gives $X + V = 5 + 3 = 8$.

Answer: (D)

23. We introduce the letter M to represent a game that Mike has won, and the letter A to represent a game that Alain has won.
We construct a tree diagram to show all possible outcomes.
The M at the far left of the tree represents the fact that Mike won the first game.
The second "column" shows the two possible outcomes for the second game - Mike could win (M), or Alain could win (A).
The third, fourth and fifth columns show the possible outcomes of games 3, 4 and 5, respectively.
A branch of the tree is linked by arrows and each branch gives the outcomes of the games which lead to one of the players becoming the champion.

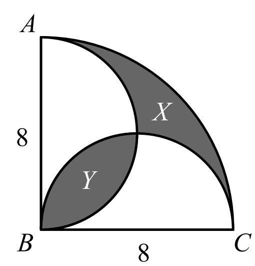


Once Mike has won 3 games, or Alain has won 3 games, the branch ends and the outcome of that final game is circled.
Since the first player to win 3 games becomes the champion, one way that Mike could become the champion is to win the second and third games (since he has already won the first game).
We represent this possibility as MMM, as shown along the top branch of the tree diagram.
In this MMM possibility, the final M is circled in the diagram, meaning that Mike has become the champion.
Since we are asked to determine the probability that Mike becomes the champion, we search

for all branches through the tree diagram which contain 3 Ms (paths ending with a circled M). The tree diagram shows six such branches: MMM, MMAM, MMAAM, MAMM, MAMAM, and MAAMM.
All other branches end with a circled A, meaning that Alain has won 3 games and becomes the champion.
Since each player is equally likely to win a game, then Mike wins a game with probability $\frac{1}{2}$, and Alain wins a game with probability $\frac{1}{2}$.
Of the six ways that Mike can win (listed above), only one of these ends after 3 games (MMM).
The probability that Mike wins in exactly 3 games is equal to the probability that Mike wins the second game, which is $\frac{1}{2}$, multiplied by the probability that Mike wins the third game, which is also $\frac{1}{2}$.
That is, the probability that Mike becomes the champion by winning the first 3 games is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.
Of the six ways that Mike can win, two of these end after 4 games (MMAM, MAMM).
The probability that Mike wins games two and four but Alain wins game three (MMAM) is equal to the probability that Mike wins the second game, which is $\frac{1}{2}$, multiplied by the probability that Alain wins the third game, which is also $\frac{1}{2}$, multiplied by probability that Mike wins the fourth game, $\frac{1}{2}$.
In this case, Mike becomes the champion with probability $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.
Similary, the probability that Mike becomes the champion by winning games three and four, but loses game two, is also $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.
Finally, we determine the probabilty that Mike becomes the champion by winning in exactly 5 games (there are three possibilities: MMAAM, MAMAM and MAAMM).
Each of these three possibilities happens with probability $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$.
Therefore, if Mike has won the first game, then the probability that he becomes champion is $\frac{1}{4} + 2 \times \frac{1}{8} + 3 \times \frac{1}{16} = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} = \frac{4+4+3}{16} = \frac{11}{16}$.

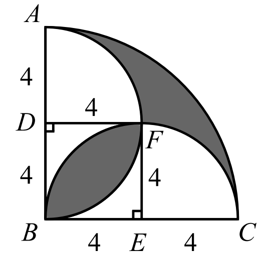
Answer: (C)

24. Let the area of the shaded region that lies outside of both semi-circles be X.



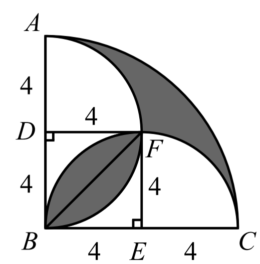
Let the area of the shaded region that lies inside of both semi-circles be Y.
The sum of the areas of both semi-circles counts the shaded area Y twice (since the area of overlap of the semi-circles is Y).
Therefore, if we subtract Y from the sum of the areas of both semi-circles, and add X, we get the area of the quarter-circle ABC.
That is, the area of quarter-circle ABC is equal to
(the area of the semi-circle drawn on AB)+(the area of the semi-circle drawn on BC)$-Y+X$.
The area of quarter-circle ABC is $\frac{1}{4}\pi(8)^2 = 16\pi$.
The area of the semi-circle drawn on AB is $\frac{1}{2}\pi(4)^2 = 8\pi$.
The area of the semi-circle drawn on BC is also 8π.
Thus, $16\pi = 8\pi + 8\pi - Y + X$ or $16\pi = 16\pi - Y + X$, and so $Y = X$.

We build square $DBEF$ so that D is 4 vertical units from B, and E is 4 horizontal units to the right of B. Next, we will show that F lies on both semi-circles.



Since ABC is a quarter of a circle, then $\angle ABC = 90^\circ$.
Beginning at point B, we move up vertically 4 units to point D, and then move right 4 units in a direction perpendicular to AB.
After these two moves, we arrive at the point labelled F.
Since the diameter of the semi-circle drawn on AB has length 8, then the radius of this semi-circle is 4.
Therefore, D is the centre of this semi-circle (since $DB = 4$), and F lies on this semi-circle (since $DF = 4$).
Beginning again at point B, we move right 4 units to point E, and then move up vertically 4 units in a direction perpendicular to BC.
Since these are the same two moves we made previously (up 4 and right 4), then we must again arrive at F.
Since the diameter of the semi-circle drawn on BC has length 8, then the radius of this semi-circle is 4.
Therefore, E is the centre of this semi-circle (since $EB = 4$), and F lies on this semi-circle (since $EF = 4$).
The two semi-circles intersect at exactly one point (other than point B).
Since we have shown that F lies on both semi-circles, then F must be this point of intersection of the two semi-circles.
Therefore, $DBEF$ is a square with side length 4, and F is the point of intersection of the two semi-circles.

Finally, we find the value of Y.
First we construct BF, the diagonal of square $DBEF$.
By symmetry, BF divides the shaded area Y into two equal areas.



Each of these equal areas, $\dfrac{Y}{2}$, is equal to the area of $\triangle BEF$ subtracted from the area of the quarter-circle BEF.
That is, $\dfrac{Y}{2} = \frac{1}{4}\pi(4)^2 - \frac{1}{2}(4)(4)$, and so $\dfrac{Y}{2} = 4\pi - 8$, or $Y = 8\pi - 16$.
The area of the shaded region is $X + Y = 2Y = 16\pi - 32$.
Of the answers given, $16\pi - 32$ is closest to 18.3.

Answer: (D)

25. *Solution 1*
Let the number of black plates, gold plates, and red plates be b, g and r, respectively (b, g and r are whole numbers).
Brady is stacking 600 plates, and so $b + g + r = 600$, where b is a multiple of 2, g is a multiple of 3, and r is a multiple of 6.
Rewrite this equation as $g = 600 - b - r$ and consider the right side of the equation.
Since 600 is a multiple of 2, and b is a multiple of 2, and r is a multiple of 2 (any multiple of 6 is a multiple of 2), then $600 - b - r$ is a multiple of 2 (the difference between even numbers is even).
Since the right side of the equation is a multiple of 2, then the left side, g, must also be a multiple of 2.
We are given that g is a multiple of 3, and since g is also a multiple of 2, then g must be an even multiple of 3 or a multiple of 6.

Similarly, rewriting the equation as $b = 600 - g - r$ and considering the right side of the equation: 600 is a multiple of 6, and g is a multiple of 6, and r is a multiple of 6, so then $600 - g - r$ is a multiple of 6 (the difference between multiples of 6 is a multiple of 6).
Since the right side of the equation is a multiple of 6, then the left side, b, must also be a multiple of 6.
That is, each of b, g and r is a multiple of 6, and so we let $b = 6B$, $g = 6G$, and $r = 6R$ where B, G and R are whole numbers.
So then the equation $b + g + r = 600$ becomes $6B + 6G + 6R = 600$, which is equivalent to $B + G + R = 100$ after dividing by 6.
Each solution to the equation $B + G + R = 100$ corresponds to a way that Brady could stack the 600 plates, and every possible way that Brady could stack the 600 plates corresponds to a solution to the equation $B + G + R = 100$.
For example, $B = 30, G = 50, R = 20$ corresponds to $b = 6 \times 30 = 180, g = 6 \times 50 = 300$, $r = 6 \times 20 = 120$, which corresponds to Brady stacking 180 black plates, below 300 gold plates, which are below 120 red plates.
Since $B + G + R = 100$, then each of B, G and R has a maximum possible value of 100.
If $B = 100$, then $G = R = 0$.
If $B = 99$, $G + R = 100 - B = 1$.
Thus, $G = 0$ and $R = 1$ or $G = 1$ and $R = 0$.
That is, once we assign values for B and G, then there is no choice for R since it is determined by the equation $R = 100 - B - G$.
Thus, to determine the number of solutions to the equation $B + G + R = 100$, we must determine the number of possible pairs (B, G) which lead to a solution.
For example, above we showed that the three pairs $(100, 0)$, $(99, 0)$, and $(99, 1)$ each correspond to a solution to the equation.
Continuing in this way, we determine all possible pairs (B, G) (which give R) that satisfy $B + G + R = 100$.

Value of B	Value of G	Number of plates: b, g, r
100	0	600,0,0
99	0	594,0,6
99	1	594,6,0
98	0	588,0,12
98	1	588,6,6
98	2	588,12,0
97	0	582,0,18
97	1	582,6,12
97	2	582,12,6
97	3	582,18,0
$\vdots$	$\vdots$	$\vdots$
$100 - n$	0	$6(100 - n), 0, 6n$
$100 - n$	1	$6(100 - n), 6, 6(n - 1)$
$100 - n$	2	$6(100 - n), 12, 6(n - 2)$
$\vdots$	$\vdots$	$\vdots$
$100 - n$	n	$6(100 - n), 6n, 0$

We see from the table that if the value of B is $100 - n$ for some whole number $n \leq 100$, then G can equal any whole number from 0 to n and so there are $n + 1$ possible choices for G.
That is, when $B = 100$, there is 1 choice for G, when $B = 99$, there are 2 choices for G, when

$B = 98$, there are 3 choices for G, and so on.
Each additional decrease of 1 in B gives 1 additional choice for G until we arrive at $B = 0$ $(n = 100)$, which gives $n + 1 = 100 + 1 = 101$ possible choices for G $(0, 1, 2, 3, \ldots, 100)$.
Therefore, the total number of solutions to $B + G + R = 100$ is given by the sum $1 + 2 + 3 + \cdots + 99 + 100 + 101$.
Using the fact that the sum of the first m positive integers $1 + 2 + 3 + \cdots + m$ is equal to $\frac{m(m+1)}{2}$, we get $1 + 2 + 3 + \cdots + 99 + 100 + 101 = \frac{101(102)}{2} = 5151$.
Since each of these solutions corresponds to a way that Brady could stack the plates, there are 5151 ways that Brady could stack the plates under the given conditions.

Solution 2
In a given way of stacking the plates, let b be the number of groups of 2 black plates, g be the number of groups of 3 gold plates, and r be the number of groups of 6 red plates.
Then there are $2b$ black plates, $3g$ gold plates, and $6r$ red plates.
Since the total number of plates in a stack is 600, then $2b + 3g + 6r = 600$.
We note that the numbers of black, gold and red plates completely determines the stack (we cannot rearrange the plates in any way), and so the number of ways of stacking the plates is the same as the number of ways of solving the equation $2b + 3g + 6r = 600$ where b, g, r are integers that are greater than or equal to 0.
Since r is at least 0 and $6r$ is at most 600, then the possible values for r are $0, 1, 2, 3, \ldots, 98, 99, 100$.

When $r = 0$, we obtain $2b + 3g = 600$.
Since g is at least 0 and $3g$ is at most 600, then g is at most 200.
Since $2b$ and 600 are even, then $3g$ is even, so g is even.
Therefore, the possible values for g are $0, 2, 4, \ldots, 196, 198, 200$.
Since $200 = 100 \times 2$, then there are 101 possible values for g.
When $g = 0$, we get $2b = 600$ and so $b = 300$.
When $g = 2$, we get $2b = 600 - 3 \times 2 = 594$ and so $b = 297$.
Each time we increase g by 2, the number of gold plates increases by 6, so the number of black plates must decrease by 6, and so b decreases by 3.
Thus, as we continue to increase g by 2s from 2 to 200, the values of b will decrease by 3s from 297 to 0.
In other words, every even value for g does give an integer value for b.
Therefore, when $r = 0$, there are 101 solutions to the equation.

When $r = 1$, we obtain $2b + 3g = 600 - 6 \times 1 = 594$.
Again, g is at least 0, is even, and is at most $594 \div 3 = 198$.
Therefore, the possible values of g are $0, 2, 4, \ldots, 194, 196, 198$.
Again, each value of g gives a corresponding integer value of b.
Therefore, when $r = 1$, there are 100 solutions to the equation.

Consider the case of an unknown value of r, which gives $2b + 3g = 600 - 6r$.
Again, g is at least 0 and is even.
Also, the maximum possible value of g is $\frac{600 - 6r}{3} = 200 - 2r$.
This means that there are $(100 - r) + 1 = 101 - r$ possible values for g. (Can you see why?)
Again, each value of g gives a corresponding integer value of b.
Therefore, for a general r between 0 and 100, inclusive, there are $101 - r$ solutions to the equation.

We make a table to summarize the possibilities:

r	g	b	# of solutions
0	$0, 2, 4, \ldots, 196, 198, 200$	$300, 297, 294, \ldots, 6, 3, 0$	101
1	$0, 2, 4, \ldots, 194, 196, 198$	$297, 294, 291 \ldots, 6, 3, 0$	100
2	$0, 2, 4, \ldots, 192, 194, 196$	$294, 291, 288 \ldots, 6, 3, 0$	99
$\vdots$	$\vdots$	$\vdots$	$\vdots$
98	$0, 2, 4$	$6, 3, 0$	3
99	$0, 2$	$3, 0$	2
100	0	0	1

Therefore, the total number of ways of stacking the plates is

$$101 + 100 + 99 + \cdots + 3 + 2 + 1$$

We note that the integers from 1 to 100 can be grouped into 50 pairs each of which has a sum of 101 $(1 + 100, 2 + 99, 3 + 98, \ldots, 50 + 51)$.
Therefore, the number of ways that Brady could stack the plates is $101 + 50 \times 101 = 5151$.

Answer: (E)

The CENTRE for EDUCATION
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2016 Gauss Contests

(Grades 7 and 8)

Wednesday, May 11, 2016
(in North America and South America)

Thursday, May 12, 2016
(outside of North America and South America)

Solutions

Centre for Education in Mathematics and Computing Faculty and Staff

Ed Anderson
Jeff Anderson
Terry Bae
Shane Bauman
Steve Brown
Carmen Bruni
Ersal Cahit
Heather Culham
Serge D'Alessio
Janine Dietrich
Jennifer Doucet
Fiona Dunbar
Mike Eden
Barry Ferguson
Judy Fox
Steve Furino
John Galbraith
Alain Gamache
Robert Garbary
Sandy Graham
Conrad Hewitt
Angie Hildebrand
Carrie Knoll
Judith Koeller
Bev Marshman
Mike Miniou
Brian Moffat
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J.P. Pretti
Kim Schnarr
Carolyn Sedore
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Lori Yee, William Dunbar P.S., Pickering, ON

Grade 7

1. Evaluating, $333 + 33 + 3 = 366 + 3 = 369$.

   Answer: (D)

2. The day on which Tanner received the most text messages will be the day with the tallest corresponding bar.
   Thus, Tanner received the most text messages on Friday.

   Answer: (A)

3. *Solution 1*
   A number is a multiple of 7 if it is the result of multiplying 7 by an integer.
   Of the answers given, only 77 results from multiplying 7 by an integer, since $77 = 7 \times 11$.

   Solution 2
   A number is a multiple of 7 if the result after dividing it by 7 is an integer.
   Of the answers given, only 77 results in an integer after dividing by 7, since $77 \div 7 = 11$.

   Answer: (C)

4. A positive fraction is larger than $\frac{1}{2}$ if its denominator is less than two times its numerator.
   Of the answers given, $\frac{4}{7}$ is the only fraction in which the denominator, 7, is less than 2 times its numerator, 4 (since $2 \times 4 = 8$).
   Therefore, $\frac{4}{7}$ is larger than $\frac{1}{2}$.

   Answer: (C)

5. Rolling the cube does not change the size of the painted triangle.
   For this reason, we can eliminate answer (A).
   Rolling the cube does not change the number of painted triangles.
   For this reason, we can eliminate answers (D) and (E).
   Rolling the cube does not change the orientation of the painted triangle with respect to the face of the cube that it is painted on.
   For this reason, we can eliminate answer (C).
   Of the given answers, the cube shown in (B) is the only cube which could be the same as the cube that was rolled.

   Answer: (B)

6. The measure of the three angles in any triangle add to 180°.
   Since two of the angles measure 25° and 70°, then the third angle in the triangle measures $180^\circ - 25^\circ - 70^\circ = 85^\circ$.
   The measure of the third angle in the triangle is 85°.

   Answer: (A)

7. Each of the 30 pieces of fruit in the box is equally likely to be chosen. Since there are 10 oranges in the box, then the probability that the chosen fruit is an orange is $\frac{10}{30}$ or $\frac{1}{3}$.

   Answer: (D)

8. *Solution 1*
Since Alex pays \$2.25 to take the bus, then 20 trips on the bus would cost Alex $20 \times \$2.25 = \$45$.
Since Sam pays \$3.00 to take the bus, then 20 trips on the bus would cost Sam $20 \times \$3.00 = \$60$.
If they each take the bus 20 times, then in total Alex would pay $\$60 - \$45 = \$15$ less than Sam.

Solution 2
Since Alex pays \$2.25 to take the bus, and Sam pays \$3.00 to take the bus, then Alex pays $\$3.00 - \$2.25 = \$0.75$ less than Sam each time they take the bus.
If they each take the bus 20 times, then in total Alex would pay $20 \times \$0.75 = \$15$ less than Sam.

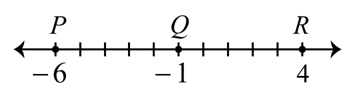
ANSWER: (C)

9. *Solution 1*
Travelling at a constant speed of 85 km/h, the entire 510 km trip would take Carrie $510 \div 85 = 6$ hours.
Since Carrie is halfway through the 510 km trip, then the remainder of the trip will take her half of the total trip time or $6 \div 2 = 3$ hours.

Solution 2
Carrie is halfway through a 510 km trip, and so she has half of the distance or $510 \div 2 = 255$ km left to travel.
Since Carrie travels at a constant speed of 85 km/h, then it will take her $255 \div 85 = 3$ hours longer to complete the trip.

ANSWER: (E)

10. Since Q is halfway between P and R, then the distance between P and Q is equal to the distance between Q and R.
The distance between P and Q is $-1 - (-6) = -1 + 6 = 5$.
Since P is 5 units to the left of Q, then R is 5 units to the right of Q.
That is, R is located at $-1 + 5 = 4$ on the number line.



ANSWER: (A)

11. In the diagram, there are 4 rows of octagons and each row contains 5 octagons.
Therefore, the total number of octagons in the diagram is $4 \times 5 = 20$.
In the diagram, there are 3 rows of squares and each row contains 4 squares.
Therefore, the total number of squares in the diagram is $3 \times 4 = 12$.
The ratio of the number of octagons to the number of squares is $20 : 12$ or $5 : 3$.

ANSWER: (E)

12. The sum of the units column is $Q + Q + Q = 3Q$.
Since Q is a single digit, and $3Q$ ends in a 6, then the only possibility is $Q = 2$.
Then $3Q = 3 \times 2 = 6$, and thus there is no carry over to the tens column.
The sum of the tens column becomes $2 + P + 2 = P + 4$, since $Q = 2$.
Since P is a single digit, and $P + 4$ ends in a 7, then the only possibility is $P = 3$.
Then $P + 4 = 3 + 4 = 7$, and thus there is no carry over to the hundreds column.
We may verify that the sum of the hundreds column is $3+3+2 = 8$, since $P = 3$ and $Q = 2$.
The value of $P + Q$ is $3 + 2 = 5$, and the final sum is shown.

$$\begin{array}{r} 322 \\ 332 \\ +\,222 \\ \hline 876 \end{array}$$

ANSWER: (B)

13. Since a cube is a rectangular prism, its volume is equal to the area of its base, $l \times w$, multiplied by its height, h.
A cube has edges of equal length and so $l = w = h$.
Thus, the volume of a cube is the product of three equal numbers.
The volume of the larger cube is $64\,\text{cm}^3$ and $64 = 4 \times 4 \times 4$, so the length of each edge of the larger cube is $4\,\text{cm}$.
The smaller cube has edges that are half the length of the edges of the larger cube, or $2\,\text{cm}$.
The volume of the smaller cube is $2 \times 2 \times 2 = 8\,\text{cm}^3$.

ANSWER: (C)

14. Ahmed could choose from the following pairs of snacks: apple and orange, apple and banana, apple and granola bar, orange and banana, orange and granola bar, or banana and granola bar.
Therefore, there are 6 different pairs of snacks that Ahmed may choose.

ANSWER: (D)

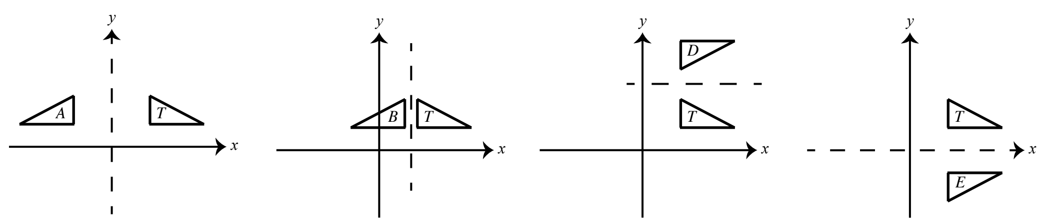
15. Sophia did push-ups for 7 days (an odd number of days), and on each day she did an equal number of push-ups more than the day before (5 more).
Therefore, the number of push-ups that Sophia did on the middle day (day 4) is equal to the average number of push-ups that she completed each day.
Sophia did 175 push-ups in total over the 7 days, and thus on average she did $175 \div 7 = 25$ push-ups each day.
Therefore, on day 4 Sophia did 25 push-ups, and so on day 5 she did $25 + 5 = 30$ push-ups, on day 6 she did $30 + 5 = 35$ push-ups and on the last day she did $35 + 5 = 40$ push-ups.
(Note: We can check that $10 + 15 + 20 + 25 + 30 + 35 + 40 = 175$, as required.)

ANSWER: (E)

16. Since $\square = \triangle + \triangle + \triangle$, then by adding a $\blacklozenge$ to each side we get $\square + \blacklozenge = \blacklozenge + \triangle + \triangle + \triangle$.
Since $\square + \blacklozenge = \blacklozenge + \triangle + \triangle + \triangle$, then by adding a $\triangle$ to each side we get that $\square + \blacklozenge + \triangle = \blacklozenge + \triangle + \triangle + \triangle + \triangle$.
(Can you explain why each of the other answers is not equal to $\square + \blacklozenge + \triangle$?)

ANSWER: (B)

17. Each of the following four diagrams shows the image of triangle T after its reflection in the dotted line.



Thus, each of the triangles labelled A, B, D, and E is a single reflection of triangle T in some line.
The triangle labelled C is the only triangle that cannot be a reflection of triangle T.

ANSWER: (C)

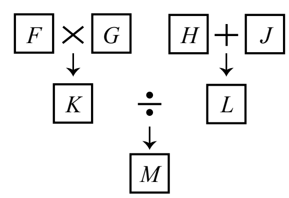
18. The mean (average) of the set of six numbers is 10, and so the sum of the set of six numbers is $6 \times 10 = 60$.
If the number 25 is removed from the set, the sum of the set of the remaining five numbers is $60 - 25 = 35$.
The mean (average) of the remaining set of five numbers is $35 \div 5 = 7$.

ANSWER: (B)

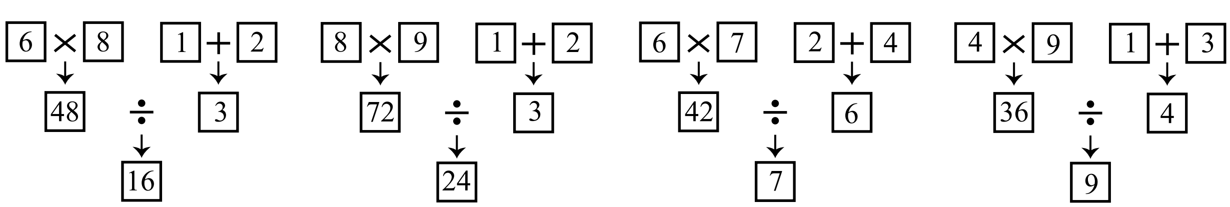
19. The shaded and unshaded sections of the ribbon have equal length.
Since there are 5 such sections, then each shaded and unshaded section has length equal to $\frac{1}{5}$ or $\frac{3}{15}$ of the length of the ribbon.
All measurements which follow are made beginning from the left end of the ribbon.
Point A is located 3 sections from the left end of the ribbon, or at a point $3 \times \frac{3}{15} = \frac{9}{15}$ along the length of the ribbon.
Point D is located 4 sections from the left end of the ribbon, or at a point $4 \times \frac{3}{15} = \frac{12}{15}$ along the length of the ribbon.
All points are equally spaced, and so points B and C divide the unshaded section between points A and D into 3 equal lengths.
Since A is located $\frac{9}{15}$ of the ribbon length from the left end, and D is located $\frac{12}{15}$ of the ribbon length from the left end, then B is located at $\frac{10}{15}$ of the ribbon length, and C is located at $\frac{11}{15}$ of the ribbon length.
Thus, if Suzy makes a vertical cut at point C, the portion of the ribbon to the left of C will be $\frac{11}{15}$ of the size of the original ribbon.
We note that no point is located more than 2 sections from the right end of the ribbon.
That is, no point is located more than $2 \times \frac{3}{15} = \frac{6}{15}$ along the length of the ribbon when measured from the right end, and so measurements are taken from the left end of the ribbon.

Answer: (C)

20. We begin by naming the boxes as shown to the right.



Of the five answers given, the integer which cannot appear in box M is 20. Why?
Since boxes F and G contain different integers, the maximum value that can appear in box K is $8 \times 9 = 72$.
Since boxes H and J contain different integers, the minimum value that can appear in box L is $1 + 2 = 3$.
Next, we consider the possibilities if 20 is to appear in box M.
If 3 appears in box L (the minimum possible value for this box), then box K must contain 60, since $60 \div 3 = 20$.
However, there are no two integers from 1 to 9 whose product is 60 and so there are no possible integers which could be placed in boxes F and G so that the product in box K is 60.
If any integer greater than or equal to 4 appears in box L, then box K must contain at least $4 \times 20 = 80$.
However, the maximum value that can appear in box K is 72.
Therefore, there are no possible integers from 1 to 9 which can be placed in boxes F, G, H, and J so that 20 appears in box M.
The diagrams below demonstrate how each of the other four answers can appear in box M.

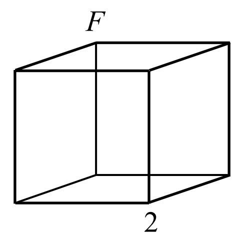


Answer: (D)

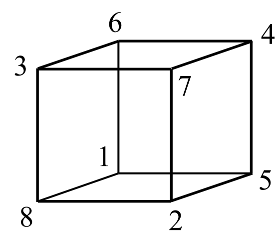
21. Line segment PQ is vertical if Q is chosen from the points in the column in which P lies. This column contains 9 points other than P which could be chosen to be Q so that PQ is vertical.
Line segment PQ is horizontal if Q is chosen from the points in the row in which P lies.
This row contains 9 points other than P which could be chosen to be Q so that PQ is horizontal.
Each of these 9 points is different from the 9 points in the column containing P.
Thus, there are $9 + 9 = 18$ points which may be chosen to be Q so that PQ is vertical or horizontal.
Since there are a total of 99 points to choose Q from, the probability that Q is chosen so that PQ is vertical or horizontal is $\frac{18}{99}$ or $\frac{2}{11}$.

ANSWER: (A)

22. First we choose to label one of the vertices 2, and then label the vertex that is farthest away from this vertex F, as shown.
(Can you explain why the vertex labelled F is the vertex that is farthest away from the vertex labelled 2?)
Each of the other six vertices of the cube lie on one of the three faces on which the vertex labelled 2 lies.



We note that the vertex labelled F is the only vertex which does not lie on a face on which the vertex labelled 2 lies.
From the six given lists, we consider those lists in which the number 2 appears.
These are: $(1, 2, 5, 8)$, $(2, 4, 5, 7)$, and $(2, 3, 7, 8)$.
Thus, the vertices labelled $1, 5$ and 8 lie on a face with the vertex labelled 2, as do the vertices labelled $4, 7$ and 3.
The only vertex label not included in this list is 6.
Thus, the vertex labelled 6 is the only vertex which does not lie on a face on which the vertex labelled 2 lies.
Therefore, the correct labelling for vertex F, the farthest vertex from the vertex labelled 2, is 6.
One possible labelling of the cube is shown.



ANSWER: (D)

23. *Solution 1*

Let the letter R represent a red marble, and the letter B represent a blue marble.
On her first draw, Angie may draw RR, RB or BB.

<u>Case 1:</u> Angie draws RR or RB on her first draw
If Angie draws RR or RB on her first draw, then she discards the R and the three remaining marbles in the jar are RBB.
On her second draw, Angie may draw RB or BB.
If she draws RB, then she discards the R and the two remaining marbles in the jar are BB.
Since there are no red marbles remaining, it is not possible for the final marble to be red in this case.
If on her second draw Angie instead draws BB, then she discards a B and the two remaining marbles in the jar are RB.
When these are both drawn on her third draw, the R is discarded and the final marble is blue.
Again in this case it is not possible for the final marble to be red.
Thus, if Angie draws RR or RB on her first draw, the probability that the final marble is red is zero.

<u>Case 2:</u> Angie draws BB on her first draw
If Angie draws BB on her first draw, then she discards a B and the three remaining marbles in the jar are RRB.
On her second draw, Angie may draw RR or RB.
If she draws RR or RB, then she discards one R and the two remaining marbles in the jar are RB.
When these are both drawn on her third draw, the R is discarded and the final marble is blue.
In this case it is not possible for the final marble to be red.
Thus, if Angie draws BB on her first draw, the probability that the final marble is red is zero.

Therefore, under the given conditions of drawing and discarding marbles, the probability that Angie's last remaining marble is red is zero.

Solution 2

Let the letter R represent a red marble, and the letter B represent a blue marble.
If the final remaining marble is R, then the last two marbles must include at least one R.
That is, the last two marbles must be RB or RR.
If the last two marbles are RB, then when they are drawn from the jar, the R is discarded and the B would remain.
Thus it is not possible for the final marble to be R if the final two marbles are RB.
So the final remaining marble is R only if the final two marbles are RR.
If the final two marbles are RR, then the last three marbles are BRR (since there are only two Rs in the jar at the beginning).
However, if the final three marbles are BRR, then when Angie draws two of these marbles from the jar, at least one of the marbles must be R and therefore one R will be discarded leaving BR as the final two marbles in the jar.
That is, it is not possible for the final two marbles in the jar to be RR.
The only possibility that the final remaining marble is R occurs when the final two marbles are RR, but this is not possible.
Therefore, under the given conditions of drawing and discarding marbles, the probability that Angie's last remaining marble is red is zero.

ANSWER: (E)

24. We begin by showing that each of 101, 148, 200, and 621 can be expressed as the sum of two or more consecutive positive integers.

$$\begin{aligned}101 &= 50+51\\148 &= 15+16+17+18+19+20+21+22\\200 &= 38+39+40+41+42\\621 &= 310+311\end{aligned}$$

We show that 512 cannot be expressed a sum of two or more consecutive positive integers. This will tell us that one of the five numbers in the list cannot be written in the desired way, and so the answer is (B).

Now, 512 cannot be written as the sum of an odd number of consecutive positive integers.
Why is this? Suppose that 512 equals the sum of p consecutive positive integers, where $p > 1$ is odd.
Since p is odd, then there is a middle integer m in this list of p integers.
Since the numbers in the list are equally spaced, then m is the average of the numbers in the list.
(For example, the average of the 5 integers $6, 7, 8, 9, 10$ is 8.)
But the sum of the integers equals the average of the integers (m) times the number of integers (p). That is, $512 = mp$.
Now $512 = 2^9$ and so does not have any odd divisors larger than 1.
Therefore, 512 cannot be written as mp since m and p are positive integers and $p > 1$ is odd.
Thus, 512 is not the sum of an odd number of consecutive positive integers.

Further, 512 cannot be written as the sum of an even number of consecutive positive integers.
Why is this? Suppose that 512 equals the sum of p consecutive positive integers, where $p > 1$ is even.
Since p is even, then there is not a single middle integer m in this list of p integers, but rather two middle integers m and $m + 1$.
Since the numbers in the list are equally spaced, then the average of the numbers in the list is the average of m and $m + 1$, or $m + \frac{1}{2}$.
(For example, the average of the 6 integers $6, 7, 8, 9, 10, 11$ is $8\frac{1}{2}$.)
But the sum of the integers equals the average of the integers $(m + \frac{1}{2})$ times the number of integers (p). That is, $512 = (m + \frac{1}{2})p$ and so $2(512) = 2(m + \frac{1}{2})p$ or $1024 = (2m + 1)p$.
Now $1024 = 2^{10}$ and so does not have any odd divisors larger than 1.
Therefore, 1024 cannot be written as $(2m+1)p$ since m and p are positive integers and $2m+1 > 1$ is odd.
Thus, 512 is not the sum of an even number of consecutive positive integers.

Therefore, 512 is not the sum of any number of consecutive positive integers.
A similar argument shows that every power of 2 cannot be written as the sum of any number of consecutive positive integers.
Returning to the original question, exactly one of the five numbers in the original list cannot be written in the desired way, and so the answer is (B).

ANSWER: (B)

25. Consider the diagonal lines that begin on the left edge of the triangle and move downward to the right.
The first number in the n^{th} diagonal line is n, and it lies in the n^{th} horizontal row.
For example, the first number in the 3^{rd} diagonal line $(\mathbf{3}, 6, 9, 12, \dots)$ is 3 and it lies in the 3^{rd} horizontal row $(\mathbf{3}, 4, 3)$.
The second number in the n^{th} diagonal line is $n + n$ or $2n$ and it lies in the horizontal row numbered $n + 1$.
The third number in the n^{th} diagonal line is $n + n + n$ or $3n$ and it lies in the horizontal row numbered $n + 2$ (each number lies one row below the previous number in the diagonal line).
Following this pattern, the m^{th} number in the n^{th} diagonal line is equal to $m \times n$ and it lies in the horizontal row numbered $n + (m - 1)$.
The table below demonstrates this for $n = 3$, the 3^{rd} diagonal line.

m	m^{th} Diagonal Number	Horizontal Row Number
1	3	3
2	$2(3) = 6$	$3 + 1 = 4$
3	$3(3) = 9$	$3 + 2 = 5$
4	$4(3) = 12$	$3 + 3 = 6$
5	$5(3) = 15$	$3 + 4 = 7$
$\vdots$	$\vdots$	$\vdots$
m	$m \times n$	$3 + (m - 1)$

The number 2016 lies in some diagonal line(s).
To determine which diagonal lines 2016 lies in, we express 2016 as a product $m \times n$ for positive integers m and n.
Further, if $2016 = m \times n$, then 2016 appears in the triangle in position m in diagonal line n, and lies in the horizontal row numbered $n + m - 1$.
We want the horizontal row in which 2016 first appears, and so we must find positive integers m and n so that $m \times n = 2016$ and $n + m$ (and therefore $n + m - 1$) is as small as possible.
In the table below, we summarize the factor pairs (m, n) of 2016 and the horizontal row number $n + m - 1$ in which each occurence of 2016 appears.

Factor Pair (m, n)	Horizontal Row Number $n + m - 1$
$(1, 2016)$	2016
$(2, 1008)$	1009
$(3, 672)$	674
$(4, 504)$	507
$(6, 336)$	341
$(7, 288)$	294
$(8, 252)$	259
$(9, 224)$	232
$(12, 168)$	179

Factor Pair (m, n)	Horizontal Row Number $n + m - 1$
$(14, 144)$	157
$(16, 126)$	141
$(18, 112)$	129
$(21, 96)$	116
$(24, 84)$	107
$(28, 72)$	99
$(32, 63)$	94
$(36, 56)$	91
$(42, 48)$	89

(Note: By recognizing that when $m \times n = 2016$, the sum $n + m$ is minimized when the positive difference between m and n is minimized, we may shorten the work shown above.)
We have included all possible pairs (m, n) so that $m \times n = 2016$ in the table above.
We see that 2016 will appear in 18 different locations in the triangle.
However, the first appearance of 2016 occurs in the horizontal row numbered 89.

ANSWER: (E)

Grade 8

1. Evaluating, $444 - 44 - 4 = 400 - 4 = 396$.

ANSWER: (A)

2. *Solution 1*
The fraction $\frac{4}{5}$ is equal to $4 \div 5$ or 0.8.
Solution 2
Since $\frac{4}{5}$ is equal to $\frac{8}{10}$, then $\frac{4}{5} = 0.8$.

ANSWER: (B)

3. Reading from the graph, we summarize the number of hours that Stan worked each day in the table below.

Day	Monday	Tuesday	Wednesday	Thursday	Friday
Number of Hours	2	0	3	1	2

Therefore, Stan worked a total of $2 + 0 + 3 + 1 + 2 = 8$ hours on the project.

ANSWER: (C)

4. Written numerically, three tenths plus four thousandths is $\frac{3}{10} + \frac{4}{1000}$ which is equal to $\frac{300}{1000} + \frac{4}{1000} = \frac{304}{1000} = 0.304$.

ANSWER: (C)

5. Folds occur along the five edges between adjoining faces in the figure shown.
Consider the face numbered 3 as being the bottom face of the completed cube.
First, fold upward along the four edges of the face numbered 3 (the edges between 3 and 5, 3 and 4, 3 and 6, and 3 and 2).
After folding upward, the faces numbered $2, 5, 4$, and 6 become the four vertical faces of the cube.
The final fold occurs along the edge between the faces numbered 1 and 2.
The face numbered 1 becomes the top face of the cube after this fold.
Since the bottom face is opposite the top face, then the face numbered 3 is opposite the face numbered 1.

ANSWER: (B)

6. Side PR is horizontal and so the y-coordinate of P is equal to the y-coordinate of R, or -2.
Side PQ is vertical and so the x-coordinate of P is equal to the x-coordinate of Q, or -11.
Therefore, the coordinates of P are $(-11, -2)$.

ANSWER: (D)

7. A rectangle with a width of $2\,\text{cm}$ and a length of $18\,\text{cm}$ has area $2 \times 18 = 36\,\text{cm}^2$.
The area of a square with side length $s\,\text{cm}$ is $s \times s\,\text{cm}^2$.
The area of the square is also $36\,\text{cm}^2$ and since $6 \times 6 = 36$, then the side length of the square, s, is $6\,\text{cm}$.

ANSWER: (A)

8. From the list $3, 4, 5, 6, 7, 8, 9$, only $3, 5$ and 7 are prime numbers.
The numbers $4, 6, 8$, and 9 are composite numbers.
The ratio of the number of prime numbers to the number of composite numbers is $3 : 4$.

ANSWER: (A)

9. Since 10% of 200 is $\frac{10}{100} \times 200 = 10 \times 2 = 20$, and 20% of 100 is $\frac{20}{100} \times 100 = 20$, then 10% of 200 is equal to 20% of 100.

ANSWER: (C)

10. The circumference of a circle with radius r is equal to $2\pi r$.
When $2\pi r = 100\pi$, $r = \frac{100\pi}{2\pi} = 50\,\text{cm}$.

ANSWER: (C)

11. In equilateral triangle QRS, each angle is equal in measure and so $\angle SQR = 60^\circ$.
Since $\angle PQR = 90^\circ$, then $\angle PQS = \angle PQR - \angle SQR = 90^\circ - 60^\circ = 30^\circ$.
In isosceles triangle PQS, $\angle QPS = \angle PQS = 30^\circ$.
Therefore, $\angle QPR = \angle QPS = 30^\circ$.

ANSWER: (E)

12. We try each of the five options:

(A): $3 + 5 \times 7 + 9 = 3 + 35 + 9 = 47$
(B): $3 + 5 + 7 \times 9 = 3 + 5 + 63 = 71$
(C): $3 \times 5 \times 7 - 9 = 15 \times 7 - 9 = 105 - 9 = 96$
(D): $3 \times 5 \times 7 + 9 = 15 \times 7 + 9 = 105 + 9 = 114$
(E): $3 \times 5 + 7 \times 9 = 15 + 63 = 78$

Therefore, the correct operations are, in order, $\times, +, \times$.

ANSWER: (E)

13. Ahmed could choose from the following pairs of snacks: apple and orange, apple and banana, apple and granola bar, orange and banana, orange and granola bar, or banana and granola bar.
Therefore, there are 6 different pairs of snacks that Ahmed may choose.

ANSWER: (D)

14. One soccer ball and one soccer shirt together cost \$100.
So then two soccer balls and two soccer shirts together cost $2 \times \$100 = \$200$.
Since we are given that two soccer balls and three soccer shirts together cost \$262, then \$200 added to the cost of one soccer shirt is \$262.
Thus, the cost of one soccer shirt is $\$262 - \$200 = \$62$, and the cost of one soccer ball is $\$100 - \$62 = \$38$.

ANSWER: (A)

15. The map's scale of $1 : 600\,000$ means that a 1 cm distance on the map represents an actual distance of 600 000 cm.
So then 2 cm measured on the map represents an actual distance of $2 \times 600\,000 = 1\,200\,000$ cm or 12 000 m or 12 km.
The actual distance between Gausstown and Piville is 12 km.

ANSWER: (A)

16. The mean (average) of the set of six numbers is 10, and so the sum of the set of six numbers is $6 \times 10 = 60$.
If the number 25 is removed from the set, the sum of the set of the remaining five numbers is $60 - 25 = 35$.
The mean (average) of the remaining set of five numbers is $35 \div 5 = 7$.

ANSWER: (B)

17. The positive integers between 10 and 2016 which have all of their digits the same are: $11, 22, 33, 44, 55, 66, 77, 88, 99, 111, 222, 333, 444, 555, 666, 777, 888, 999$, and 1111.
To be divisible by 3, the sum of the digits of the positive integer must equal a multiple of 3.
From the list above, the only 2-digit numbers whose digit sum is a multiple of 3 are $33, 66$ and 99 (with digit sums of 6, 12 and 18, respectively).
(We may verify that each of the other digit sums, $2, 4, 8, 10, 14$, and 16 are not multiples of 3.)
A 3-digit positive integer with all digits equal to d has digit sum $d + d + d = 3d$ (which is a multiple of 3).
Thus, all 3-digit positive integers with equal digits are divisible by 3.
That is, all 9 of the 3-digit integers listed above are divisible by 3.
Finally, the number 1111 has digit sum 4 and thus is not divisible by 3.
There are $3 + 9 = 12$ positive integers between 10 and 2016, having all of their digits the same, that are divisible by 3.

Answer: (B)

18. Joe used $\frac{3}{8}$ of the gas in the tank after travelling 165 km, and so he used $\frac{3}{8} \div 3 = \frac{1}{8}$ of the gas in the tank after travelling $165 \div 3 = 55$ km.
Since Joe used $\frac{1}{8}$ of the gas in the tank to travel 55 km, he used all the gas in the tank to travel $55 \times 8 = 440$ km.
If Joe has already travelled 165 km, then he can travel another $440 - 165 = 275$ km before the gas tank is completely empty.

Answer: (E)

19. The first scale shows that 2 ◯'s balance 6 □'s and so 1 ◯ balances 3 □'s.
Thus, we may eliminate answer (C).
The second scale shows that 2 ◯'s and 6 □'s balance 4 △'s and so 1 ◯ and 3 □'s balance 2 △'s.
Thus, we may eliminate answer (B).
Since 1 ◯ and 3 □'s balance 2 △'s and 1 ◯ balances 3 □'s, then 6 □'s balance 2 △'s or 3 □'s balance 1 △.
Thus, we may eliminate answer (E).
Since 1 ◯ balances 3 □'s, and 1 △ balances 3 □'s, then 1 ◯ balances 1 △.
Thus, we may eliminate answer (A).
Finally, we are left with answer (D).
Since 1 ◯ balances 3 □'s and 1 △ balances 3 □'s, then 1 ◯ and 1 △ balance 6 □'s, not 4 □'s.
Thus, answer (D) is the only answer which is not true.

Answer: (D)

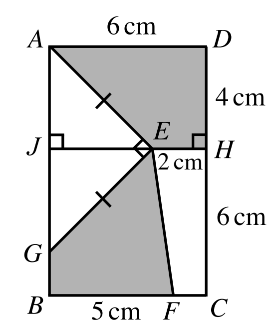
20. Points D and C have equal y-coordinates, -3, and so side DC is parallel to the x-axis and has length $3 - (-2) = 5$.
Points B and C have equal x-coordinates, 3, and so side BC is parallel to the y-axis and has length $9 - (-3) = 12$.
That is, in $\triangle BCD$, sides DC and BC are perpendicular or $\angle BCD = 90^\circ$ with $BC = 9$ and $DC = 5$.
Using the Pythagorean Theorem, $BD^2 = DC^2 + BC^2 = 5^2 + 12^2$ and so $BD^2 = 25 + 144 = 169$ or $BD = \sqrt{169} = 13$ (since $BD > 0$).

Answer: (A)

21. If the ten thousands digits of the two numbers differ by more than 1, then the two numbers will differ by more than 10 000. (For example, a number of the form $5x\,xxx$ is at least 50 000 and a number of the form $3x\,xxx$ is less than 40 000 so these numbers differ by more than 10 000.)
Since all of the given answers are less than 1000 and since the two ten thousands digits cannot be equal, then the ten thousands digits must differ by 1. We will determine the exact ten thousands digits later, so we let the smaller of the two ten thousands digits be d and the larger be D.
To make the difference between $Dx\,xxx$ and $dx\,xxx$ as small as possible, we try to simultaneously make $Dx\,xxx$ as close to $D0\,000$ as possible and $dx\,xxx$ as close to $d9\,999$ as possible while using all different digits.
In other words, we try to make $Dx\,xxx$ as small as possible and $dx\,xxx$ as large as possible while using all different digits.
To make $Dx\,xxx$ as small as possible, we use the smallest possible digits in the places with the highest value.
Since all of the digits must be different, then the minimum possible value of $Dx\,xxx$ is $D0\,123$.
To make $dx\,xxx$ as large as possible, we use the largest possible digits in the places with the highest value.
Since all of the digits must be different, then the maximum possible value of $dx\,xxx$ is $d9\,876$.
Since we have made $Dx\,xxx$ as small as possible and $dx\,xxx$ as large as possible and used completely different sets of digits, then doing these two things will make their difference as small as possible, assuming that there are digits remaining to assign to D and d that differ by 1.
The digits that have not been used are 5 and 4; thus, we set $D = 5$ and $d = 4$.
This gives numbers 50 123 and 49 876.
Their difference is $50\,123 - 49\,876 = 247$, which is the minimum possible difference.

ANSWER: (C)

22. We find the area of the shaded region by determining the area of the unshaded region and subtracting this from the total area of the rectangle.



We begin by extending HE to J on side AB, as shown.
Since HE is perpendicular to DH, then HJ is perpendicular to DH.
Since DH is parallel to AJ, then HJ is also perpendicular to AJ, and so $ADHJ$ is a rectangle.
Since $ADHJ$ is a rectangle, then $AJ = DH = 4$ cm.
Also, $AD = JH = 6$ cm.
Since $JH = 6$ cm, then $HE + EJ = 6$ cm or $2\text{ cm} + EJ = 6$ cm and so $EJ = 4$ cm.
Since $\triangle AEG$ is isosceles with $AE = GE$, then altitude EJ divides base AG into two equal lengths.
Since $AJ = 4$ cm, then $GJ = AJ = 4$ cm.
Therefore, $\triangle AEG$ has base $AG = 8$ cm and height $EJ = 4$ cm and so its area is $\frac{1}{2}(8)(4) = 16\text{ cm}^2$.
Since $BC = AD = 6$ cm, then $BF + CF = 6$ cm or $5\text{ cm} + CF = 6$ cm and so $CF = 1$ cm.
In quadrilateral $EHCF$, sides HE and CF are both perpendicular to DC, and so they are parallel to each other.
That is, quadrilateral $EHCF$ is a trapezoid with parallel sides $EH = 2$ cm, $CF = 1$ cm, and height $HC = 6$ cm (since HC is perpendicular to both HE and CF).
The area of trapezoid $EHCF$ is $\frac{1}{2}(6)(2 + 1) = 9\text{ cm}^2$.
The total area of the shaded region is found by subtracting the area of $\triangle AEG$ and the area of trapezoid $EHCF$ from the area of rectangle $ABCD$.

The area of rectangle $ABCD$ is $6(6+4) = 6(10) = 60\,\text{cm}^2$, and so the total area of the shaded region is $60 - 16 - 9 = 35\,\text{cm}^2$.

Answer: (D)

23. For Zeus to arrive at the point $(1056, 1007)$, he must travel up and to the right from his starting point $(0, 0)$.
More specifically, Zeus will need to make at least 1056 moves to the right (R).
Since Zeus cannot move in the same direction twice in a row, then no two moves R can be next to each other. In other words, there must be at least one move between each pair of Rs.
Since there are 1056 moves R and there must be at least one move between each pair, then there are at least 1055 more moves (at least one between the 1^{st} and 2^{nd} R, between the 2^{nd} and 3^{rd} R, and so on).
Therefore, at least $1056 + 1055 = 2111$ moves are needed.
We will show that there is actually a sequence of 2111 moves that obey the given rules and that take Zeus to $(1056, 1007)$.
For Zeus to end at a point with y-coordinate 1007, he will need to make at least 1007 moves up (U).
So we put one U between the 1^{st} and 2^{nd} R, between the 2^{nd} and 3^{rd} R, and so on up to between the 1007^{th} and the 1008^{th} R.
This leaves us with $1055 - 1007 = 48$ spaces between Rs to fill.
We want to fill these spaces with moves that do not affect Zeus' position (since we already have him moving 1056 moves R and 1007 moves U) and so that none of the moves are R (since the spaces to be filled are between adjacent moves R).
We can do this by making the first 24 of these moves U and the remaining 24 of these moves down (D). This results in Zeus moving an additional 24 units up but then an additional 24 moves down, which is no net change in position.
We have shown that Zeus needs at least 2111 moves to get to $(1056, 1007)$ and that he can actually get to $(1056, 1007)$ in 2111 moves following the given rules, so the smallest number of moves that he needs is 2111.

Answer: (D)

24. When two integers are multiplied together, the final two digits (the tens digit and the units digits) of the product are determined by the final two digits of each of the two numbers that are multiplied.
This is true since the place value of any digit contributes to its equal place value (and possibly also to a greater place value) in the product.
That is, the hundreds digit of each number being multiplied contributes to the hundreds digit (and possibly to digits of higher place value) in the product.
Thus, to determine the tens digit of any product, we need only consider the tens digits and the units digits of each of the two numbers that are being multiplied.
For example, to determine the final two digits of the product 1215×603 we consider the product $15 \times 03 = 45$. We may verify that the tens digit of the product $1215 \times 603 = 732\,645$ is indeed 4 and the units digit is indeed 5.
Since $3^5 = 243$, then the final two digits of $3^{10} = 3^5 \times 3^5 = 243 \times 243$ are given by the product $43 \times 43 = 1849$ and thus are 49.
Since the final two digits of 3^{10} are 49 and $3^{20} = 3^{10} \times 3^{10}$, then the final two digits of 3^{20} are given by $49 \times 49 = 2401$, and thus are 01.
Then $3^{40} = 3^{20} \times 3^{20}$ ends in 01 also (since $01 \times 01 = 01$).
Further, 3^{20} multiplied by itself 100 times, or $(3^{20})^{100} = 3^{2000}$ also ends with 01.

Since 3^{10} ends with 49 and 3^5 ends with 43, then $3^{15} = 3^{10} \times 3^5$ ends with $49 \times 43 = 2107$ and thus has final two digits 07.
This tells us that $3^{16} = 3^{15} \times 3^1$ ends with $07 \times 03 = 21$.
Finally, $3^{2016} = 3^{2000} \times 3^{16}$ and thus ends in $01 \times 21 = 21$, and so the tens digit of 3^{2016} is 2.

ANSWER: (B)

25. We begin by adding variables to some of the blanks in the grid to make it easier to refer to specific entries:

				18
	43	f	g	h
		40		j
		k		m
x	n	p	26	q

Since the numbers in each row form an arithmetic sequence and the numbers in each column form an arithmetic sequence, we will refer in several sequences to the *common difference*.
Let the common difference between adjacent numbers as we move down column 3 be d.
Therefore, $k = 40 + d$ and $p = k + d = 40 + 2d$.
Also, $40 = f + d$, or $f = 40 - d$.
Moving from left to right along row 2, the common difference between adjacent numbers must be $(40 - d) - 43 = -3 - d$.
Therefore,

$$g = f + (-3 - d) = (40 - d) + (-3 - d) = 37 - 2d$$
$$h = g + (-3 - d) = (37 - 2d) + (-3 - d) = 34 - 3d$$

This gives:

				18
	43	$40 - d$	$37 - 2d$	$34 - 3d$
		40		j
		$40 + d$		m
x	n	$40 + 2d$	26	q

Moving from the top to the bottom of column 5, the common difference between adjacent numbers can be found by subtracting 18 from $34 - 3d$.
That is, the common difference between adjacent numbers in column 5 is $(34-3d)-18 = 16-3d$.
Moving down column 5, we can see that

$$j = (34 - 3d) + (16 - 3d) = 50 - 6d$$
$$m = (50 - 6d) + (16 - 3d) = 66 - 9d$$
$$q = (66 - 9d) + (16 - 3d) = 82 - 12d$$

In each case, we have added the common difference to the previous number to obtain the next number.

This gives the following updated grid:

				18
	43	$40-d$	$37-2d$	$34-3d$
		40		$50-6d$
		$40+d$		$66-9d$
x	n	$40+2d$	26	$82-12d$

In row 5, the difference between the fourth and third entries must equal the difference between the fifth and fourth entries.
In other words,

$$\begin{aligned} 26-(40+2d) &= (82-12d)-26 \\ -14-2d &= 56-12d \\ 10d &= 70 \\ d &= 7 \end{aligned}$$

We can then substitute $d=7$ into our grid to obtain:

				18
	43	33	23	13
		40		8
		47		3
x	n	54	26	-2

This allows us to determine the value of x by moving along row 5.
We note that $26-54=-28$ so the common difference in row 5 is -28.
Therefore, $n+(-28)=54$ or $n=54+28=82$.
Similarly, $x+(-28)=n=82$ and so $x=82+28=110$.
Therefore, the sum of the digits of the value of $x=110$ is $1+1+0=2$.

Answer: (B)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2015 Gauss Contests

(Grades 7 and 8)

Wednesday, May 13, 2015
(in North America and South America)

Thursday, May 14, 2015
(outside of North America and South America)

Solutions

Centre for Education in Mathematics and Computing Faculty and Staff

Ed Anderson
Jeff Anderson
Terry Bae
Steve Brown
Ersal Cahit
Heather Culham
Serge D'Alessio
Frank DeMaio
Janine Dietrich
Jennifer Doucet
Fiona Dunbar
Mike Eden
Barry Ferguson
Judy Fox
Steve Furino
John Galbraith
Sandy Graham
Conrad Hewitt
Angie Hildebrand
Carrie Knoll
Judith Koeller
Bev Marshman
Mike Miniou
Dean Murray
Jen Nelson
J.P. Pretti
Kim Schnarr
Carolyn Sedore
Ian VanderBurgh
Troy Vasiga
JoAnn Vincent
Tim Zhou

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Chris Wu, Zion Heights J.H.S., Toronto, ON
Lori Yee, William Dunbar P.S., Pickering, ON

Grade 7

1. The circle is divided into 4 equal regions. Since 1 of these 4 regions is shaded, then the fraction of the circle that is shaded is $\frac{1}{4}$.

   ANSWER: (C)

2. Evaluating, $10 \times (5 - 2) = 10 \times 3 = 30$.

   ANSWER: (D)

3. Reading from the graph, Phil ran 4 km, Tom ran 6 km, Pete ran 2 km, Amal ran 8 km, and Sanjay ran 7 km. Therefore, Pete ran the least distance.

   ANSWER: (C)

4. The equal-arm balance shows that 2 rectangles have the same mass as 6 circles.
   If we organize these shapes into two equal piles on both sides of the balance, then we see that 1 rectangle has the same mass as 3 circles.

   ANSWER: (B)

5. Of the possible answers, the length of your thumb is closest to 5 cm.

   ANSWER: (E)

6. There are 100 centimetres in 1 metre. Therefore, there are $3.5 \times 100 = 350$ cm in 3.5 metres.

   ANSWER: (A)

7. The length of the side not labelled is equal to the sum of the two horizontal lengths that are labelled, or $2 + 3 = 5$. Thus, the perimeter of the figure shown is $5 + 5 + 2 + 3 + 3 + 2 = 20$.

   ANSWER: (D)

8. The average (mean) number of points scored per game multiplied by the number of games played is equal to the total number of points scored during the season.
   Therefore, the number of games that Hannah played is equal to the total number of points she scored during the season divided by her average (mean) number of points scored per game, or $312 \div 13 = 24$.

   ANSWER: (A)

9. The positive divisors of 20 are: $1, 2, 4, 5, 10$, and 20.
   Therefore, the number 20 has exactly 6 positive divisors.

   ANSWER: (B)

10. Using the digits $4, 7$ and 9 without repeating any digit in a given number, the following 3-digit whole numbers can be formed: $479, 497, 749, 794, 947$, and 974.
    There are exactly 6 different 3-digit whole numbers that can be formed in the manner described.

    ANSWER: (A)

11. *Solution 1*
    At Gaussville School, 40% or $\frac{40}{100} = \frac{4}{10}$ of the 480 total students voted for math.
    Therefore, the number of students who voted for math is $\frac{4}{10} \times 480 = 4 \times 48 = 192$.

    Solution 2
    At Gaussville School, 40% or 0.4 of the 480 total students voted for math.
    Therefore, the number of students who voted for math is $0.4 \times 480 = 192$.

    ANSWER: (B)

12. The first fold creates 2 layers of paper. The second fold places 2 sets of 2 layers together, for a total of 4 layers of paper. Similarly, the third fold places 2 sets of 4 layers of paper together, for a total of 8 layers of paper.
That is, each new fold places 2 sets of the previous number of layers together, thereby doubling the previous number of layers.
The results of the first five folds are summarized in the table below.

Number of folds	0	1	2	3	4	5
Number of layers	1	2	4	8	16	32

After the sheet has been folded in half five times, the number of layers in the folded sheet is 32.

Answer: (B)

13. *Solution 1*
The multiples of 5 between 1 and 99 are:

$$5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95.$$

Of these, only $10, 20, 30, 40, 50, 60, 70, 80,$ and 90 are even.
Therefore, there are 9 even whole numbers between 1 and 99 that are multiples of 5.

Solution 2
To create an even multiple of 5, we must multiply 5 by an even whole number (since 5 is odd, multiplying 5 by an odd whole number creates an odd result).
The smallest positive even multiple of 5 is $5 \times 2 = 10$.
The largest even multiple of 5 less than 99 is $5 \times 18 = 90$.
That is, multiplying 5 by each of the even numbers from 2 to 18 results in the only even multiples of 5 between 1 and 99.
Since there are 9 even numbers from 2 to 18 (inclusive), then there are 9 even whole numbers between 1 and 99 that are multiples of 5.

Answer: (C)

14. Consider the value of U in the diagram shown.
Since a 3 already occurs in the second row, then U cannot equal 3 (each of the numbers $1, 2, 3$, occur only once in each row).
Since a 1 already occurs in the third column, then U cannot equal 1 (each of the numbers $1, 2, 3$, occur only once in each column).
Since U cannot equal 3 or 1, then $U = 2$.
Therefore, a 2 and a 3 already occur in the second row and so $X = 1$.
At this point, a 2 and a 1 already occur in the third column and so $Y = 3$.
The value of $X + Y = 1 + 3 = 4$.

		1
3	X	U
		Y

Answer: (E)

15. The rectangle has area $5 \times 12 = 60\,\text{cm}^2$.
Each of the two congruent unshaded triangles has area $\frac{1}{2} \times 2 \times 5 = 5\,\text{cm}^2$.
The area of the shaded region is equal to the area of the rectangle minus the areas of the two unshaded triangles, which is $60 - 5 - 5 = 50\,\text{cm}^2$.

Answer: (E)

16. The total value of one quarter, one dime and one nickel is $25 + 10 + 5 = 40$ ¢.
Since you have equal numbers of quarters, dimes and nickels, you can separate your coins into piles, each containing exactly 1 quarter, 1 dime, and 1 nickel.
Each pile has a value of 40¢, and since $440 \div 40 = 11$, then you must have 11 quarters, 11 dimes and 11 nickels.
Therefore, you have 11 dimes.
Note: You can check that $11 \times (25$ ¢ $+ 10$ ¢ $+ 5$ ¢$) = 11 \times 40$ ¢ $= 440$ ¢ $= \$4.40$, as required.
Answer: (B)

17. The original cube (before the corner was cut off) had 12 edges.
Cutting off the corner does not eliminate any of the 12 edges of the original cube.
However, cutting off the corner does add 3 edges that were not present originally, the 3 edges of the new triangular face. Since no edges of the original cube were lost, but 3 new edges were created, then the new solid has $12 + 3 = 15$ edges.
Answer: (D)

18. To find the image of PQ, we reflect points P and Q across the x-axis, then join them.
Since P is 3 units above the x-axis, then the reflection of P across the x-axis is 3 units below the x-axis at the same x-coordinate.
That is, point T is the image of P after it is reflected across the x-axis.
Similarly, after a reflection across the x-axis, the image of point Q will be 6 units below the x-axis but have the same x-coordinate as Q.
That is, point U is the image of Q after it is reflected across the x-axis.
Therefore, the line segment TU is the image of PQ after it is reflected across the x-axis.
Answer: (B)

19. Since the number of digits that repeat is 6, then the digits 142857 begin to repeat again after 120 digits (since $120 = 6 \times 20$).
That is, the 121^{st} digit is a 1, the 122^{nd} digit is a 4, and the 123^{rd} digit is a 2.
Answer: (C)

20. Since the sum of the measures of the three angles in any triangle is $180°$, then the sum of the measures of the two unknown angles in the given triangle is $180° - 45° = 135°$.
The measures of the two unknown angles are in the ratio $4 : 5$, and so one of the two angle measures is $\frac{5}{4+5} = \frac{5}{9}$ of the sum of the two angles, while the other angle measures $\frac{4}{4+5} = \frac{4}{9}$ of the sum of the two angles.
That is, the larger of the two unknown angles measures $\frac{5}{9} \times 135° = 75°$, and the smaller of the unknown angles measures $\frac{4}{9} \times 135° = 60°$.
We may check that $60° + 75° + 45° = 180°$.
The largest angle in the triangle measures $75°$.
Answer: (C)

21. We begin by choosing the largest number in each row, $5, 10, 15, 20, 25$, and calling this list L.
The sum of the five numbers in L is $5 + 10 + 15 + 20 + 25 = 75$ and this sum satisfies the condition that no two numbers come from the same row.
However, the numbers in L are taken from columns 1 and 5 only, and the numbers must be chosen so that no two come from the same column.
Thus, the largest of the five answers given, 75, is not possible.
Note: In assuring that we take one number from each row, this choice of numbers, L, is the only way to obtain a sum of 75 (since we chose the largest number in each row).

Of the five answers given, the next largest answer is 73.
Since L uses the largest number in each row and has a sum of 75, we can obtain a sum of 73 either by replacing one of the numbers in L with a number that is two less, or by replacing two of the numbers in L with numbers that are each one less.
For example, the list $3, 10, 15, 20, 25$ (one change to L) has sum 73 as does the list $4, 9, 15, 20, 25$ (two changes to L).
That is, to obtain a sum of 73 while choosing exactly one number from each row, we must choose at least three of the numbers from L.
However, since two numbers in L lie in column 1 and three numbers from L lie in column 5, it is not possible to choose at least three numbers from L so that no two of the numbers are from the same column.
Any other replacement would give a sum less than 73, which would require the replacement of a number with a larger number in another row to compensate. This is impossible since each row is represented in L by the largest number in the row.
Therefore, it is not possible to obtain a sum of 73.

Of the five answers given, the next largest answer is 71.
By choosing the numbers, $3, 9, 14, 20, 25$ we obtain the sum $3 + 9 + 14 + 20 + 25 = 71$ while satisfying the condition that no two numbers come from the same row and no two numbers come from the same column.
Thus, 71 is the largest possible sum that satisfies the given conditions.
Note: There are other choices of five numbers which also give a sum of 71 and satisfy the given conditions.

ANSWER: (C)

22. Since the perimeter of the square is P and the 4 sides of a square are equal in length, then each side of the square has length $\frac{1}{4}P$. We now work backward to determine the width and length of the rectangle.
The width of the rectangle was doubled to produced the side of the square with length $\frac{1}{4}P$.
Therefore, the width of the rectangle is half of the side length of the square, or $\frac{1}{2} \times \frac{1}{4}P = \frac{1}{8}P$.
The length of the rectangle was halved to produce the side of the square with length $\frac{1}{4}P$.
Therefore, the length of the rectangle is twice the side length of the square, or $2 \times \frac{1}{4}P = \frac{1}{2}P$.
Finally, we determine the perimeter of the rectangle having width $\frac{1}{8}P$ and length $\frac{1}{2}P$, obtaining $2 \times \left(\frac{1}{8}P + \frac{1}{2}P\right) = 2 \times \left(\frac{1}{8}P + \frac{4}{8}P\right) = 2 \times \left(\frac{5}{8}P\right) = \frac{5}{4}P$.

ANSWER: (D)

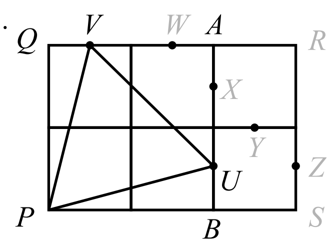
23. *Solution 1*
Every 4-digit palindrome is of the form $abba$, where a is a digit between 1 and 9 inclusive and b is a digit between 0 and 9 inclusive (and b is not necessarily different than a).
Every 5-digit palindrome is of the form $abcba$, where a is a digit between 1 and 9 inclusive, b is a digit between 0 and 9 inclusive (and b is not necessarily different than a), and c is a digit between 0 and 9 inclusive (and c is not necessarily different than a and b).
That is, for every 4-digit palindrome $abba$ there are 10 possible digits c so that $abcba$ is a 5-digit palindrome.
For example if $a = 2$ and $c = 3$, then the 4-digit palindrome 2332 can be used to create the 10 5-digit palindromes: $23032, 23132, 23232, 23332, 23432, 23532, 23632, 23732, 23832, 23932$.
Thus, for every 4-digit palindrome $abba$, there are exactly 10 5-digit palindromes $abcba$ and so the ratio of the number of 4-digit palindromes to the number of 5-digit palindromes is $1 : 10$.

Solution 2

Every 4-digit palindrome is of the form $abba$, where a is a digit between 1 and 9 inclusive and b is a digit between 0 and 9 inclusive (and b is not necessarily different than a).
There are 9 choices for the first digit a and, for each of these choices, there are 10 choices for the second digit b or $9 \times 10 = 90$ choices for the first two digits ab.
Once the first two digits of the 4-digit palindrome are chosen, then the third and fourth digits are also determined (since the third digit must equal the second and the fourth must equal the first).
That is, there are 90 4-digit palindromes.
Every 5-digit palindrome is of the form $defed$, where d is a digit from 1 to 9 inclusive and e is a digit from 0 to 9 inclusive (and e is not necessarily different than d) and f is a digit from 0 to 9 inclusive (and f is not necessarily different than d and e).
There are 9 choices for the first digit d and 10 choices for the second digit e and 10 choices for the third digit f or $9 \times 10 \times 10 = 900$ choices for the first three digits def.
Once the first three digits of the 5-digit palindrome are chosen, then the fourth and fifth digits are also determined (since the fourth digit must equal the second and the fifth must equal the first).
That is, there are 900 5-digit palindromes.
Thus, the ratio of the number of 4-digit palindromes to the number of 5-digit palindromes is $90 : 900$ or $1 : 10$.

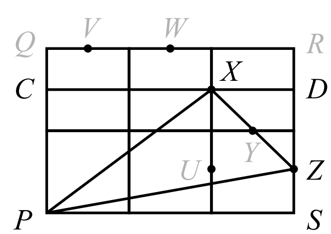
ANSWER: (E)

24. We can determine which triangle has the greatest area by using a fixed side length of 4 for each of the identical squares and using this to calculate the unknown areas.

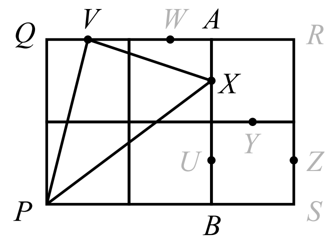


We begin by constructing $\triangle PVU$ and noticing that it is contained within square $QABP$, as shown. The area of $\triangle PVU$ is determined by subtracting the areas of triangles PQV, VAU and PBU from the area of square $QABP$.
Since $QA = 8$ and $AB = 8$, then the area of square $QABP$ is $8 \times 8 = 64$.
Since $PQ = 8$ and $QV = 2$, then the area of $\triangle PQV$ is $\frac{1}{2} \times 8 \times 2 = 8$.
Since $VA = 6$ and $AU = 6$, then the area of $\triangle VAU$ is $\frac{1}{2} \times 6 \times 6 = 18$.
Since $PB = 8$ and $UB = 2$, then the area of $\triangle PBU$ is $\frac{1}{2} \times 8 \times 2 = 8$.
Therefore, the area of $\triangle PVU$ is $64 - 8 - 18 - 8 = 30$.

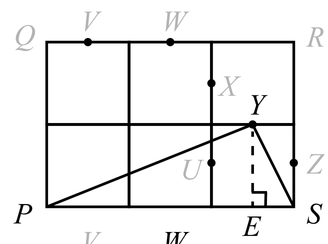
Next, we construct $\triangle PXZ$ and then construct rectangle $CDSP$ by drawing CD parallel to PS through X. Further, X is the midpoint of the side of a square and so C and D are also midpoints of the sides of their respective squares.



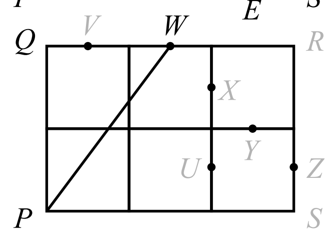
The area of $\triangle PXZ$ is determined by subtracting the areas of triangles PCX, XDZ and PSZ from the area of rectangle $CDSP$.
Since $CD = 12$ and $DS = 6$, then the area of rectangle $CDSP$ is $12 \times 6 = 72$.
Since $PC = 6$ and $CX = 8$, then the area of $\triangle PCX$ is $\frac{1}{2} \times 6 \times 8 = 24$.
Since $XD = 4$ and $DZ = 4$, then the area of $\triangle XDZ$ is $\frac{1}{2} \times 4 \times 4 = 8$.
Since $PS = 12$ and $ZS = 2$, then the area of $\triangle PSZ$ is $\frac{1}{2} \times 12 \times 2 = 12$.
Therefore, the area of $\triangle PXZ$ is $72 - 24 - 8 - 12 = 28$.



Construct $\triangle PVX$ and notice that it is contained within square $QABP$, as shown. The area of $\triangle PVX$ is determined by subtracting the areas of triangles PQV, VAX and PBX from the area of square $QABP$.
As we previously determined, the area of square $QABP$ is 64 and the area of $\triangle PQV$ is 8.
Since $VA = 6$ and $AX = 2$, then the area of $\triangle VAX$ is $\frac{1}{2} \times 6 \times 2 = 6$.
Since $PB = 8$ and $XB = 6$, then the area of $\triangle PBX$ is $\frac{1}{2} \times 8 \times 6 = 24$.
Therefore, the area of $\triangle PVX$ is $64 - 8 - 6 - 24 = 26$.



Construct $\triangle PYS$ and the perpendicular from Y to E on PS, as shown. Since $PS = 12$ and $YE = 4$ (YE is parallel to RS and thus equal in length to the side of the square), then the area of $\triangle PYS$ is $\frac{1}{2} \times 12 \times 4 = 24$.



Construct $\triangle PQW$, as shown. Since $PQ = 8$ and $QW = 6$, then the area of $\triangle PQW$ is $\frac{1}{2} \times 8 \times 6 = 24$.

The areas of the 5 triangles are $30, 28, 26, 24$, and 24. The triangle with greatest area, 30, is $\triangle PVU$.

Answer: (A)

25. All 2-digit prime numbers are odd numbers, so to create a reversal pair, both digits of each prime must be odd (so that both the original number and its reversal are odd numbers).
We also note that the digit 5 cannot appear in either prime number of the reversal pair since any 2-digit number ending in 5 is not prime.
Combining these two facts together leaves only the following list of prime numbers from which to search for reversal pairs: $11, 13, 17, 19, 31, 37, 71, 73, 79$, and 97.
This allows us to determine that the only reversal pairs are: 13 and 31, 17 and 71, 37 and 73, and 79 and 97.
(Note that the reversal of 11 does not produce a different prime number and the reversal of 19 is 91, which is not prime since $7 \times 13 = 91$.)

Given a reversal pair, we must determine the prime numbers (different than each prime of the reversal pair) whose product with the reversal pair is a positive integer less than 10 000.

The product of the reversal pair 79 and 97 is $79 \times 97 = 7663$.
Since the smallest prime number is 2 and $2 \times 7663 = 15\,326$, which is greater than 10 000, then the reversal pair 79 and 97 gives no possibilities that satisfy the given conditions.

We continue in this way, analyzing the other 3 reversal pairs, and summarize our results in the table below.

Prime Number	Product of the Prime Number with the Reversal Pair			
	13 and 31	17 and 71	37 and 73	79 and 97
2	$2 \times 13 \times 31 = 806$	$2 \times 17 \times 71 = 2414$	$2 \times 37 \times 73 = 5402$	greater than 10 000
3	$3 \times 13 \times 31 = 1209$	$3 \times 17 \times 71 = 3621$	$3 \times 37 \times 73 = 8103$	
5	$5 \times 13 \times 31 = 2015$	$5 \times 17 \times 71 = 6035$	greater than 10 000	
7	$7 \times 13 \times 31 = 2821$	$7 \times 17 \times 71 = 8449$		
11	$11 \times 13 \times 31 = 4433$	greater than 10 000		
13	can't use 13 twice			
17	$17 \times 13 \times 31 = 6851$			
19	$19 \times 13 \times 31 = 7657$			
23	$23 \times 13 \times 31 = 9269$			
29	greater than 10 000			
Total	8	4	2	0

In any column, once we obtain a product that is greater than 10 000, we may stop evaluating subsequent products since they use a larger prime number and thus will exceed the previous product.
In total, there are $8 + 4 + 2 = 14$ positive integers less than 10 000 which have the required property.

Answer: (B)

Grade 8

1. Evaluating, $1000 + 200 - 10 + 1 = 1200 - 10 + 1 = 1190 + 1 = 1191$.

   Answer: (A)

2. Since there are 60 minutes in an hour, then 40 minutes after 10:20 it is 11:00.
   Therefore, 45 minutes after 10:20 it is 11:05.

   Answer: (E)

3. Of the possible answers, the length of your thumb is closest to 5 cm.

   Answer: (E)

4. Reading from the graph, Phil ran 4 km, Tom ran 6 km, Pete ran 2 km, Amal ran 8 km, and Sanjay ran 7 km.
   Ordering these distances from least to greatest, we get Pete ran 2 km, Phil ran 4 km, Tom ran 6 km, Sanjay ran 7 km, and Amal ran 8 km.
   In this ordered list of 5 distances, the median distance is in the middle, the third greatest.
   Therefore, Tom ran the median distance.

   Answer: (B)

5. *Solution 1*
   Since $x + 3 = 10$, then $x = 10 - 3 = 7$.
   When $x = 7$, the value of $5x + 15$ is $5(7) + 15 = 35 + 15 = 50$.

   Solution 2
   When multiplying $x + 3$ by 5, we get $5 \times (x + 3) = 5 \times x + 5 \times 3 = 5x + 15$.
   Since $x + 3 = 10$, then $5 \times (x + 3) = 5 \times 10 = 50$.
   Therefore, $5 \times (x + 3) = 5x + 15 = 50$.

   Answer: (E)

6. The two equal widths, each of length 4, contribute $2 \times 4 = 8$ to the perimeter of the rectangle.
   The two lengths contribute the remaining $42 - 8 = 34$ to the perimeter.
   Since the two lengths are equal, they each contribute $34 \div 2 = 17$ to the perimeter.
   Therefore, the length of the rectangle is 17.

   Answer: (B)

7. To begin, there are 4 circles and 2 rectangles on the left arm, balanced by 10 circles on the right arm.
   If we remove 4 circles from each side of the equal-arm scale, the scale will remain balanced (since we are removing the same mass from each side).
   That is, the 2 rectangles that will remain on the left arm are equal in mass to the 6 circles that will remain on the right arm.
   Since 2 rectangles are equal in mass to 6 circles, then 1 rectangle has the same mass as 3 circles.

   Answer: (B)

8. *Solution 1*
   A 5% increase in 160 is equal to $\frac{5}{100} \times 160$ or $0.05 \times 160 = 8$.
   Therefore, Aidan's height increased by 8 cm over the summer.
   His height at the end of the summer was $160 + 8 = 168$ cm.

   Solution 2
   Since Aidan's 160 cm height increased by 5%, then his height at the end of the summer was $(1 + \frac{5}{100}) \times 160 = (1 + 0.05) \times 160 = 1.05 \times 160 = 168$ cm.

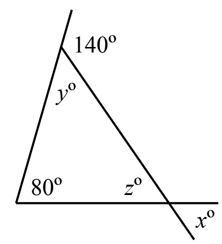
   Answer: (A)

9. When $x = 4$ and $y = 2$, $x + y = 4 + 2 = 6$, $xy = 4 \times 2 = 8$, $x - y = 4 - 2 = 2$, $x \div y = 4 \div 2 = 2$, and $y \div x = 2 \div 4 = \frac{1}{2}$.
Therefore, the expression which gives the smallest value when $x = 4$ and $y = 2$ is $y \div x$.
ANSWER: (E)

10. *Solution 1*
Evaluating using a denominator of 12, $\frac{1}{2} + \frac{1}{4} = \frac{6}{12} + \frac{3}{12} = \frac{9}{12}$ and so the number represented by $\square$ is 9.

Solution 2
Since $\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$ and $\frac{3}{4} = \frac{9}{12}$, then $\frac{1}{2} + \frac{2}{4} = \frac{9}{12}$.
The number represented by $\square$ is 9.
ANSWER: (C)

11. Straight angles measure 180°.
Therefore, $y^\circ + 140^\circ = 180^\circ$, and so $y = 180 - 140 = 40$.
The three interior angles of any triangle add to 180°.
Thus, $40^\circ + 80^\circ + z^\circ = 180^\circ$, and so $z = 180 - 40 - 80 = 60$.
Opposite angles have equal measures.
Since the angle measuring z° is opposite the angle measuring x°, then $x = z = 60$.



ANSWER: (C)

12. Since Zara's bicycle tire has a circumference of 1.5 m, then each full rotation of the tire moves the bike 1.5 m forward.
If Zara travels 900 m on her bike, then her tire will make $900 \div 1.5 = 600$ full rotations.
ANSWER: (C)

13. To find the image of PQ, we reflect points P and Q across the x-axis, then join them.
Since P is 3 units above the x-axis, then the reflection of P across the x-axis is 3 units below the x-axis at the same x-coordinate.
That is, point T is the image of P after it is reflected across the x-axis.
Similarly, after a reflection across the x-axis, the image of point Q will be 6 units below the x-axis but have the same x-coordinate as Q.
That is, point U is the image of Q after it is reflected across the x-axis.
Therefore, the line segment TU is the image of PQ after it is reflected across the x-axis.
ANSWER: (B)

14. In the table below, we determine the total value of the three bills that remain in Carolyn's wallet when each of the four bills is removed.

Bill Removed	Sum of the Bills Remaining
\$5	\$10+\$20+\$50=\$80
\$10	\$5+\$20+\$50=\$75
\$20	\$5+\$10+\$50=\$65
\$50	\$5+\$10+\$20=\$35

It is equally likely that any one of the four bills is removed from the wallet and therefore any of the four sums of the bills remaining in the wallet is equally likely.
Of the four possible sums, \$80, \$75, \$65, and \$35, two are greater than \$70.
Therefore, the probability that the total value of the three bills left in Carolyn's wallet is greater than \$70, is $\frac{2}{4}$ or 0.5.
ANSWER: (A)

15. In the table below, we list the mass of each dog at the end of each month.

Month	0	1	2	3	4	5	6	7	8	9	10	11	12
Walter's mass (in kg)	12	14	16	18	20	22	24	26	28	30	32	34	36
Stanley's mass (in kg)	6	8.5	11	13.5	16	18.5	21	23.5	26	28.5	31	33.5	36

After 12 months have passed, Stanley's mass is 36 kg and is equal to Walter's mass.
(Note that since Stanley's mass is increasing at a greater rate than Walter's each month, this is the only time that the two dogs will have the same mass.)

ANSWER: (D)

16. First, we must determine the perimeter of the given triangle.
Let the unknown side length measure x cm.
Since the triangle is a right-angled triangle, then by the Pythagorean Theorem we get $x^2 = 8^2 + 6^2$ or $x^2 = 64 + 36 = 100$ and so $x = \sqrt{100} = 10$ (since $x > 0$).
Therefore the perimeter of the triangle is $10+8+6 = 24$ cm and so the perimeter of the square is also 24 cm.
Since the 4 sides of the square are equal in length, then each measures $\frac{24}{4} = 6$ cm.
Thus, the area of the square is $6 \times 6 = 36$ cm^2.

ANSWER: (D)

17. Since the number of digits that repeat is 6, then the digits 142857 begin to repeat again after 120 digits (since $120 = 6 \times 20$).
That is, the 121^{st} digit is a 1, the 122^{nd} digit is a 4, and the 123^{rd} digit is a 2.

ANSWER: (C)

18. Using the definition of Δ, we see that $p\Delta 3 = p \times 3 + p + 3 = 3p + p + 3 = 4p + 3$.
Since $p\Delta 3 = 39$, then $4p + 3 = 39$ or $4p = 39 - 3 = 36$ and so $p = \frac{36}{4} = 9$.

ANSWER: (C)

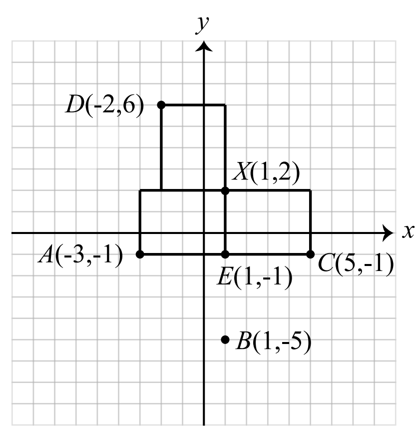
19. *Solution 1*
Originally there are 3 times as many boys as girls, so then for every 3 boys there is 1 girl and $3 + 1 = 4$ children in the room.
That is, the number of boys in the room is $\frac{3}{4}$ of the number of children in the room.
Next we consider each of the 5 possible answers, in turn, to determine which represents the total number of children in the room originally.
If the original number of children in the room is 15 (as in answer (A)), the number of boys is $\frac{3}{4} \times 15 = \frac{45}{4} = 11.25$.
Since it is not possible to have 11.25 boys in the room, then we know that 15 is not the correct answer.
If the original number of children in the room is 20 (as in answer (B)), the number of boys is $\frac{3}{4} \times 20 = \frac{60}{4} = 15$.
If the number of boys in the room was originally 15, then the number of girls was $20 - 15 = 5$.
Next we must check if there will be 5 times as many boys as girls in the room once 4 boys and 4 girls leave the room.
If 4 boys leave the room, there are 11 boys remaining. If 4 girls leave the room, there is 1 girl remaining and since there are not 5 times as many boys as girls, then 20 is not the correct answer.
If the original number of children in the room is 24 (as in answer (C)), the number of boys is $\frac{3}{4} \times 24 = \frac{72}{4} = 18$.
If the number of boys in the room was originally 18, then the number of girls was $24 - 18 = 6$.

If 4 boys leave the room, there are 14 boys left and if 4 girls leave the room, then there are 2 girls left.
Since there are not 5 times as many boys as girls, then 24 is not the correct answer.
If the original number of children in the room is 32 (as in answer (D)), the number of boys is $\frac{3}{4} \times 32 = \frac{96}{4} = 24$.
If the number of boys in the room was originally 24, then the number of girls was $32 - 24 = 8$.
If 4 boys leave the room, there are 20 left and if 4 girls leave the room, then there are 4 left.
Since there are 5 times as many boys as girls, then we know that the original number of children is 32.
(Note: We may check that the final answer, 40, gives 30 boys and 10 girls originally and when 4 boys and 4 girls leave the room there are 26 boys and 6 girls which again does not represent 5 times as many boys as girls.)

Solution 2
Originally there are 3 times as many boys as girls, so if there are x girls in the room, then there are $3x$ boys.
If 4 boys leave the room, there are $3x - 4$ boys remaining.
If 4 girls leave the room, there are $x - 4$ girls remaining.
At this point, there are 5 times as many boys as girls in the room.
That is, $5 \times (x - 4) = 3x - 4$ or $5x - 20 = 3x - 4$ and so $5x - 3x = 20 - 4$ or $2x = 16$ and so $x = 8$.
Therefore, the original number of girls in the room is 8 and the original number of boys is $3 \times 8 = 24$.
The original number of students in the room is $24 + 8 = 32$.

ANSWER: (D)

20. *Solution 1*
Call the given vertex of the rectangle $(1, 2)$ point X and name each of the 5 answers to match the letters A through E, as shown.



Point $E(1, -1)$ is 3 units below point X (since their x-coordinates are equal and their y-coordinates differ by 3). Thus, $E(1, -1)$ could be the coordinates of a vertex of a 3 by 4 rectangle having vertex X (X and E would be adjacent vertices of the rectangle).

Point $C(5, -1)$ is 3 units below and 4 units right of point X (since their y-coordinates differ by 3 and their x-coordinates differ by 4). Thus, $C(5, -1)$ could be the coordinates of a vertex of a 3 by 4 rectangle having vertex X (X and C would be opposite vertices of the rectangle).

Point $A(-3, -1)$ is 3 units below and 4 units left of point X (since their y-coordinates differ by 3 and their x-coordinates differ by 4). Thus, $A(-3, -1)$ could be the coordinates of a vertex of a 3 by 4 rectangle having vertex X (X and A would be opposite vertices of the rectangle).

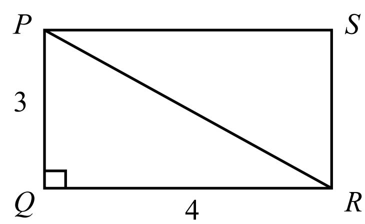
Point $D(-2, 6)$ is 4 units above and 3 units left of point X (since their y-coordinates differ by 4 and their x-coordinates differ by 3). Thus, $D(-2, 6)$ could be the coordinates of a vertex of a 3 by 4 rectangle having vertex X (X and D would be opposite vertices of the rectangle).

The only point remaining is $B(1, -5)$ and since it is possible for each of the other 4 answers to

be one of the other vertices of the rectangle, then it must be $(1, -5)$ that can not be.
Point $B(1, -5)$ is 7 units below point X (since their y-coordinates differ by 7).
How might we show that no two vertices of a 3 by 4 rectangle are 7 units apart?
(See Solution 2).

Solution 2

The distance between any two adjacent vertices of a 3 by 4 rectangle $PQRS$ is either 3 or 4 units (such as P and Q or Q and R, as shown).



The distance between any two opposite vertices of a rectangle (such as P and R) can be found using the Pythagorean Theorem.
In the right-angled triangle PQR, we get $PR^2 = 3^2 + 4^4$ or $PR^2 = 9 + 16 = 25$ and so $PR = \sqrt{25} = 5$ (since $PR > 0$).
That is, the greatest distance between any two vertices of a 3 by 4 rectangle is 5 units.
As shown and explained in Solution 1, the distance between $X(1, 2)$ and $B(1, -5)$ is 7 units.
Therefore $(1, -5)$ could not be the coordinates of one of the other vertices of a 3 by 4 rectangle having vertex $X(1, 2)$.

Answer: (B)

21. In square $PQRS$, $PS = SR$ and since M and N are midpoints of these sides having equal length, then $MS = SN$.
The area of $\triangle SMN$ is $\frac{1}{2} \times MS \times SN$.
Since this area equals 18, then $\frac{1}{2} \times MS \times SN = 18$ or $MS \times SN = 36$ and so $MS = SN = 6$ (since they are equal in length).
The side of the square, PS, is equal in length to $PM + MS = 6 + 6 = 12$ (since M is the midpoint of PS) and so $PS = SR = RQ = QP = 12$.
The area of $\triangle QMN$ is equal to the area of square $PQRS$ minus the combined areas of the three right-angled triangles, $\triangle SMN$, $\triangle NRQ$ and $\triangle QPM$.
Square $PQRS$ has area $PS \times SR = 12 \times 12 = 144$.
$\triangle SMN$ has area 18, as was given in the question.
$\triangle NRQ$ has area $\frac{1}{2} \times QR \times RN = \frac{1}{2} \times 12 \times 6 = 36$ (since $SN = RN = 6$).
$\triangle QPM$ has area $\frac{1}{2} \times QP \times PM = \frac{1}{2} \times 12 \times 6 = 36$.
Thus the area of $\triangle QMN$ is $144 - 18 - 36 - 36 = 54$.

Answer: (E)

22. Let the number of adult tickets sold be a.
Since the price for each adult ticket is \$12, then the revenue from all adult tickets sold (in dollars) is $12 \times a$ or $12a$.
Since the number of child tickets sold is equal to the number of adult tickets sold, we can let the number of child tickets sold be a, and the total revenue from all \$6 child tickets be $6a$ (in dollars).
In dollars, the combined revenue of all adult tickets and all child tickets is $12a + 6a = 18a$.

Since the total number of tickets sold is 120, and a adult tickets were sold and a child tickets were sold, then the remaining $120 - 2a$ tickets were sold to seniors.
Since the price for each senior ticket is \$10, then the revenue from all senior tickets sold (in dollars) is $10 \times (120 - 2a)$.
Thus the combined revenue from all ticket sales is $10 \times (120 - 2a) + 18a$, dollars.

The total revenue from the ticket sales was \$1100 and so $10 \times (120 - 2a) + 18a = 1100$.
Solving this equation, we get $10 \times 120 - 10 \times 2a + 18a = 1100$ or $1200 - 20a + 18a = 1100$ or $1200 - 2a = 1100$ and so $2a = 100$ or $a = 50$.
Therefore, the number of senior tickets sold for the concert was
$120 - 2a = 120 - 2(50) = 120 - 100 = 20$.

We may check that the number of tickets sold to each of the three groups gives the correct total revenue.
Since the number of adult tickets sold was equal to the number of child tickets sold which was equal to a, then 50 of each were sold.
The revenue from 50 adult tickets is $50 \times \$12 = \$600$.
The revenue from 50 child tickets is $50 \times \$6 = \$300$.
The revenue from 20 senior tickets is $20 \times \$10 = \$200$.
The total revenue from all tickets sold was $\$600 + \$300 + \$200 = \$1100$, as required.

ANSWER: (B)

23. The list of integers $4, 4, x, y, 13$ has been arranged from least to greatest, and so $4 \leq x$ and $x \leq y$ and $y \leq 13$.
The sum of the 5 integers is $4 + 4 + x + y + 13 = 21 + x + y$ and so the average is $\dfrac{21 + x + y}{5}$.
Since this average is a whole number, then $21 + x + y$ must be divisible by 5 (that is, $21 + x + y$ is a multiple of 5).
How small and how large can the sum $21 + x + y$ be?
We know that $4 \leq x$ and $x \leq y$, so the smallest that $x + y$ can be is $4 + 4 = 8$.
Since $x + y$ is at least 8, then $21 + x + y$ is at least $21 + 8 = 29$.
Using the fact that $x \leq y$ and $y \leq 13$, the largest that $x + y$ can be is $13 + 13 = 26$.
Since $x + y$ is at most 26, then $21 + x + y$ is at most $21 + 26 = 47$.
The multiples of 5 between 29 and 47 are $30, 35, 40$, and 45.
When $21 + x + y = 30$, we get $x + y = 30 - 21 = 9$.
The only ordered pair (x, y) such that $4 \leq x$ and $x \leq y$ and $y \leq 13$, and $x + y = 9$ is $(x, y) = (4, 5)$.
Continuing in this way, we determine all possible values of x and y that satisfy the given conditions in the table below.

Value of $21 + x + y$	Value of $x + y$	Ordered Pairs (x, y) with $4 \leq x$ and $x \leq y$ and $y \leq 13$
30	$30 - 21 = 9$	$(4, 5)$
35	$35 - 21 = 14$	$(4, 10), (5, 9), (6, 8), (7, 7)$
40	$40 - 21 = 19$	$(6, 13), (7, 12), (8, 11), (9, 10)$
45	$45 - 21 = 24$	$(11, 13), (12, 12)$

The number of ordered pairs (x, y) such that the average of the 5 integers $4, 4, x, y, 13$ is itself an integer is 11.

ANSWER: (E)

24. The two joggers meet every 36 seconds.
Therefore, the combined distance that the two joggers run every 36 seconds is equal to the total distance around one lap of the oval track, which is constant.
Thus the greater the first jogger's constant speed, the greater the distance that they run every 36 seconds, meaning the second jogger runs less distance in the same time (their combined

distance is constant) and hence the smaller the second jogger's constant speed.
Conversely, the slower the first jogger's constant speed, the less distance that they run every 36 seconds, meaning the second jogger must run a greater distance in this same time and hence the greater the second jogger's constant speed.

This tells us that if the first jogger completes one lap of the track as fast as possible, which is in 80 seconds, then the second jogger's time to complete one lap of the track is as slow as possible.
We will call this time t_{max}, the maximum possible time that it takes the second jogger to complete one lap of the track.
Similarly, if the first jogger completes one lap of the track as slowly as possible, which is in 100 seconds, then the second jogger's time to complete one lap of the track is as fast as possible.
We will call this time t_{min}, the minimum possible time that it takes the second jogger to complete one lap of the track.

Finding the value of t_{max}

Recall that t_{max} is the time it takes the second jogger to complete one lap when the first jogger completes one lap of the track in 80 seconds.
If the first jogger can complete one lap of the track in 80 seconds, then in 36 seconds of running, the first jogger will complete $\frac{36}{80} = \frac{9}{20}$ of a complete lap of the track.
In this same 36 seconds, the two joggers combined distance running is 1 lap, and so the second jogger runs $1 - \frac{9}{20} = \frac{11}{20}$ of a complete lap.
If the second jogger runs $\frac{11}{20}$ of a complete lap in 36 seconds, then the second jogger runs $\frac{20}{11} \times \frac{11}{20} = 1$ complete lap in $\frac{20}{11} \times 36 = \frac{720}{11}$ seconds. Thus, $t_{max} = \frac{720}{11} = 65.\overline{45}$ seconds.

Finding the value of t_{min}

Recall that t_{min} is the time it takes the second jogger to complete one lap when the first jogger completes one lap of the track in 100 seconds.
If the first jogger can complete one lap of the track in 100 seconds, then in 36 seconds of running, the first jogger will complete $\frac{36}{100} = \frac{9}{25}$ of a complete lap of the track.
In this same 36 seconds, the two joggers combined distance running is 1 lap, and so the second jogger runs $1 - \frac{9}{25} = \frac{16}{25}$ of a complete lap.
If the second jogger runs $\frac{16}{25}$ of a complete lap in 36 seconds, then the second jogger runs $\frac{25}{16} \times \frac{16}{25} = 1$ complete lap in $\frac{25}{16} \times 36 = \frac{900}{16}$ seconds. Thus, $t_{min} = \frac{900}{16} = 56.25$ seconds.

Determining the product of the smallest and largest integer values of t

Since the second jogger completes 1 lap of the track in at most $65.\overline{45}$ seconds, then the largest possible integer value of t is 65 seconds.
The second jogger completes 1 lap of the track in at least 56.25 seconds, so then the smallest possible integer value of t is 57 seconds.
Finally, the product of the smallest and largest integer values of t is $57 \times 65 = 3705$.

ANSWER: (A)

25. Let the alternating sum of the digits be S.
If the 7-digit integer is $abcdefg$, then $S = a - b + c - d + e - f + g$.
This sum can be grouped into the digits which contribute positively to the sum, and those which contribute negatively to the sum.
Rewriting the sum in this way, we get $S = (a + c + e + g) - (b + d + f)$.
Taking the 4 digits which contribute positively to S (there are always 4), we let $P = a+c+e+g$.
Similarly, taking the 3 digits which contribute negatively to S (there are always 3), we let $N = b + d + f$.
Thus, it follows that $S = (a + c + e + g) - (b + d + f) = P - N$.

We determine the largest possible value of S by choosing the 4 largest integers, $4, 5, 6, 7$ (in any order), to make up P, and choosing the 3 smallest integers, $1, 2, 3$ (in any order), to make up N.
That is, the largest possible alternating sum is $S = (4 + 5 + 6 + 7) - (1 + 2 + 3) = 16$.
We determine the smallest possible value of S by choosing the 4 smallest integers, $1, 2, 3, 4$ (in any order), to make up P, and choosing the 3 largest integers, $5, 6, 7$ (in any order), to make up N.
That is, the smallest possible alternating sum is $S = (1 + 2 + 3 + 4) - (5 + 6 + 7) = -8$.
Since S must be divisible by 11 (with $S \geq -8$ and $S \leq 16$), then either $S = 11$ or $S = 0$.
The sum of the first 7 positive integers is $1+2+3+4+5+6+7 = 28$, and since each of these 7 integers must contribute to either P or to N, then $P + N = 28$.

Case 1: The alternating sum of the digits is 11, or $S = 11$

If $S = 11$, then $S = P - N = 11$. If $P - N$ is 11 (an odd number), then either P is an even number and N is odd, or the opposite is true (they can't both be odd and they can't both be even).
That is, the difference between two integers is odd only if one of the integers is even and the other is odd (we say that P and N have *different parity*).
However, if one of P or N is even and the other is odd, then their sum $P + N$ is also odd.
But we know that $P + N = 28$, an even number.
Therefore, it is not possible that $S = 11$.
There are no 7-digit integers formed from the integers 1 through 7 that have an alternating digit sum of 11 and are divisible by 11.

Case 2: The alternating sum of the digits is 0, or $S = 0$

If $S = 0$, then $S = P - N = 0$ and so $P = N$.
Since $P + N = 28$, then $P = N = 14$.
We find all groups of 3 digits, chosen from the digits 1 to 7, such that their sum $N = 14$.
There are exactly 4 possibilities: $(7, 6, 1), (7, 5, 2), (7, 4, 3)$, and $(6, 5, 3)$.
In each of these 4 cases, the digits from 1 to 7 that were not chosen, $(2, 3, 4, 5), (1, 3, 4, 6), (1, 2, 5, 6)$, and $(1, 2, 4, 7)$, respectively, represent the 4 digits whose sum is $P = 14$.
We summarize this in the table below.

4 digits whose sum is $P = 14$	3 digits whose sum is $N = 14$	2 examples of 7-digit integers created from these
$2, 3, 4, 5$	$7, 6, 1$	$2736415, 3126475$
$1, 3, 4, 6$	$7, 5, 2$	$1735426, 6745321$
$1, 2, 5, 6$	$7, 4, 3$	$2714536, 5763241$
$1, 2, 4, 7$	$6, 5, 3$	$4615237, 7645231$

Consider the first row of numbers in this table above.
Each arrangement of the 4 digits $2, 3, 4, 5$ combined with each arrangement of the 3 digits $7, 6, 1$ (in the required way) gives a new 7-digit integer whose alternating digit sum is 0.
Two such arrangements are shown (you may check that $S = 0$ for each).
Since there are $4 \times 3 \times 2 \times 1 = 24$ ways to arrange the 4 digits (4 choices for the first digit, 3 choices for the second, 2 choices for the third and 1 choice for the last digit), and $3 \times 2 \times 1 = 6$ ways to arrange the 3 digits, then there are $24 \times 6 = 144$ ways to arrange the 4 digits and the 3 digits.
Each of these 144 arrangements is different from the others, and since $P = N = 14$ for each, then $S = P - N = 0$ and so each of the 144 7-digit numbers is divisible by 11.
Similarly, there are also 144 arrangements that can be formed with each of the other 3 groups of integers that are shown in the final 3 rows of the table.
That is, there are a total of $144 \times 4 = 576$ 7-digit integers (formed from the integers 1 through 7) which are divisible by 11.

The total number of 7-digit integers that can be formed from the integers 1 through 7 is equal to the total number of arrangements of the integers 1 through 7, or $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$.
Therefore, when the digits 1 through 7 are each used to form a random 7-digit integer, the probability that the number formed is divisible by 11 is $\dfrac{576}{5040} = \dfrac{4}{35}$.

Answer: (E)

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2014 Gauss Contests

(Grades 7 and 8)

Wednesday, May 14, 2014
(in North America and South America)

Thursday, May 15, 2014
(outside of North America and South America)

Solutions

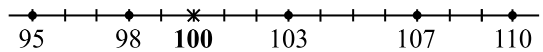
Grade 7

1. Evaluating, $(4 \times 3) + 2 = 12 + 2 = 14$.

ANSWER: (C)

2. *Solution 1*

We place each of the five answers and 100 on a number line.
Of the five answers given, the two closest numbers to 100 are 98 and 103.
Since 98 is 2 units away from 100 and 103 is 3 units away from 100, then 98 is closest to 100.



Solution 2

We calculate the positive difference between 100 and each of the five possible answers.
The number closest to 100 on the number line will produce the smallest of these positive differences.

Possible Answers	98	95	103	107	110
Positive Differences	$100 - 98 = 2$	$100 - 95 = 5$	$103 - 100 = 3$	$107 - 100 = 7$	$110 - 100 = 10$

Since the smallest positive difference is 2, then 98 is the closest to 100 on the number line.

ANSWER: (A)

3. Since five times the number equals one hundred, then the number equals one hundred divided by five. Therefore the number is $100 \div 5 = 20$.

ANSWER: (E)

4. The spinner has 6 sections in total and 2 of these sections contain the letter Q.
Sections are equal to one another in size and thus they are each equally likely to be landed on.
Therefore, the probability of landing on a section that contains the letter Q is $\frac{2}{6}$.

ANSWER: (D)

5. Each scoop of fish food can feed 8 goldfish.
Therefore, 4 scoops of fish food can feed $4 \times 8 = 32$ goldfish.

ANSWER: (E)

6. Both the numerator and the denominator are divisible by 5.
Dividing, we get $\frac{15}{25} = \frac{15 \div 5}{25 \div 5} = \frac{3}{5}$. Therefore, $\frac{15}{25}$ is equivalent to $\frac{3}{5}$.

ANSWER: (C)

7. The largest two-digit number that is a multiple of 7 is $7 \times 14 = 98$.
Thus, there are 14 positive multiples of 7 that are less than 100.
However, this includes $7 \times 1 = 7$ which is not a two-digit number.
Therefore, there are $14 - 1 = 13$ positive two-digit numbers which are divisible by 7.
(Note that these 13 numbers are $14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91$, and 98.)

ANSWER: (E)

8. *Solution 1*

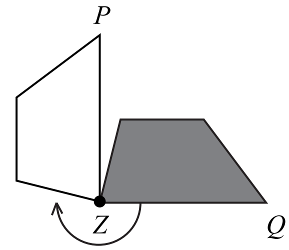
Evaluating the left side of the equation, we get $9210 - 9124 = 86$.
Therefore the right side of the equation, $210 - \square$, must also equal 86.
Since $210 - 124 = 86$, then the value represented by the $\square$ is 124.

Solution 2

Since $9210 - 9124 = (9000 + 210) - (9000 + 124) = 9000 - 9000 + 210 - 124 = 210 - 124$, then the value represented by the $\square$ is 124.

ANSWER: (D)

9. The measure of $\angle PZQ$ formed by the bottom edge of the shaded quadrilateral, ZQ, and this same edge after the rotation, ZP, is approximately 90°.
The transformation of the shaded quadrilateral to the unshaded quadrilateral is a clockwise rotation around point Z through an angle equal to the measure of reflex angle PZQ.
The measure of $\angle PZQ$ added to the measure of reflex $\angle PZQ$ is equal to the measure of one complete rotation, or 360°.
Therefore, the measure of reflex angle PZQ is approximately $360^\circ - 90^\circ$ or 270°.
Thus the clockwise rotation around point Z is through an angle of approximately 270°.



ANSWER: (B)

10. In the table below, each of the five expressions is evaluated using the correct order of operations.

Expression	Value
(A) $3 - 4 \times 5 + 6$	$3 - 20 + 6 = -17 + 6 = -11$
(B) $3 \times 4 + 5 \div 6$	$12 + 5 \div 6 = 12 + \frac{5}{6} = 12\frac{5}{6}$
(C) $3 + 4 \times 5 - 6$	$3 + 20 - 6 = 23 - 6 = 17$
(D) $3 \div 4 + 5 - 6$	$\frac{3}{4} + 5 - 6 = 5\frac{3}{4} - 6 = \frac{23}{4} - \frac{24}{4} = -\frac{1}{4}$
(E) $3 \times 4 \div 5 + 6$	$12 \div 5 + 6 = \frac{12}{5} + 6 = 2\frac{2}{5} + 6 = 8\frac{2}{5}$

The only expression that is equal to 17 is $3 + 4 \times 5 - 6$, or (C).

ANSWER: (C)

11. Since each of the numbers in the set is between 0 and 1, then the tenths digit of each number contributes more to its value than any of its other digits.
The largest tenths digit of the given numbers is 4, and so 0.43 is the largest number in the set.
The smallest tenths digit of the given numbers is 0, so 0.034 is the smallest number in the set.
Therefore, the sum of the smallest number in the set and the largest number in the set is $0.034 + 0.43 = 0.464$.

ANSWER: (D)

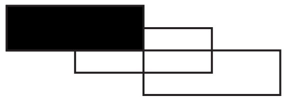
12. The two diagonals of a square bisect one another (divide each other into two equal lengths) at the centre of the square.
Therefore, the two diagonals divide the square into four identical triangles.
One of these four triangles is the shaded region which has area equal to one quarter of the area of the square.
Since the area of the square is $8 \times 8 = 64$ cm^2, the area of the shaded region is $64 \div 4 = 16$ cm^2.

ANSWER: (C)

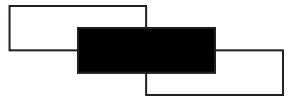
13. The sum of the three numbers in the first column is $13 + 14 + 9 = 36$.
The sum of the numbers in each column and in each row in the square is the same and so the sum of the three numbers in the second row is also 36.
That is, $14 + x + 10 = 36$ or $x + 24 = 36$, and so $x = 36 - 24 = 12$.

ANSWER: (E)

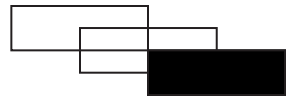
14. We systematically count rectangles by searching for groups of rectangles that are of similar size. The largest rectangles in the diagram are all roughly the same size and overlap in pairs. There are 3 of these; each is shaded black and shown below.



Rectangle 1

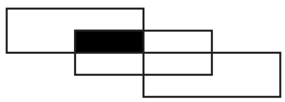


Rectangle 2

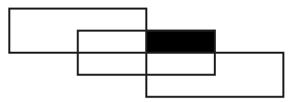


Rectangle 3

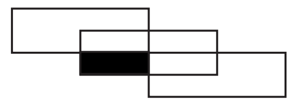
Rectangle 2 (shown above) consists of 4 small rectangles.
We shade these rectangles black and label them $4, 5, 6, 7$, as shown below.



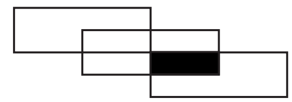
Rectangle 4



Rectangle 5

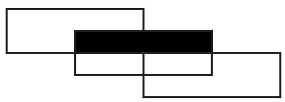


Rectangle 6

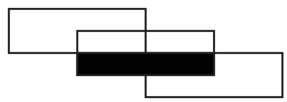


Rectangle 7

Together, Rectangle 4 and Rectangle 5 (shown above) create Rectangle 8, shown below.
Similarly, Rectangle 6 and Rectangle 7 together create Rectangle 9, shown below.

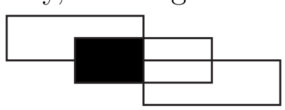


Rectangle 8

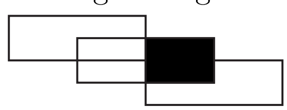


Rectangle 9

Finally, Rectangle 4 and Rectangle 6 (shown above) together create Rectangle 10, shown below.
Similarly, Rectangle 5 and Rectangle 7 together create Rectangle 11, shown below.



Rectangle 10

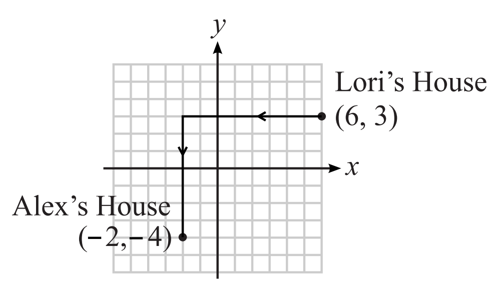


Rectangle 11

There are no other rectangles that can be formed.
In total, there are 11 rectangles in the given diagram.

Answer: (A)

15. The horizontal translation needed to get from Lori's house to Alex's house is the difference between the x-coordinate of Lori's house, 6, and the x-coordinate of Alex's house, -2, or $6 - (-2) = 6 + 2 = 8$.
The vertical translation needed to get from Lori's house to Alex's house is the difference between the y-coordinate of Lori's house, 3, and the y-coordinate of Alex's house, -4, or $3 - (-4) = 3 + 4 = 7$.
From Lori's house, Alex's house is left and down.
Therefore the translation needed to get from Lori's house to Alex's house is 8 units left and 7 units down.



Answer: (D)

16. Reading from the graph, Riley-Ann scored 8 points in Game 1, 7 points in Game 2, 20 points in Game 3, 7 points in Game 4, and 18 points in Game 5.
Therefore the mean number of points that she scored per game is $\frac{8+7+20+7+18}{5} = \frac{60}{5} = 12$.
Since the ordered list (smallest to largest) of the number of points scored per game is $7, 7, 8, 18, 20$, then the median is 8, the number in the middle of this ordered list.
The difference between the mean and the median of the number of points that Riley-Ann scored is $12 - 8 = 4$.

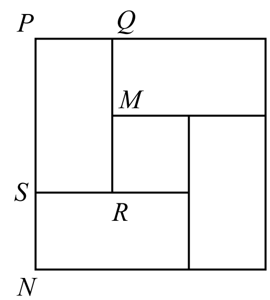
Answer: (D)

17. *Solution 1*
Since PQR is a straight line segment, then $\angle PQR = 180°$.
Since $\angle SQP + \angle SQR = 180°$, then $\angle SQR = 180° - \angle SQP = 180° - 75° = 105°$.
The three angles in a triangle add to $180°$, so $\angle QSR + \angle SQR + \angle QRS = 180°$, or $\angle QSR = 180° - \angle SQR - \angle QRS = 180° - 105° - 30° = 45°$.

Solution 2
The exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles of the triangle.
Since $\angle SQP$ is an exterior angle of $\triangle SQR$, and the two opposite interior angles are $\angle QSR$ and $\angle QRS$, then $\angle SQP = \angle QSR + \angle QRS$.
Thus, $75° = \angle QSR + 30°$ or $\angle QSR = 75° - 30° = 45°$.

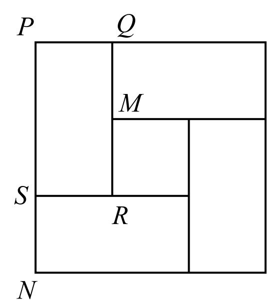
ANSWER: (E)

18. *Solution 1*



The outer square has an area of 9 cm^2, so the sides of this outer square have length 3 cm (since $3 \times 3 = 9$), and thus $PN = 3$ cm.
The inner square has an area of 1 cm^2, so the sides of this inner square have length 1 cm (since $1 \times 1 = 1$), and thus $MR = 1$ cm.
Since $PN = 3$ cm, then $PS + SN = 3$ cm and so $QR + SN = 3$ cm (since $QR = PS$).
But $QR = QM + MR$, so then $QM + MR + SN = 3$ cm or $QM + 1 + SN = 3$ cm (since $MR = 1$ cm).
From this last equation we get $QM + SN = 2$ cm.
Since each of QM and SN is the width of an identical rectangle, then $QM = SN = 1$ cm.
Using $PS + SN = 3$ cm, we get $PS + 1 = 3$ cm and so $PS = 2$ cm.
Since the rectangles are identical, then $SN = PQ = 1$ cm.
The perimeter of rectangle $PQRS$ is $2 \times (PS + PQ) = 2 \times (2 + 1) = 2 \times 3 = 6$ cm.

Solution 2



The outer square has an area of 9 cm^2, so the sides of this outer square have length 3 cm (since $3 \times 3 = 9$), and thus $PN = 3$ cm.
Since $PN = 3$ cm, then $PS + SN = 3$ cm.
Since each of PQ and SN is the width of an identical rectangle, then $PQ = SN$ and so $PS + SN = PS + PQ = 3$ cm.
The perimeter of $PQRS$ is $2 \times (PS + PQ) = 2 \times 3 = 6$ cm.

ANSWER: (A)

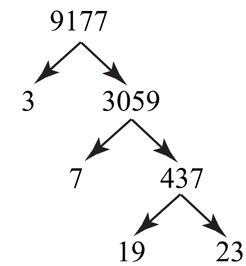
19. The width of Sarah's rectangular floor is 18 hand lengths.
Since each hand length is 20 cm, then the width of the floor is $18 \times 20 = 360$ cm.
The length of Sarah's rectangular floor is 22 hand lengths.
Since each hand length is 20 cm, then the length of the floor is $22 \times 20 = 440$ cm.
Thus the area of the floor is $360 \times 440 = 158\,400$ cm^2.
Of the given answers, the closest to the area of the floor is $160\,000$ cm^2.

ANSWER: (A)

20. *Solution 1*
Since $20 \times 20 \times 20 = 8000$ and $30 \times 30 \times 30 = 27\,000$, then we might guess that the three consecutive odd numbers whose product is 9177 are closer to 20 than they are to 30.
Using trial and error, we determine that $21 \times 23 \times 25 = 12\,075$, which is too large.

The next smallest set of three consecutive odd numbers is $19, 21, 23$ and the product of these three numbers is $19 \times 21 \times 23 = 9177$, as required.
Thus, the sum of the three consecutive odd numbers whose product is 9177 is $19+21+23 = 63$.

Solution 2
We begin by determining the prime numbers whose product is 9177. (This is called the *prime factorization* of 9177.)
This prime factorization of 9177 is shown in the factor tree to the right. That is, $9177 = 3 \times 3059 = 3 \times 7 \times 437 = 3 \times 7 \times 19 \times 23$.
Since $3 \times 7 = 21$, then $9177 = 21 \times 19 \times 23$ and so the three consecutive numbers whose product is 9177 are $19, 21, 23$.
Thus, the sum of the three consecutive odd numbers whose product is 9177 is $19 + 21 + 23 = 63$.

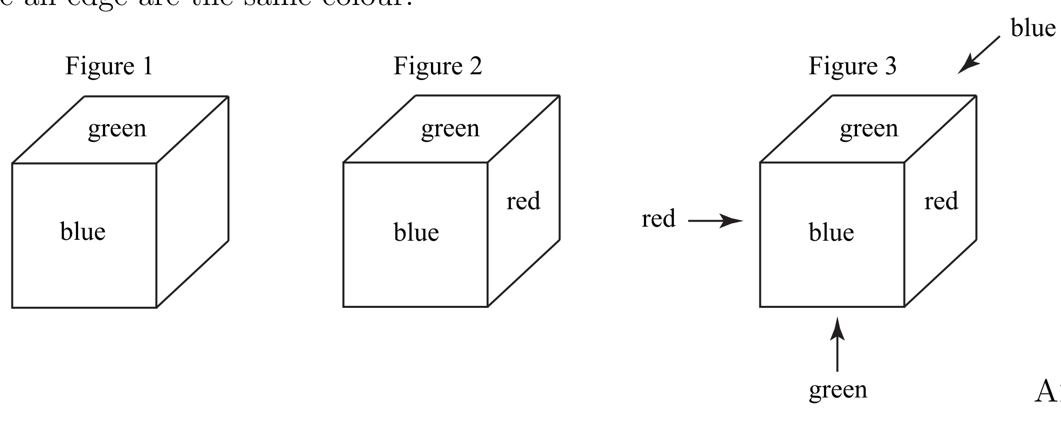


Answer: (D)

21. At Store Q, the bicycle's regular price is 15% more than the price at Store P, or 15% more than \$200.
Since 15% of 200 is $\frac{15}{100} \times 200 = 0.15 \times 200 = 30$, then 15% more than \$200 is \$200 + \$30 or \$230.
This bicycle is on sale at Store Q for 10% off of the regular price, \$230.
Since 10% of 230 is $\frac{10}{100} \times 230 = 0.10 \times 230 = 23$, then 10% off of \$230 is \$230 − \$23 or \$207.
The sale price of the bicycle at Store Q is \$207.

Answer: (D)

22. Assume the top face of the cube is coloured green.
Since the front face of the cube shares an edge with the top face, it cannot be coloured green. Thus, we need at least two colours.
Thus, we assume that the front face is coloured blue, as shown in Figure 1.
Since the right face shares an edge with the top face and with the front face, it cannot be coloured green or blue. Thus, we need at least three colours.
Thus, we assume that the right face is coloured red, as shown in Figure 2.
We have shown that at least 3 colours are needed. In fact, the cube can be coloured with exactly 3 colours by colouring the left face red, the back face blue, and the bottom face green (Figure 3).
In this way, the cube is coloured with exactly 3 colours and no two faces that share an edge are the same colour.
Therefore, 3 is the smallest number of colours needed to paint a cube so that no two faces that share an edge are the same colour.



Answer: (B)

23. *Solution 1*

For each of the 6 possible outcomes that could appear on the red die, there are 6 possible outcomes that could appear on the blue die.

That is, the total number of possible outcomes when a standard six-sided red die and a standard six-sided blue die are rolled is $6 \times 6 = 36$.

These 36 outcomes are shown in the table below.

When a number that appears on the red die is greater than a number that appears on the blue die, a checkmark has been placed in the appropriate cell, corresponding to the intersection of the column and row.

For example, the table cell containing the double checkmark ✓✓represents the outcome of a 4 appearing on the red die and a 2 appearing on the blue die.

Number on the Red Die

Number on the Blue Die

Red / Blue	1	2	3	4	5	6
1		✓	✓	✓	✓	✓
2			✓	✓✓	✓	✓
3				✓	✓	✓
4					✓	✓
5						✓
6						

Of the 36 possible outcomes, $1 + 2 + 3 + 4 + 5$ or 15 have a number appearing on the red die that is larger than the number appearing on the blue die.

The probability that the number appearing on the red die is greater than the number appearing on the blue die is $\frac{15}{36}$.

Solution 2

As in Solution 1, we determine the total number of possible outcomes to be 36.

Each of these 36 outcomes can be grouped into one of three possibilities; the number appearing on the red die is greater than the number appearing on the blue die, the number appearing on the red die is less than the number appearing on the blue die, or the numbers appearing on the two dice are equal.

There are 6 possible outcomes in which the numbers appearing on the two dice are equal (both numbers are 1, both numbers are 2, and so on).

Of the 36 total outcomes, this leaves $36 - 6 = 30$ outcomes in which either the number appearing on the red die is greater than the number appearing on the blue die, or the number appearing on the red die is less than the number appearing on the blue die.

These two possibilities are equally likely to happen (since both dice are identical except for colour), and so the number appearing on the red die will be greater than the number appearing on the blue die in half of the 30 outcomes, or 15 outcomes.

Thus, the probability that the number appearing on the red die is greater than the number appearing on the blue die is $\frac{15}{36}$.

Answer: (C)

24. We begin by joining Q to P.
Since Q and P are the midpoints of ST and UV, then QP is parallel to both SV and TU and rectangles $SQPV$ and $QTUP$ are identical.
In rectangle $SQPV$, VQ is a diagonal.
Similarly, since PR is parallel to VQ then PR extended to T is a diagonal of rectangle $QTUP$, as shown in Figure 1.

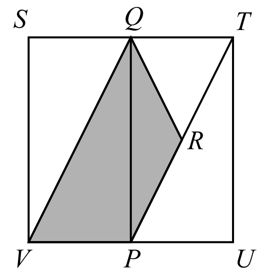


Figure 1

In Figure 2, we label points A, B, C, D, E, and F, the midpoints of SQ, QT, TU, UP, PV, and VS, respectively.
We join A to E, B to D and F to C, with FC intersecting QP at the centre of the square O, as shown.
Since $PR = QR$ and R lies on diagonal PT, then both FC and BD pass through R. (That is, R is the centre of $QTUP$.)
The line segments AE, QP, BD, and FC divide square $STUV$ into 8 identical rectangles.
In one of these rectangles, $QBRO$, diagonal QR divides the rectangle into 2 equal areas.
That is, the area of $\triangle QOR$ is half of the area of rectangle $QBRO$.
Similarly, the area of $\triangle POR$ is half of the area of rectangle $PORD$.

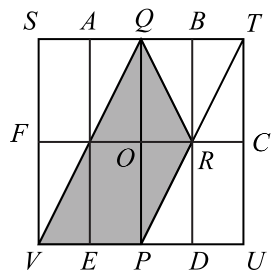


Figure 2

Rectangle $SQPV$ has area equal to 4 of the 8 identical rectangles.
Therefore, $\triangle QPV$ has area equal to 2 of the 8 identical rectangles (since diagonal VQ divides the area of $SQPV$ in half).
Thus the total shaded area, which is $\triangle QOR + \triangle POR + \triangle QPV$, is equivalent to the area of $\frac{1}{2} + \frac{1}{2} + 2$ or 3 of the identical rectangles.
Since square $STUV$ is divided into 8 of these identical rectangles, and the shaded area is equivalent to the area of 3 of these 8 rectangles, then the unshaded area occupies an area equal to that of the remaining $8 - 3$ or 5 rectangles.
Therefore, the ratio of the shaded area to the unshaded area is $3 : 5$.

Answer: (B)

25. We begin by listing the first few line segment endpoints as determined by the number of segments that Paul has drawn.

Line Segment Number	Endpoint of the Line Segment
1	$(1, 0)$
2	$(1, 2)$
3	$(4, 2)$
4	$(4, 6)$
5	$(9, 6)$
6	$(9, 12)$
7	$(16, 12)$
8	$(16, 20)$

Since each new line segment is drawn either vertically or horizontally from the endpoint of the previous line segment, then only one of the x-coordinate or y-coordinate changes from one endpoint to the next endpoint.
Further, since the odd numbered line segments are horizontal, the x-coordinate of the endpoint of these segments changes from the previous endpoint while the y-coordinate remains the same.

Similarly, since the even numbered line segments are vertical, the y-coordinate of the endpoint of these segments changes from the previous endpoint while the x-coordinate remains the same.

We are given that one of the line segments ends at the point $(529, 506)$.
We will begin by determining the number of segments that must be drawn for the x-coordinate of the endpoint of the final line segment to be 529.
The first line segment is drawn horizontally from the origin, to the right, has length 1, and thus ends with an x-coordinate of 1.
The third line segment is drawn horizontally to the right, has length 3 and thus ends with an x-coordinate of $1 + 3 = 4$.
The fifth line segment is drawn horizontally to the right, has length 5 and thus ends with an x-coordinate of $1 + 3 + 5 = 9$.
To get an idea of how many line segments are required before the x-coordinate of the endpoint is 529, we begin by considering the first 21 line segments.
The endpoint of the 21st line segment has x-coordinate equal to $1 + 3 + 5 + \cdots + 17 + 19 + 21$.
To make this sum easier to determine, we rearrange the terms to get
$(1+21)+(3+19)+(5+17)+(7+15)+(9+13)+11 = 22+22+22+22+22+11 = 22\times 5+11 = 121$.
Since 121 is much smaller than 529, we continue to try larger numbers of line segments until we finally reach 45 line segments.
The endpoint of the 45th line segment has x-coordinate equal to $1 + 3 + 5 + \cdots + 41 + 43 + 45$.
To make this sum easier to determine, we rearrange the terms to get

$$(1 + 45) + (3 + 43) + (5 + 41) + \cdots + (19 + 27) + (21 + 25) + 23 = 46 \times 11 + 23 = 529.$$

Therefore, the 45th line segment ends with an x-coordinate of 529.
The next line segment (the 46th) will also have an endpoint with this same x-coordinate, 529, since even numbered line segments are vertical (thus, only the y-coordinate will change).
This tells us that $(529, 506)$ is the endpoint of either the 45th or the 46th line segment.

At this point we may confirm that 506 is the y-coordinate of the endpoint of the 45th line segment (and hence the 44th line segment as well).
The second line segment is drawn vertically from the x-axis, upward, has length 2, and thus ends with y-coordinate 2.
The fourth line segment is drawn vertically upward, has length 4 and thus ends with a y-coordinate of $2 + 4 = 6$.
The sixth line segment is drawn vertically upward, has length 6 and thus ends with a y-coordinate of $2 + 4 + 6 = 12$.
The endpoint of the 45th line segment has y-coordinate equal to $2 + 4 + 6 + \cdots + 40 + 42 + 44$.
Rearranging the terms of this sum, we get

$$(2 + 44) + (4 + 42) + (6 + 40) + \cdots + (20 + 26) + (22 + 24) = 46 \times 11 = 506.$$

Thus, $(529, 506)$ is the endpoint of the 45th line segment.
The 46th line segment is vertical and has length 46.
Therefore the y-coordinate of the endpoint of the 46th line segment is

$$2 + 4 + 6 + \cdots + 40 + 42 + 44 + 46 = 506 + 46 = 552.$$

Since the x-coordinate of the endpoint of the 46th line segment is the same as that of the 45th, then the endpoint of the next line segment that Paul draws is $(529, 552)$.

Answer: (A)

Grade 8

1. The number 10 101 is ten thousand one hundred one.
   Therefore, it is equal to $10\,000 + 100 + 1$.

   Answer: (D)

2. Since one scoop can feed 8 goldfish, then 4 scoops can feed $4 \times 8 = 32$ goldfish.

   Answer: (E)

3. Evaluating, $(2014 - 2013) \times (2013 - 2012) = 1 \times 1 = 1$.

   Answer: (B)

4. The measure of the three angles in any triangle add to 180°.
   Since two of the angles measure 90° and 55°, then the third angle in the triangle measures $180^\circ - 90^\circ - 55^\circ = 35^\circ$.
   The measure of the smallest angle in the triangle is 35°.

   Answer: (D)

5. The positive value of any number is equal to the distance that the number is away from zero along a number line.
   The smaller the distance a number is away from zero along a number line, the closer the number is to zero.
   Ignoring negative signs (that is, we consider all 5 answers to be positive), the number that is closest to zero is the smallest number.
   The smallest number from the list $\{1101, 1011, 1010, 1001, 1110\}$ is 1001.
   Of the given answers, the number -1001 is the closest number to zero.

   Answer: (D)

6. Since $5y - 100 = 125$, then $5y = 125 + 100 = 225$, and so $y = \frac{225}{5} = 45$.

   Answer: (A)

7. Each prime number is divisible by exactly two numbers, 1 and itself.
   Each of the even numbers $\{12, 14, 16, 18, 20, 22, 24, 26, 28\}$ is divisible by 1 and itself and also by 2.
   Therefore the numbers $\{12, 14, 16, 18, 20, 22, 24, 26, 28\}$ are not prime.
   Each of the numbers $\{15, 21, 27\}$ is divisible by 1 and itself and also by 3.
   Therefore the numbers $\{15, 21, 27\}$ are not prime.
   The number 25 is divisible by 1 and itself and also by 5, so it is not prime.
   Each of the remaining numbers is divisible by exactly 1 and itself.
   Thus, the list of prime numbers between 10 and 30 is $\{11, 13, 17, 19, 23, 29\}$.
   There are 6 prime numbers between 10 and 30.

   Answer: (C)

8. The given triangle is isosceles, so the unmarked side also has length x cm.
   The perimeter of the triangle is 53 cm, so $x + x + 11 = 53$ or $2x + 11 = 53$ or $2x = 53 - 11 = 42$, and so $x = \frac{42}{2} = 21$.

   Answer: (B)

9. *Solution 1*
To order the fractions $\left\{\frac{3}{7}, \frac{3}{2}, \frac{6}{7}, \frac{3}{5}\right\}$ from smallest to largest, we first express each fraction with a common denominator of $7 \times 2 \times 5 = 70$.
The set $\left\{\frac{3}{7}, \frac{3}{2}, \frac{6}{7}, \frac{3}{5}\right\}$ is equivalent to the set $\left\{\frac{3\times10}{7\times10}, \frac{3\times35}{2\times35}, \frac{6\times10}{7\times10}, \frac{3\times14}{5\times14}\right\}$ or to the set $\left\{\frac{30}{70}, \frac{105}{70}, \frac{60}{70}, \frac{42}{70}\right\}$.
Ordered from smallest to largest, the set is $\left\{\frac{30}{70}, \frac{42}{70}, \frac{60}{70}, \frac{105}{70}\right\}$, so ordered from smallest to largest the original set is $\left\{\frac{3}{7}, \frac{3}{5}, \frac{6}{7}, \frac{3}{2}\right\}$.

Solution 2
Since the numerators of the given fractions are each either 3 or 6, we rewrite each fraction with a common numerator of 6.
The set $\left\{\frac{3}{7}, \frac{3}{2}, \frac{6}{7}, \frac{3}{5}\right\}$ is equivalent to the set $\left\{\frac{3\times2}{7\times2}, \frac{3\times2}{2\times2}, \frac{6}{7}, \frac{3\times2}{5\times2}\right\}$ or to the set $\left\{\frac{6}{14}, \frac{6}{4}, \frac{6}{7}, \frac{6}{10}\right\}$.
Since the numerators are equal, then the larger the denominator, the smaller the fraction.
Ordered from smallest to largest, the set is $\left\{\frac{6}{14}, \frac{6}{10}, \frac{6}{7}, \frac{6}{4}\right\}$, so ordered from smallest to largest the original set is $\left\{\frac{3}{7}, \frac{3}{5}, \frac{6}{7}, \frac{3}{2}\right\}$.

ANSWER: (A)

10. *Solution 1*
The ratio of the number of girls to the number of boys is $3 : 5$, so for every 3 girls there are 5 boys and $3 + 5 = 8$ students.
That is, the number of girls in the class is $\frac{3}{8}$ of the number of students in the class.
Since the number of students in the class is 24, the number of girls is $\frac{3}{8} \times 24 = \frac{72}{8} = 9$.
The remaining $24 - 9 = 15$ students in the class must be boys.
Therefore, there are $15 - 9 = 6$ fewer girls than boys in the class.

Solution 2
The ratio of the number of girls to the number of boys is $3 : 5$, so if there are $3x$ girls in the class then there are $5x$ boys.
In total, there are 24 students in the class, so $3x + 5x = 24$ or $8x = 24$ and so $x = 3$.
The number of girls in the class is $3x = 3(3) = 9$ and the number of boys in the class is $5x = 5(3) = 15$.
Therefore, there are $15 - 9 = 6$ fewer girls than boys in the class.

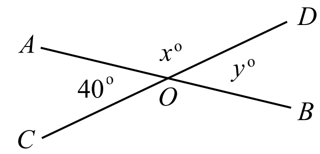
Solution 3
The ratio of the number of girls to the number of boys is $3 : 5$, so for every 3 girls there are 5 boys and $3 + 5 = 8$ students.
That is, $\frac{3}{8}$ of the number of students in the class are girls and $\frac{5}{8}$ of the number of students are boys.
Thus, the difference between the number of boys in the class and the number of girls in the class is $\frac{5}{8} - \frac{3}{8} = \frac{2}{8} = \frac{1}{4}$ of the number of students in the class.
Therefore, there are $\frac{1}{4} \times 24 = 6$ fewer girls than boys in the class.

ANSWER: (D)

11. Since there are 7 days in a week, any multiple of 7 days after a Wednesday is also a Wednesday.
Since 70 is a multiple of 7, then 70 days after John was born, the day is also a Wednesday.
Therefore, 71 days after John was born is a Thursday and 72 days after John was born is a Friday.
Alison was born on a Friday.

ANSWER: (E)

12. Opposite angles have equal measures.
Since $\angle AOC$ and $\angle DOB$ are opposite angles, then $y = 40$.
Straight angle COD measures $180°$.
Since $\angle AOC + \angle AOD = \angle COD$, then $40° + x° = 180°$ and so $x = 140$.
The difference $x - y$ is $140 - 40 = 100$.



Answer: (E)

13. *Solution 1*
Since the scores in each of the 5 sets are listed in order, from smallest to largest, the number in the middle (the 3rd number) of the 5 numbers is the median.
The mean is calculated and also shown below for each of the 5 sets of scores.

Set of Scores	Median	Mean
$10, 20, 40, 40, 40$	40	$\dfrac{10+20+40+40+40}{5} = \dfrac{150}{5} = 30$
$40, 50, 60, 70, 80$	60	$\dfrac{40+50+60+70+80}{5} = \dfrac{300}{5} = 60$
$20, 20, 20, 50, 80$	20	$\dfrac{20+20+20+50+80}{5} = \dfrac{190}{5} = 38$
$10, 20, 30, 100, 200$	30	$\dfrac{10+20+30+100+200}{5} = \dfrac{360}{5} = 72$
$50, 50, 50, 50, 100$	50	$\dfrac{50+50+50+50+100}{5} = \dfrac{300}{5} = 60$

From the table above, we see that the first set of scores is the only set in which the median, 40, is greater than the mean, 30.

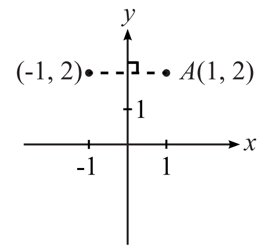
Solution 2
The first set of 5 scores, $10, 20, 40, 40, 40$, is ordered from smallest to largest and thus the median is the 3rd (middle) number of this set, 40.
The mean of the final 3 numbers of this set is also 40, since each is equal to 40.
Since the first 2 numbers of the set are both less than the mean of the other 3 numbers (40), then the mean of all 5 numbers in the set must be less than 40.
Using a similar argument for the remaining four answers, we see that each of the other four sets has its mean equal to or greater than its median.
Therefore, the first set of scores is the only set in which the median is greater than the mean.

Answer: (A)

14. Betty has 3 equally likely choices for the flavour of ice cream (chocolate or vanilla or strawberry).
For each of these 3 choices, she has 2 equally likely choices for the syrup (butterscotch or fudge).
Thus, Betty has 3×2 or 6 equally likely choices for the flavour and syrup of the sundae.
For each of these 6 choices, Betty has 3 equally likely choices for the topping (cherry or banana or pineapple).
This makes 6×3 or 18 equally likely choices for her sundae.
Since only 1 of these 18 choices is the sundae with vanilla ice cream, fudge syrup and banana topping, then the probability that Betty randomly chooses this sundae is $\frac{1}{18}$.

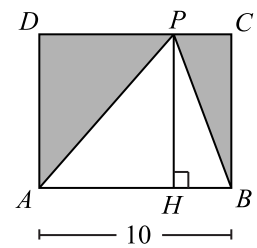
Answer: (A)

15. After reflecting the point $A(1,2)$ in the y-axis, the y-coordinate of the image will be the same as the y-coordinate of point A, $y=2$.
Point A is a distance of 1 to the right of the y-axis.
The image will be the same distance from the y-axis, but to the left of the y-axis. Thus, the image has x-coordinate -1.
The coordinates of the reflected point are $(-1,2)$.



Answer: (B)

16. *Solution 1*
Construct PH perpendicular to AB, as shown.
The area of $\triangle ABP$ is 40 and so $\frac{1}{2} \times AB \times PH = 40$, or $AB \times PH = 80$, and since $AB = 10$, then $PH = 8$.
Since $CB = PH = 10$, the area of $ABCD$ is $10 \times 8 = 80$.
The shaded area equals the area of $\triangle ABP$ subtracted from the area of $ABCD$, or $80 - 40 = 40$.



Solution 2
As in Solution 1, we construct PH perpendicular to AB.
Since both DA and CB are perpendicular to AB, then PH is parallel to DA and to CB.
That is, $DAHP$ and $PHBC$ are rectangles.
Diagonal PA divides the area of rectangle $DAHP$ into two equal areas, $\triangle PAH$ and $\triangle PAD$.
Diagonal PB divides the area of rectangle $PHBC$ into two equal areas, $\triangle PBH$ and $\triangle PBC$.
Therefore, the area of $\triangle PAH$ added to the area of $\triangle PBH$ is equal to the area of $\triangle PAD$ added to the area of $\triangle PBC$.
However, the area of $\triangle PAH$ added to the area of $\triangle PBH$ is equal to the area of $\triangle ABP$, which is 40.
So the area of $\triangle PAD$ added to the area of $\triangle PBC$ is also 40.
Therefore, the area of the shaded region is 40.

Answer: (B)

17. Of the 10 multiple choice questions, Janine got 80% or $0.80 \times 10 = 8$ correct.
Of the 30 short answer questions, Janine got 70% or $0.70 \times 30 = 21$ correct.
In total, Janine answered $8+21 = 29$ of the 40 questions correct, or $\frac{29}{40} \times 100\% = 72.5\%$ correct.

Answer: (B)

18. The area of the rectangle, 48 cm^2, is given by the product of the rectangle's length and width, and so we must consider all possible pairs of whole numbers whose product is 48.
In the table below, we systematically examine all possible factors of 48 in order to determine the possible whole number side lengths of the rectangle and its perimeter.

Factors of 48	Side Lengths of Rectangle	Perimeter of Rectangle
$48 = 1 \times 48$	1 and 48	$2 \times (1+48) = 2 \times 49 = 98$
$48 = 2 \times 24$	2 and 24	$2 \times (2+24) = 2 \times 26 = 52$
$48 = 3 \times 16$	3 and 16	$2 \times (3+16) = 2 \times 19 = 38$
$48 = 4 \times 12$	4 and 12	$2 \times (4+12) = 2 \times 16 = 32$
$48 = 6 \times 8$	6 and 8	$2 \times (6+8) = 2 \times 14 = 28$

The side lengths of the rectangle having area 48 cm^2 and perimeter 32 cm, are 4 and 12.
In cm, the positive difference between the length and width of the rectangle is $12 - 4 = 8$.

Answer: (D)

19. At Store Q, the bicycle's regular price is 15% more than the price at Store P, or 15% more than \$200.
Since 15% of 200 is $\frac{15}{100} \times 200 = 0.15 \times 200 = 30$, then 15% more than \$200 is \$200+\$30 or \$230.
This bicycle is on sale at Store Q for 10% off of the regular price, \$230.
Since 10% of 230 is $\frac{10}{100} \times 230 = 0.10 \times 230 = 23$, then 10% off of \$230 is \$230 − \$23 or \$207.
The sale price of the bicycle at Store Q is \$207.
Answer: (D)

20. Using 7 of the 5¢ stamps (35¢) and 1 of the 8¢ stamps (8¢) makes 43¢ in postage.
Also, 3 of the 5¢ stamps (15¢) and 3 of the 8¢ stamps (24¢) makes 39¢ in postage.
This eliminates the choices (D) and (E).
Of the five given answers, the next largest is 27¢.
To make 27¢ in postage, $0, 1, 2$, or 3 of the 8¢ stamps must be used (using more than 3 of the 8¢ stamps would exceed 27¢ in postage).
If 0 of the 8¢ stamps are used, then the remaining 27¢ must be made from 5¢ stamps.
This is not possible since 27 is not a multiple of 5.
If 1 of the 8¢ stamps are used, then the remaining 27¢ − 8¢ = 19¢ must be made from 5¢ stamps. This is not possible since 19 is not a multiple of 5.
If 2 of the 8¢ stamps are used, then the remaining 27¢ − 16¢ = 11¢ must be made from 5¢ stamps. This is not possible since 11 is not a multiple of 5.
If 3 of the 8¢ stamps are used, then the remaining 27¢ − 24¢ = 3¢ must be made from 5¢ stamps. This is clearly not possible.
Therefore of the five given answers, 27¢ is the largest amount of postage that cannot be made using only 5¢ and 8¢ stamps.
It is worth noting that all amounts of postage larger than 27¢ can be made using only 5¢ and 8¢ stamps.
To see why this is true, first consider that all five postage amounts from 28¢ to 32¢ can be made as shown in the table below.

Number of 5¢ Stamps	Number of 8¢ Stamps	Value of Stamps
4	1	(4 × 5¢)+(1 × 8¢)= 28¢
1	3	(1 × 5¢)+(3 × 8¢)= 29¢
6	0	(6 × 5¢)+(0 × 8¢)= 30¢
3	2	(3 × 5¢)+(2 × 8¢)= 31¢
0	4	(0 × 5¢)+(4 × 8¢)= 32¢

The next five amounts of postage, 33¢ to 37¢, can be made by adding 1 additional 5¢ stamp to each of the previous five amounts of postage.
That is, 28¢ + 5¢ = 33¢, and 29¢ + 5¢ = 34¢, and so on.
We can make every postage amount larger than 27¢ by continuing in this way to add 1 additional 5¢ stamp to each of the five amounts from the previous group.
Answer: (C)

21. The shaded area in the top three rows of the diagram contains 6 circles with radius 1 cm.
The shaded area in the bottom row of the diagram contains 4 semi-circles with radius 1 cm, whose combined area is equal to the area of 2 circles with radius 1 cm.
In total, the shaded area of the diagram contains $6 + 2 = 8$ circles with radius 1 cm.
In cm^2, the total shaded area is $8 \times \pi \times 1^2 = 8\pi$.
Answer: (E)

22. We first determine the surface area of the original cube, before the two cubes were cut from its corners.
The original cube had 6 identical square faces, each of whose area was $3 \times 3 = 9$ cm^2.
Thus, the surface area of the original cube was $6 \times 9 = 54$ cm^2.
Next we will explain why the resulting solid in question has surface area equal to that of the original cube, 54 cm^2.

Consider the front, bottom right corner of the resulting solid, as shown and labeled in Figure 1.
Cutting out a 1 cm by 1 cm by 1 cm cube from this corner exposes 3 new square faces, $RSPQ$, $RSTU$ and $SPWT$, which were not a part of the surface area of the original cube.
Next we consider what surface area was part of the original cube that is not present in the resulting solid.

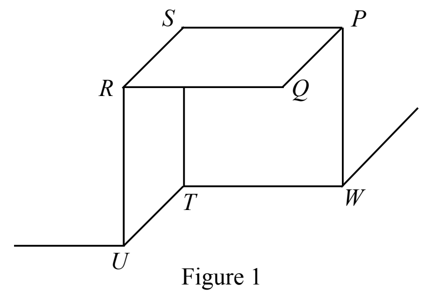


Figure 1

To visualize this, we replace the vertex V of the original cube and reconstruct segments QV, UV and WV, as shown in Figure 2.
We note that $PQRSTUVW$ is the 1 cm by 1 cm by 1 cm cube that was cut from the corner of the original cube.
In Figure 2, we see that the surface areas that were part of the original cube that are not present in the resulting solid are the 3 square faces $UTWV$, $QPWV$ and $RQVU$.

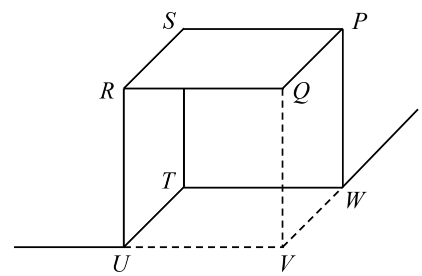


Figure 2

To summarize, removal of the 1 cm by 1 cm by 1 cm cube from the corner of the original cube exposes 3 new square faces, $RSPQ$, $RSTU$ and $SPWT$, which were not a part of the surface area of the original cube.
These 3 faces represent half of the surface area of cube $PQRSTUVW$ (since they are 3 of the 6 identical faces of the cube).
However, in removing the 1 cm by 1 cm by 1 cm cube from the corner of the original cube, the 3 faces, $UTWV$, $QPWV$ and $RQVU$, are lost.
These 3 faces represent the other half of the surface area of cube $PQRSTUVW$.
That is, the total surface area gained by removal of the 1 cm by 1 cm by 1 cm cube is equal to the total surface area lost by removing this same cube.
This same argument can be made for the removal of the 2 cm by 2 cm by 2 cm cube from the top, back left corner of the original cube.
Thus the surface area of the resulting solid is equal to the surface area of the original cube, which was 54 cm^2.

Answer: (E)

23. The first positive odd integer is 1, the second is $2(2) - 1 = 3$, the third is $2(3) - 1 = 5$, the fourth is $2(4) - 1 = 7$.
That is, the one hundredth positive odd integer is $2(100) - 1 = 199$ and the sum that we are being asked to determine is $1 + 3 + 5 + \cdots + 195 + 197 + 199$.

Solution 1
Since each odd integer can be expressed as the sum of two consecutive integers, we rewrite $1+3+5+\cdots+195+197+199$ as $1+(1+2)+(2+3)+\cdots+(97+98)+(98+99)+(99+100)$.
Rearranging the terms in the previous sum, we get
$(1 + 2 + 3 + \cdots + 98 + 99 + 100) + (1 + 2 + 3 + \cdots + 97 + 98 + 99)$.

The sum in the first set of brackets, $1 + 2 + 3 + \cdots + 98 + 99 + 100$ is equal to 5050.
The sum in the second set of brackets, $1 + 2 + 3 + \cdots + 98 + 99$ is 100 less than 5050 or 4950.
Therefore, $1 + 3 + 5 + \cdots + 195 + 197 + 199 = 5050 + 4950 = 10\,000$.

Solution 2
Doubling each term on the left of the equality $1 + 2 + 3 + \cdots + 98 + 99 + 100 = 5050$, doubles the result on the right.
That is, $2 + 4 + 6 + \cdots + 196 + 198 + 200 = 10\,100$.
Subtracting 1 from each term on the left side of this equality gives
$(2-1)+(4-1)+(6-1)+\cdots+(196-1)+(198-1)+(200-1)$ or $1+3+5+\cdots+195+197+199$, the required sum.
Since there are 100 terms on the left side of the equality, then subtracting one 100 times reduces the left side of the equality by 100.
Subtracting 100 from the right side, we get $10\,100 - 100 = 10\,000$, and so
$1 + 3 + 5 + \cdots + 195 + 197 + 199 = 10\,000$.

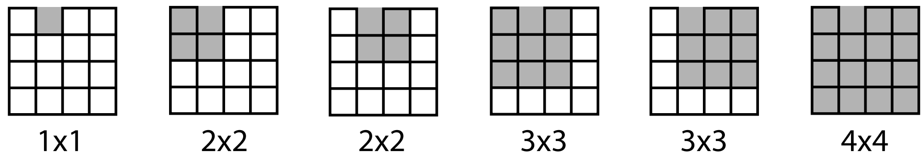
Solution 3
It is also possible to determine the sum of the first 100 positive odd integers,
$1 + 3 + 5 + \cdots + 195 + 197 + 199$, without using the result given in the question.
Reorganizing the terms into sums of pairs, first the largest number with the smallest, then the next largest number with the next smallest, and so on,
$1+3+5+\cdots+195+197+199 = (1+199)+(3+197)+(5+195)+\cdots+(97+103)+(99+101)$.
The sum of the first pair, $1 + 199$ is 200.
Each successive pair following $1 + 199$ includes a number that is 2 more than a number in the previous pair, and also a number that is 2 less than a number in the previous pair, so then the sum of every pair is also 200.
The total number of terms in the sum is 100, and thus there are 50 pairs of numbers each of which has a sum of 200.
Therefore, $1 + 3 + 5 + \cdots + 195 + 197 + 199 = 50 \times 200 = 10\,000$.

Answer: (B)

24. The original 4×4 grid contains exactly 30 squares, of which 16 are of size 1×1, 9 are of size 2×2, 4 are of size 3×3, and 1 is of size 4×4 ($16 + 9 + 4 + 1 = 30$).
In each of the 5 answers given, exactly one or two of the 1×1 squares are missing from the original grid.
We determine the number of these 30 squares that cannot be formed as a result of these missing 1×1 squares, and subtract this total from 30 to find the total number of squares in each of the 5 given configurations.

	Number of Missing Squares				Total Number
	1×1	2×2	3×3	4×4	of Squares
A	1	1	1	1	$30 - 4 = 26$
B	1	2	2	1	$30 - 6 = 24$
C	2	2	2	1	$30 - 7 = 23$
D	2	2	2	1	$30 - 7 = 23$
E	2	2	2	1	$30 - 7 = 23$

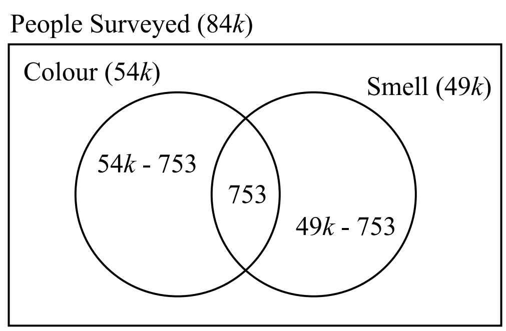
For clarity, the 6 squares that exist in the original grid but that are missing from the grid in answer B are shown below by size.



The grid given in answer B contains exactly 24 squares.

Answer: (B)

25. A total of N people participated in the survey.
Exactly $\frac{9}{14}$ of those surveyed, or $\frac{9}{14} \times N$ people, said that the colour of the flower was important.
Exactly $\frac{7}{12}$ of those surveyed, or $\frac{7}{12} \times N$ people, said that the smell of the flower was important.
Since the number of people surveyed must be a whole number, then N must be a multiple of 14 and N must also be a multiple of 12.
The lowest common multiple of 14 and 12 is $2 \times 7 \times 6$ or 84, and so N is a multiple of 84.
Since N is a multiple of 84, let $N = 84k$ where k is some positive integer.
That is, $\frac{9}{14} \times N = \frac{9}{14}(84k) = 54k$ people said that the colour of the flower was important and $\frac{7}{12} \times N = \frac{7}{12}(84k) = 49k$ people said that the smell of the flower was important.
The information can be represented with a Venn diagram, as shown below.



Since $N = 84k$, we must find both a maximum and a minimum value for k in order to determine the number of possible values for N.

Finding a Minimum Value for k

We know that 753 people said that both the colour and the smell were important.
We also know that $54k$ said that the colour was important and that $49k$ said that the smell was important.
The 753 people who said that both were important are included among both the $54k$ people and among the $49k$ people.
Therefore, $54k$ must be at least 753 and $49k$ must also be at least 753.
Since $54k$ is larger than $49k$, then we only need to find k for which $49k$ is larger than or equal to 753.
We note that $49 \times 15 = 735$ and that $49 \times 16 = 784$.
Therefore, the smallest value of k for which $49k$ is at least 753 is $k = 16$.
As noted above, $k = 16$ will also make $54k$ larger than 753.
Therefore, the minimum possible value for k is 16.

Finding a Maximum Value for k

There were $84k$ people surveyed of whom $54k$ said that the colour was important.
This means that there are $84k - 54k = 30k$ people who said that the colour was not important.
We note that $49k$ people said that the smell was important. These $49k$ people include some or all of the $30k$ people who said that the colour was not important.
In other words, at most $30k$ of the $49k$ people said that the smell was important and the colour was not important, which means that at least $49k - 30k = 19k$ people said that the smell was important and that the colour was important.
Since we know that 753 people said that both the smell and colour were important, then 753 is at least as large as $19k$.
Since $19 \times 40 = 760$ and $19 \times 39 = 741$, then k must be at most 39.

The minimum value of k is 16 and the maximum value of k is 39.
Thus, there are $39 - 16 + 1 = 24$ possible values for k and since $N = 84k$, there are also 24 possible values for N.

Note: We should also justify that each value of k between 16 and 39 is possible.
Using the Venn diagram above, we see that the total of $84k$ people surveyed means that $84k - (54k - 753) - 753 - (49k - 753) = 753 - 19k$ people fall outside the two circles.
From our work in finding the minimum above, we can see that for each k between 16 and 39, inclusive, each of the three integers $54k - 753$, 753, and $49k - 753$ is positive, since k is at least 16. Also, from our work in finding the maximum above, we can see that for each k between 16 and 39, inclusive, the integer $753 - 19k$ is positive, since k is at most 39.
Therefore, all four quantities are positive integers for each of these values of k, which means that we can construct a Venn diagram with these numbers, as required.

Answer: (D)

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Grade 8 solutions
follow the
Grade 7 solutions

2013 Gauss Contests

(Grades 7 and 8)

Wednesday, May 15, 2013
(in North America and South America)

Thursday, May 16, 2013
(outside of North America and South America)

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Grade 7

1. Evaluating, $(5 \times 3) - 2 = 15 - 2 = 13$.

   ANSWER: (E)

2. *Solution 1*
   A number is a multiple of 9 if it is the result of multiplying 9 by an integer.
   Of the answers given, only 45 results from multiplying 9 by an integer, since $45 = 9 \times 5$.

   Solution 2
   A number is a multiple of 9 if the result after dividing it by 9 is an integer.
   Of the answers given, only 45 results in an integer after dividing by 9, since $45 \div 9 = 5$.

   ANSWER: (D)

3. Thirty-six hundredths equals $\frac{36}{100}$ or 0.36.

   ANSWER: (A)

4. By grouping terms using brackets as shown, $(1 + 1 - 2) + (3 + 5 - 8) + (13 + 21 - 34)$, we can see that the result inside each set of brackets is 0.
   Thus the value of $1 + 1 - 2 + 3 + 5 - 8 + 13 + 21 - 34$ is 0.

   ANSWER: (D)

5. Since PQ is a straight line segment, the three angles given sum to 180°.
   That is, $90^\circ + x^\circ + 20^\circ = 180^\circ$ or $x^\circ = 180^\circ - 90^\circ - 20^\circ$ or $x = 70$.

   ANSWER: (B)

6. Nick has 6 nickels and each nickel is worth 5¢. So Nick has 6×5¢ or 30¢ in nickels.
   Nick has 2 dimes and each dime is worth 10¢. So Nick has 2×10¢ or 20¢ in dimes.
   Nick has 1 quarter and each quarter is worth 25¢. So Nick has 1×25¢ or 25¢ in quarters.
   In total, Nick has 30¢ + 20¢ + 25¢ or 75¢.

   ANSWER: (B)

7. *Solution 1*
   To determine the smallest number in the set $\left\{\frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{5}{6}, \frac{7}{12}\right\}$, we express each number with a common denominator of 12. The set $\left\{\frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{5}{6}, \frac{7}{12}\right\}$ is equivalent to the set $\left\{\frac{1\times6}{2\times6}, \frac{2\times4}{3\times4}, \frac{1\times3}{4\times3}, \frac{5\times2}{6\times2}, \frac{7}{12}\right\}$ or to the set $\left\{\frac{6}{12}, \frac{8}{12}, \frac{3}{12}, \frac{10}{12}, \frac{7}{12}\right\}$.
   The smallest number in this set is $\frac{3}{12}$, so $\frac{1}{4}$ is the smallest number in the original set.

   Solution 2
   With the exception of $\frac{1}{4}$, each number in the set is greater than or equal to $\frac{1}{2}$.
   We can see this by recognizing that the numerator of each fraction is greater than or equal to one half of its denominator.
   Thus, $\frac{1}{4}$ is the only number in the list that is less than $\frac{1}{2}$ and so it must be the smallest number in the set.

   ANSWER: (C)

8. Since Ahmed stops to talk with Kee one quarter of the way to the store, then the remaining distance to the store is $1 - \frac{1}{4} = \frac{3}{4}$ of the total distance.
   Since $\frac{3}{4} = 3 \times \frac{1}{4}$, then the distance that Ahmed travelled from Kee to the store is 3 times the distance that Ahmed travelled from the start to reach Kee.
   That is, 12 km is 3 times the distance between the start and Kee. So the distance between the start and Kee is $\frac{12}{3} = 4$ km. Therefore, the total distance travelled by Ahmed is $4 + 12$ or 16 km.

   ANSWER: (B)

9. When $n = 4$, we are looking for an expression that produces a value of 7.
The results of substituting $n = 4$ into each expression and evaluating are shown below.

Expression	Value
(A) $3n - 2$	$3(4) - 2 = 12 - 2 = 10$
(B) $2(n - 1)$	$2(4 - 1) = 2(3) = 6$
(C) $n + 4$	$4 + 4 = 8$
(D) $2n$	$2(4) = 8$
(E) $2n - 1$	$2(4) - 1 = 8 - 1 = 7$

Since $2n - 1$ is the only expression which gives a value of 7 when $n = 4$, it is the only possible answer. We check that the expression $2n - 1$ does give the remaining values, $1, 3, 5, 9$, when $n = 1, 2, 3, 5$, respectively.
(Alternately, we may have began by substituting $n = 1$ and noticing that this eliminates answers (B), (C) and (D). Substituting $n = 2$ eliminates (A). Substituting $n = 3$, $n = 4$ and $n = 5$ confirms the answer (E).)

Answer: (E)

10. To make the difference $UVW - XYZ$ as large as possible, we make UVW as large as possible and XYZ as small as possible.
The hundreds digit of a number contributes more to its value than its tens digit, and its tens digit contributes more to its value than its units digit.
Thus, we construct the largest possible number UVW by choosing 9 (the largest digit) to be its hundreds digit, U, and by choosing 8 (the second largest digit) to be its tens digit, V, and by choosing 7 (the third largest digit) to be the units digit, W.
Similarly, we construct the smallest possible number XYZ by choosing 1 (the smallest allowable digit) to be its hundreds digit, X, and 2 (the second smallest allowable digit) to be its tens digit, Y, and by choosing 3 (the third smallest allowable digit) to be its units digit, Z.
The largest possible difference is $UVW - XYZ$ or $987 - 123$ or 864.

Answer: (B)

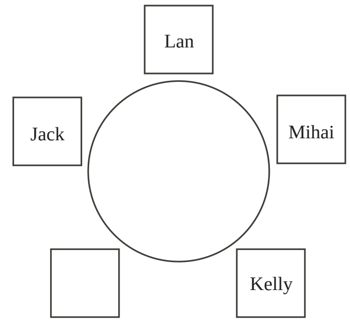
11. Each face of a cube is a square. The dimensions of each face of the cube are 1 cm by 1 cm. Thus, the area of each face of the cube is $1 \times 1 = 1\,\text{cm}^2$.
Since a cube has 6 identical faces, the surface area of the cube is $6 \times 1 = 6\,\text{cm}^2$.

Answer: (E)

12. The greatest common factor of two numbers is the largest positive integer which divides into both numbers with no remainder.
For answer (B), 50 divided by 20 leaves a remainder, so we may eliminate (B) as a possible answer.
Similarly for answer (D), 25 divided by 20 leaves a remainder, so we may eliminate (D) as a possible answer.
For answer (A), since 200 divides 200 and 200 divides 2000, the greatest common factor of 200 and 2000 cannot be 20.
For answer (E), since 40 divides 40 and 40 divides 80, the greatest common factor of 40 and 80 cannot be 20.
The largest positive integer which divides both 20 and 40 is 20, and so (C) is the correct answer.

Answer: (C)

13. Since Lan and Mihai are seated beside each other, while Jack and Kelly are not, the only possible location for the remaining chair (Nate's chair) is between Jack and Kelly, as shown. Therefore, the 2 people who are seated on either side of Nate are Jack and Kelly.



Answer: (B)

14. Substituting $x = 4$ into $3x + 2y$ we get, $3(4) + 2y = 12 + 2y$.
Since this expression $12 + 2y$ is equal to 30, then $2y$ must equal $30 - 12$ or 18.
If $2y = 18$, then y is $18 \div 2$ or 9.

Answer: (E)

15. Each time Daniel reaches into the jar, he removes half of the coins that are in the jar.
Since he removes half of the coins, then the other half of the coins remain in the jar.
We summarize Daniel's progress in the table below.

Number of Times Coins are Removed	0	1	2	3	4	5	6
Number of Coins Remaining in the Jar	64	32	16	8	4	2	1

For exactly 1 coin to remain in the jar, Daniel must reach in and remove coins from the jar 6 times.

Answer: (C)

16. *Solution 1*
Consider the set of five consecutive even numbers $8, 10, 12, 14, 16$.
The mean of these five numbers is $\frac{8+10+12+14+16}{5} = \frac{60}{5}$ or 12.
If the five consecutive even numbers were smaller, their mean would be less than 12.
If the five consecutive even numbers were larger, their mean would be greater than 12.
Therefore, this is the set of five consecutive even numbers that we seek.
The mean of the smallest and largest of these five numbers is $\frac{8+16}{2} = \frac{24}{2}$ or 12.

Solution 2
The mean of five consecutive even numbers is the middle (third largest) number.
To see this, consider that the smallest of the five numbers is 4 less than the middle number, while the largest of the five numbers is 4 more than the middle number.
Thus, the mean of the smallest and largest numbers is the middle number.
Similarly, the second smallest of the five numbers is 2 less than the middle number, while the fourth largest of the five numbers is 2 more than the middle number.
Thus, the mean of these two numbers is also the middle number.
Since the mean of the five numbers is 12, then the middle number is 12.
Therefore, the mean of the largest and smallest of the five numbers is also 12.

Answer: (A)

17. For every 3 chocolates that Claire buys at the regular price, she buys a fourth for 25 cents.
Consider dividing the 12 chocolates that Claire buys into 3 groups of 4 chocolates.
In each group of 4, Claire buys 3 chocolates at the regular price and the fourth chocolate is

purchased for 25 cents.
That is, of the 12 chocolates that Claire buys, 3 are bought at 25 cents each while the remaining 9 are purchased at the regular price.
The total cost to purchase 3 chocolates at 25 cents each is $3 \times 25 = 75$ cents.
Since Claire spent \$6.15 and the total cost of the 25 cent chocolates was 75 cents, then the cost of the regular price chocolates was $\$6.15 - \$0.75 = \$5.40$.
Since 9 chocolates were purchased at the regular price for a total of \$5.40, then the regular price of one chocolate is $\frac{\$5.40}{9} = \$0.60$ or 60 cents.

ANSWER: (C)

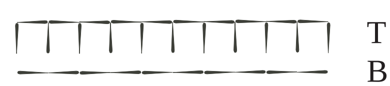
18. The total area of the shaded regions is the difference between the area of square $JKLM$ and the area of the portion of rectangle $PQRS$ that overlaps $JKLM$.
Since $JK = 8$, then the area of square $JKLM$ is 8×8 or 64.
Since JK is parallel to PQ, then the portion of $PQRS$ that overlaps $JKLM$ is a rectangle, and has length equal to JK or 8, and width equal to PS or 2.
So the area of the portion of $PQRS$ that overlaps $JKLM$ is $8 \times 2 = 16$.
Therefore the total area of the shaded regions is $64 - 16 = 48$.

ANSWER: (D)

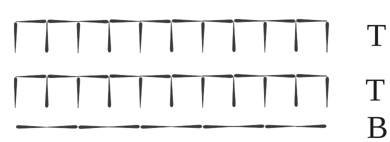
19. Using the special six-sided die, the probability of rolling a number that is a multiple of three is $\frac{1}{2}$.
Since $\frac{1}{2}$ of 6 is 3, then exactly 3 numbers on the die must be multiples of 3.
Since the probability of rolling an even number is $\frac{1}{3}$ and $\frac{1}{3}$ of 6 is 2, then exactly 2 numbers on the die must be even.
The die in (A) has only 2 numbers that are multiples of 3 (3 and 6), and thus may be eliminated.
The die in (C) has 4 numbers that are even $(2, 4, 6, 6)$, and thus may be eliminated.
The die in (D) has 3 numbers that are even $(2, 4, 6)$, and thus may be eliminated.
The die in (E) has 4 numbers that are multiples of 3 $(3, 3, 3, 6)$, and thus may be eliminated.
The die in (B) has exactly 3 numbers that are multiples of 3 $(3, 3, 6)$, and exactly 2 even numbers (2 and 6), and is therefore the correct answer.

ANSWER: (B)

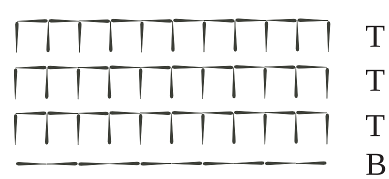
20. In the diagram shown, the 31 identical toothpicks used in the 1×10 grid are separated into 2 sections.
The top section, T, is made from 11 vertical toothpicks and 10 horizontal toothpicks, or 21 toothpicks in total.
The bottom section, B, is made of 10 horizontal toothpicks.
The 2×10 and 3×10 grids are similarly separated into top and bottom sections, as shown.
We observe that the 1×10 grid consists of 1 top section and 1 bottom.
The 2×10 grid consists of 2 top sections and 1 bottom.
The 3×10 grid consists of 3 top sections and 1 bottom.
Continuing in this way, a grid of size $n \times 10$ will consist of n top sections and 1 bottom section, for any positive integer n.
So then a grid of size 43×10 consists of 43 top sections and 1 bottom section.
Each top section is made from 21 toothpicks and each bottom section is made from 10 toothpicks.
Thus, the total number of toothpicks in a 43×10 grid is $(43 \times 21) + (1 \times 10)$ or $903 + 10$ or 913.



1 x 10



2 x 10



3 x 10

Answer: (A)

21. The sum of the units column is $P + P + P = 3P$.
Since P is a single digit, and $3P$ ends in a 7, then the only possibility is $P = 9$.
This gives:

$$\begin{array}{cccc} & 7 & 7 & 9 \\ & 6 & Q & 9 \\ + & Q & Q & 9 \\ \hline 1 & 9 & 9 & 7 \end{array}$$

Then $3P = 3 \times 9 = 27$, and thus 2 is carried to the tens column.
The sum of the tens column becomes $2 + 7 + Q + Q$ or $9 + 2Q$.
Since $9 + 2Q$ ends in a 9 (since $P = 9$), then $2Q$ ends in $9 - 9 = 0$.
Since Q is a single digit, there are two possibilities for Q such that $2Q$ ends in 0.
These are $Q = 0$ and $Q = 5$.
If $Q = 0$, then the sum of the tens column is 9 with no carry to the hundreds column.
In this case, the sum of the hundreds column is $7 + 6 + Q$ or 13 (since $Q = 0$); the units digit of this sum does not match the 9 in the total.
Thus, we conclude that Q cannot equal 0 and thus must equal 5.
Verifying that $Q = 5$, we check the sum of the tens column again.
Since $2 + 7 + 5 + 5 = 19$, then 1 is carried to the hundreds column.
The sum of the hundreds column is $1 + 7 + 6 + 5 = 19$, as required.
Thus, $P + Q = 9 + 5 = 14$ and the completed addition is shown below.

$$\begin{array}{cccc} & 7 & 7 & 9 \\ & 6 & 5 & 9 \\ + & 5 & 5 & 9 \\ \hline 1 & 9 & 9 & 7 \end{array}$$

Answer: (C)

22. We use labels, m and n, in the fourth row of the grid, as shown.
Then, $10, m, 36, n$ are four terms of an arithmetic sequence.
Since 10 and 36 are two terms apart in this sequence, and their difference is $36 - 10 = 26$, the constant added to one term to obtain the next term in the fourth row is $\frac{26}{2}$ or 13.
That is, $m = 10 + 13 = 23$, and $n = 36 + 13 = 49$.
(We confirm that the terms $10, 23, 36, 49$ do form an arithmetic sequence.)
In the fourth column, 25 and n (which equals 49) are two terms apart in this sequence, and their difference is $49 - 25 = 24$.
Thus, the constant added to one term to obtain the next term in the fourth column is $\frac{24}{2}$ or 12.
That is, $x = 25 + 12 = 37$ (or $x = 49 - 12 = 37$).
The completed grid is as shown.

1			
4			25
7			x
10	m	36	n

1	5	9	13
4	11	18	25
7	17	27	37
10	23	36	49

Answer: (A)

23. Since $\triangle PQR$ is isosceles with $PQ = QR$ and $\angle PQR = 90^\circ$, then $\angle QPR = \angle QRS = 45^\circ$.
Also in $\triangle PQR$, altitude QS bisects PR ($PS = SR$) forming two identical triangles, SQP and SQR.

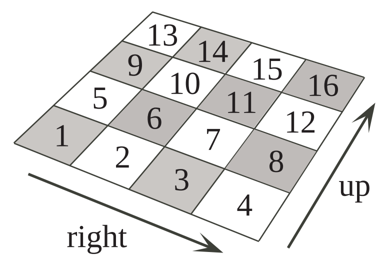
Since these two triangles are identical, each has $\frac{1}{2}$ of the area of $\triangle PQR$.

In $\triangle SQR$, $\angle QSR = 90^\circ$, $\angle QRS = 45^\circ$, and so $\angle SQR = 45^\circ$.
Thus, $\triangle SQR$ is also isosceles with $SQ = SR$.
Then similarly, altitude ST bisects QR ($QT = TR$) forming two identical triangles, SQT and SRT.
Since these two triangles are identical, each has $\frac{1}{2}$ of the area of $\triangle SQR$ or $\frac{1}{4}$ of the area of $\triangle PQR$.

Continuing in this way, altitude TU divides $\triangle STR$ into two identical triangles, STU and RTU.
Each of these two triangles has $\frac{1}{2}$ of $\frac{1}{4}$ or $\frac{1}{8}$ of the area of $\triangle PQR$.
Continuing, altitude UV divides $\triangle RTU$ into two identical triangles, RUV and TUV.
Each of these two triangles has $\frac{1}{2}$ of $\frac{1}{8}$ or $\frac{1}{16}$ of the area of $\triangle PQR$.
Finally, altitude VW divides $\triangle RUV$ into two identical triangles, UVW and RVW.
Each of these two triangles has $\frac{1}{2}$ of $\frac{1}{16}$ or $\frac{1}{32}$ of the area of $\triangle PQR$.
Since the area of $\triangle STU$ is $\frac{1}{8}$ of the area of $\triangle PQR$, and the area of $\triangle UVW$ is $\frac{1}{32}$ of the area of $\triangle PQR$, then the total fraction of $\triangle PQR$ that is shaded is $\frac{1}{8} + \frac{1}{32} = \frac{4+1}{32}$ or $\frac{5}{32}$.

Answer: (D)

24. We begin by numbering the checkerboard squares from 1 to 16, as shown, so that we may refer to each of them specifically.
We denote a move "up" by the letter U and a move "right" by R.
We will begin by determining which of the 16 squares will not be touched by the face with the circle and then proceed to show that the remaining squares will be touched by the face with the circle.
Since the cube begins on square 1 and the circle is facing out, square 1 will not be touched by the face with the circle.



Similarly, each of the squares 2, 3, 4 can only be reached by moving the cube right (the sequence of moves to reach each of these three squares is R, RR and RRR, respectively), and in each case the circle remains facing out.
Squares 2, 3 and 4 will not be touched by the face with the circle.
Squares 5 and 9 can only be reached by moving the cube up (the sequence of moves to reach each of these two squares is U and UU, respectively).
In either case, the face with the circle will not touch squares 5 and 9.
Square 6 can be reached with two different sequences of moves, RU or UR.
In both cases, the face with the circle will not touch square 6.
Square 10 can be reached with three different sequences of moves, UUR, URU or RUU.
In all three cases, the face with the circle will not touch square 10.
In turns out that these eight squares $(1, 2, 3, 4, 5, 6, 9, 10)$ are the only squares that will not be touched by the face with the circle on any path.
The table below lists sequences of moves that demonstrate how each of the remaining eight squares will be touched by the face with the circle.
The second column lists the sequence of moves, while the third column lists the position of the face with the circle as the cube progresses through the sequence of moves.
We have used the letters F for front, B for back, T for top, O for bottom, L for left, and R for right to indicate the location of the face containing the circle.

Square	Sequence of Moves	Position of the Circle
7	URR	TRO
8	$RURR$	$FTRO$
11	$URUR$	$TRRO$
12	$RURUR$	$FTRRO$
13	UUU	TBO
14	$RUUU$	$FTBO$
15	$RRUUU$	$FFTBO$
16	$RRRUUU$	$FFFTBO$

Therefore, the number of different squares that will not be contacted by the face with the circle on any path is 8.

Answer: (C)

25. We denote the number of tickets of each of the five colours by the first letter of the colour.
We are given that $b : g : r = 1 : 2 : 4$ and that $g : y : o = 1 : 3 : 6$.
Through multiplication by 2, the ratio $1 : 3 : 6$ is equivalent to the ratio $2 : 6 : 12$.
Thus, $g : y : o = 2 : 6 : 12$.
We chose to scale this ratio by a factor of 2 so that the only colour common to the two given ratios, green, now has the same number in both of these ratios.
That is, $b : \mathbf{g} : r = 1 : \mathbf{2} : 4$ and $\mathbf{g} : y : o = \mathbf{2} : 6 : 12$ and since the term g is 2 in each ratio, then we can combine these to form a single ratio, $b : g : r : y : o = 1 : 2 : 4 : 6 : 12$.
This ratios tells us that for every blue ticket, there are 2 green, 4 red, 6 yellow, and 12 orange tickets.
Thus, if there was only 1 blue ticket, then there would be $1 + 2 + 4 + 6 + 12 = 25$ tickets in total.
However, we are given that the box contains 400 tickets in total.
Therefore, the number of blue tickets in the box is $\frac{400}{25} = 16$.
Through multiplication by 16, the ratio $b : g : r : y : o = 1 : 2 : 4 : 6 : 12$ becomes $b : g : r : y : o = 16 : 32 : 64 : 96 : 192$.
(Note that there are $16 + 32 + 64 + 96 + 192 = 400$ tickets in total.)
Next, we must determine the smallest number of tickets that must be drawn to ensure that at least 50 tickets of one colour have been selected.
It is important to consider that up to 49 tickets of any one colour could be selected without being able to *ensure* that 50 tickets of one colour have been selected.
That is, it is possible that the first 195 tickets selected could include exactly 49 orange, 49 yellow, 49 red, all 32 green, and all 16 blue tickets $(49 + 49 + 49 + 32 + 16 = 195)$.
Since all green and blue tickets would have been drawn from the box, the next ticket selected would have to be the 50^{th} orange, yellow or red ticket.
Thus, the smallest number of tickets that must be drawn to ensure that at least 50 tickets of one colour have been selected is 196.

Answer: (D)

Grade 8

1. Evaluating, $10^2 + 10 + 1 = 10 \times 10 + 10 + 1 = 100 + 10 + 1 = 111$.

   ANSWER: (D)

2. Evaluating, $15 - 3 - 15 = 12 - 15 = -3$.

   ANSWER: (D)

3. *Solution 1*
   To determine the smallest number in the set $\left\{\frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{5}{6}, \frac{7}{12}\right\}$, we express each number with a common denominator of 12. The set $\left\{\frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{5}{6}, \frac{7}{12}\right\}$ is equivalent to the set $\left\{\frac{1\times 6}{2\times 6}, \frac{2\times 4}{3\times 4}, \frac{1\times 3}{4\times 3}, \frac{5\times 2}{6\times 2}, \frac{7}{12}\right\}$ or to the set $\left\{\frac{6}{12}, \frac{8}{12}, \frac{3}{12}, \frac{10}{12}, \frac{7}{12}\right\}$.
   The smallest number in this set is $\frac{3}{12}$, so $\frac{1}{4}$ is the smallest number in the original set.

   Solution 2
   With the exception of $\frac{1}{4}$, each number in the set is greater than or equal to $\frac{1}{2}$.
   We can see this by recognizing that the numerator of each fraction is greater than or equal to one half of its denominator.
   Thus, $\frac{1}{4}$ is the only number in the list that is less than $\frac{1}{2}$ and so it must be the smallest number in the set.

   ANSWER: (C)

4. Since Ahmed stops to talk with Kee one quarter of the way to the store, then the remaining distance to the store is $1 - \frac{1}{4} = \frac{3}{4}$ of the total distance.
   Since $\frac{3}{4} = 3 \times \frac{1}{4}$, then the distance Ahmed travelled from Kee to the store, is 3 times the distance Ahmed travelled from the start to reach Kee.
   That is, 12 km is 3 times the distance between the start and Kee.
   So the distance between the start and Kee is $\frac{12}{3} = 4$ km.
   Therefore, the total distance travelled by Ahmed altogether is $4 + 12$ or 16 km.

   ANSWER: (B)

5. Since Jarek multiplies a number by 3 and gets an answer of 90, then the number must be $\frac{90}{3} = 30$. (We may check that $30 \times 3 = 90$.)
   If Jarek instead divides the number 30 by 3, then the answer he gets is $\frac{30}{3} = 10$.

   ANSWER: (B)

6. *Solution 1*
   We first evaluate the product on the left side of the equation.
   $10 \times 20 \times 30 \times 40 \times 50 = 200 \times 30 \times 40 \times 50 = 6000 \times 40 \times 50 = 240\,000 \times 50 = 12\,000\,000$.
   Similarly, we evaluate the product on the right side of the equation.
   $100 \times 2 \times 300 \times 4 \times \square = 200 \times 300 \times 4 \times \square = 60\,000 \times 4 \times \square = 240\,000 \times \square$.
   The left side of the equation is equal to the right side of the equation, so then
   $12\,000\,000 = 240\,000 \times \square$.
   Therefore, the number that goes in the box is $12\,000\,000 \div 240\,000 = 50$.

   Solution 2
   The product of the first two numbers on the left side of the equation is equal to the product of the first two numbers on the right side of the equation.
   That is, $10 \times 20 = 200 = 100 \times 2$.
   Similarly, the product of the next two numbers (the third and fourth numbers) on the left

side of the equation is equal to the product of the next two numbers on the right side of the equation.
That is, $30 \times 40 = 1200 = 300 \times 4$.
Since the products of the first four numbers on each side of the equation are equal, then the final (fifth) number on each side of the equation must be equal.
Therefore, the number that goes in the box is equal to the fifth number on the left side of the equation or 50.

Answer: (C)

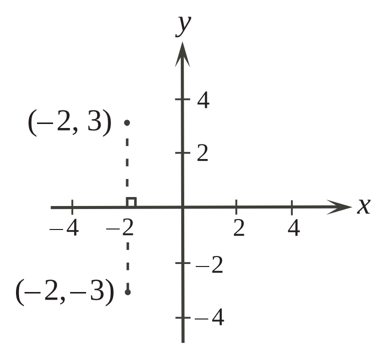
7. There are 26 letters in the English alphabet.
There are 5 different letters, a, l, o, n, s, in Alonso's name.
If one letter is randomly drawn from the bag, then the probability that it is a letter in Alonso's name is $\frac{5}{26}$.

Answer: (C)

8. When Mathy Manuel's autograph dropped 30% in value, it lost $\$100 \times 0.30 = \$30$ of its value.
After this drop, the autograph was worth $\$100 - \$30 = \$70$.
If the autograph then increased by 40% in value, the increase would be $\$70 \times 0.40 = \$28$.
After this increase, the autograph would be worth $\$70 + \$28 = \$98$.

Answer: (A)

9. After reflecting the point $(-2, -3)$ in the x-axis, the x-coordinate of the image will be the same as the x-coordinate of the original point, $x = -2$.
The original point is a distance of 3 below the x-axis.
The image will be the same distance from the x-axis, but above the x-axis.
Thus, the image has y-coordinate 3.
The coordinates of the image point are $(-2, 3)$.



Answer: (E)

10. The value of four nickels is $4 \times 5¢ = 20¢$.
The value of six dimes is $6 \times 10¢ = 60¢$.
The value of two quarters is $2 \times 25¢ = 50¢$.
The ratio of the value of four nickels to six dimes to two quarters is $20 : 60 : 50 = 2 : 6 : 5$.

Answer: (B)

11. Substituting $x = 4$ into $3x + 2y$ we get $3(4) + 2y = 12 + 2y$.
Since this expression $12 + 2y$ is equal to 30, then $2y$ must equal $30 - 12$ or 18.
If $2y = 18$, then y is $18 \div 2$ or 9.

Answer: (E)

12. *Solution 1*
Using order of operations to evaluate, $(2^3)^2 - 4^3 = 8^2 - 4^3 = 64 - 64 = 0$.

Solution 2
We first express 4 as 2^2 and then use exponent rules to evaluate.

$$\begin{aligned}(2^3)^2 - 4^3 &= (2^3)^2 - (2^2)^3 \\ &= 2^{3\times 2} - 2^{2\times 3} \\ &= 2^6 - 2^6 \\ &= 0\end{aligned}$$

ANSWER: (A)

13. Two consecutive Summer Olympics are held 4 years apart.
Each successive Summer Olympics requires an additional 4 years before it is held.
We use this to summarize the minimum time required for the largest number of Summer Olympics to be held, as shown.

Number of Summer Olympics	Minimum Number of Years Apart
2	4
3	8
4	12
5	16
6	20

From the table above, we see that at least 20 years are required to host 6 Summer Olympics, while only 16 years are required to host 5.
(For example, the 5 olympics could be held in years $1, 5, 9, 13$, and 17.)
Therefore, the maximum number of Summer Olympics that can be held during an 18 year period is 5.

ANSWER: (C)

14. Let s be the side length of the cube.
The surface area of a cube is made up of 6 identical squares.
Since the surface area is 54 cm^2, then each of the 6 squares has area $(54 \div 6)\,\text{cm}^2 = 9\,\text{cm}^2$.
The area of a square with side length s is s^2, so $s^2 = 9$ or $s = \sqrt{9} = 3$ cm.
The volume of a cube is given by the product of its length, width and height, which are all equal to s, 3 cm.
Thus the volume of the cube with surface area 54 cm^2 is $3 \times 3 \times 3 = 27$ cm^3.

ANSWER: (D)

15. *Solution 1*
When 10 000 is divided by 13 with the help of a calculator, we get $10\,000 \div 13 = 769.230\ldots$.
Since $769 \times 13 = 9997$ and $10\,000 - 9997 = 3$, then we have a quotient of 769 and a remainder of 3.
That is, $10\,000 = 769 \times 13 + 3$.
This is called a *division statement*.
Similarly, we may determine the remainder when each of the five possible answers (*the dividend*) is divided by 13 (*the divisor*).
We summarize this work in the table below.

Answer	Division Statement	Remainder
(A)	$9\,997 = 769 \times 13 + 0$	0
(B)	$10\,003 = 769 \times 13 + 6$	6
(C)	$10\,013 = 770 \times 13 + 3$	3
(D)	$10\,010 = 769 \times 13 + 0$	0
(E)	$10\,016 = 770 \times 13 + 6$	6

Of the five possible answers, only 10 013 gives a remainder of 3 when divided by 13.

Solution 2
When 10 000 is divided by 13, the remainder is 3.
Thus, adding any multiple of 13 to 10 000 will give the same remainder, 3, upon division by 13.
Of the five choices given, the only answer that differs from 10 000 by a multiple of 13 is 10 013.
When 10 013 is divided by 13, the remainder is also 3.

ANSWER: (C)

16. Since it is equally likely that a child is a boy as it is that a child is a girl, then the probability that any child is a girl is $\frac{1}{2}$.
The probability of any child being born a girl is independent of the number or gender of any children already born into the family.
That is, the probability of the second child being a girl is also $\frac{1}{2}$, as is the probability of the third child being a girl.
Therefore, the probability that all 3 children are girls is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

ANSWER: (E)

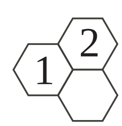
17. Since $PQRS$ is a rectangle, then $\angle PQR = 90^\circ$.
Since $\angle PQR = \angle PQS + \angle RQS$, then $90^\circ = (5x)^\circ + (4x)^\circ$ or $90 = 9x$ and so $x = 10$.
In $\triangle SRQ$, $\angle SRQ = 90^\circ$ and $\angle RQS = (4x)^\circ = 40^\circ$.
Therefore, $\angle QSR = y^\circ = 180^\circ - 90^\circ - 40^\circ = 50^\circ$.
So $y = 50$.

ANSWER: (D)

18. Sally's answer is $\frac{2}{3} \times 1\frac{1}{2} = \frac{2}{3} \times \frac{3}{2} = \frac{6}{6} = 1$.
Jane's answer is $\frac{2}{3} + 1\frac{1}{2} = \frac{2}{3} + \frac{3}{2} = \frac{4}{6} + \frac{9}{6} = \frac{13}{6} = 2\frac{1}{6}$.
The difference between Sally's answer and Jane's answer is $2\frac{1}{6} - 1$ or $1\frac{1}{6}$.

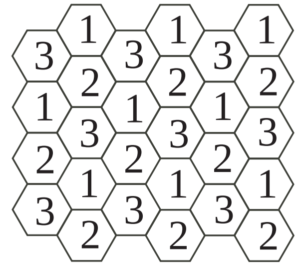
ANSWER: (B)

19. Since we seek the minimum number of colours that Serena can use to colour the hexagons, we first determine if it is possible for her to use only two colours (using only one colour is not possible). We will use the numbers $1, 2, 3$ to represent distinct (different) colours.
We begin by choosing any group of three hexagons in which each pair of hexagons share a side, as shown.
We colour two of the hexagons shown with colours 1 and 2 (since they share a side).



Each of these two coloured hexagons share a side with the third hexagon which therefore can not be coloured 1 or 2.
Thus, the minimum number of colours that Serena can use is at least three.

Next, we determine if the entire tiling can be coloured using only three colours.
One possible colouring of the tiles that uses only three colours is shown to the right.



While other colourings of the tiles are possible, Serena is able to use only three colours and ensure that no two hexagons that share a side are the same colour.
There are many nice patterns of the colours in this tiling.
Can you find a different colouring of the tiles that uses only three colours?

Answer: (E)

20. *Solution 1*
Suppose that the cost of one book, in dollars, is C.
Then Christina has $\frac{3}{4}C$ and Frieda has $\frac{1}{2}C$.
Combining their money, together Christina and Frieda have $\frac{3}{4}C + \frac{1}{2}C = \frac{3}{4}C + \frac{2}{4}C = \frac{5}{4}C$.
If the book was \$3 cheaper, then the cost to buy one book would be $C - 3$.
If the cost of one book was $C - 3$, then the cost to buy two at this price would be $2(C - 3)$ or $2C - 6$.
Combined, Christina and Frieda would have enough money to buy exactly two books at this reduced price.
Thus, $2C - 6 = \frac{5}{4}C$.
Solving,

$$\begin{aligned} 2C - 6 &= \tfrac{5}{4}C \\ 2C - \tfrac{5}{4}C &= 6 \\ \tfrac{8}{4}C - \tfrac{5}{4}C &= 6 \\ \tfrac{3}{4}C &= 6 \end{aligned}$$

Since $\frac{3}{4}$ of 8 is 6, then $C = 8$.
Therefore, the original price of the book is \$8.

Solution 2
We proceed by systematically trying the five multiple choice answers given.

Initial cost of the book	Combined money for Christina and Frieda	Reduced cost of the book	Number of books they may buy
\$4	$\frac{3}{4}$ of \$4 + $\frac{1}{2}$ of \$4 = \$3 + \$2 = \$5	\$1	$5 \div 1 = 5$
\$16	$\frac{3}{4}$ of \$16 + $\frac{1}{2}$ of \$16 = \$12 + \$8 = \$20	\$13	$20 \div 13 = 1.53\ldots$
\$12	$\frac{3}{4}$ of \$12 + $\frac{1}{2}$ of \$12 = \$9 + \$6 = \$15	\$9	$15 \div 9 = 1.66\ldots$
\$10	$\frac{3}{4}$ of \$10 + $\frac{1}{2}$ of \$10 = \$7.50 + \$5 = \$12.50	\$7	$12.50 \div 7 = 1.78\ldots$
\$8	$\frac{3}{4}$ of \$8 + $\frac{1}{2}$ of \$8 = \$6 + \$4 = \$10	\$5	$10 \div 5 = 2$

We see that if the original price of the book is \$8, Christina and Frieda are able to buy exactly two copies of the book at the reduced price.
Thus, the initial cost of the book is \$8.

Answer: (E)

21. We use labels m and n in the first column, as shown in the top grid.
Then, $5, m, n, 23$ are four terms of an arithmetic sequence.
Since 5 and 23 are three terms apart in this sequence, and their difference is $23 - 5 = 18$, the constant added to one term to obtain the next term in the first column is $\frac{18}{3}$ or 6.
That is, $m = 5 + 6 = 11$, and $n = 11 + 6 = 17$.
(We confirm that the terms $5, 11, 17, 23$ do form an arithmetic sequence.)

5			
m			1211
n		1013	
23	x		

We use labels p and q in the second row, as shown in the middle grid.
Then, $11, p, q, 1211$ are four terms of an arithmetic sequence.
Since 11 and 1211 are three terms apart in this sequence, and their difference is $1211 - 11 = 1200$, the constant added to one term to obtain the next term in the second row is $\frac{1200}{3}$ or 400.
That is, $p = 11 + 400 = 411$, and $q = 411 + 400 = 811$.
(We confirm that the terms $11, 411, 811, 1211$ do form an arithmetic sequence.)

5			
11	p	q	1211
17		1013	
23	x		

We use label r in the third row, as shown in the third grid.
Then, $17, r, 1013$ are three terms of an arithmetic sequence.
Since 17 and 1013 are two terms apart in this sequence, and their difference is $1013 - 17 = 996$, the constant added to one term to obtain the next term in the third row is $\frac{996}{2}$ or 498.
That is, $r = 17 + 498 = 515$.
(We confirm that the terms $17, 515, 1013$ do form an arithmetic sequence.)

5			
11	411	811	1211
17	r	1013	
23	x		

Finally, in the second column the terms $411, 515, x$ are three terms of an arithmetic sequence.
Since 411 and 515 are one term apart in this sequence, the constant added to one term to obtain the next term in the second column is $515 - 411 = 104$.
Then, $x = 515 + 104 = 619$.
The completed grid is as shown.

5	307	609	911
11	411	811	1211
17	515	1013	1511
23	619	1215	1811

ANSWER: (B)

22. In $\triangle FGH$, $FG = GH = x$ since they are both radii of the same circle.
By the Pythagorean Theorem, $FH^2 = FG^2 + GH^2 = x^2 + x^2$, or $FH^2 = 2x^2$, and so $(\sqrt{8})^2 = 2x^2$ or $2x^2 = 8$ and $x^2 = 4$, so then $x = 2$ (since $x > 0$).
FG, GH and arc FH form a sector of a circle with centre G and radius GH.
Since $\angle FGH = 90^\circ$, which is $\frac{1}{4}$ of 360°, then the area of this sector is one quarter of the area of the circle with centre G and radius $GH = FG = 2$.
The shaded area is equal to the area of sector FGH minus the area of $\triangle FGH$.
The area of sector FGH is $\frac{1}{4}\pi(2)^2$ or $\frac{1}{4}\pi(4)$ or π.
The area of $\triangle FGH$ is $\frac{FG \times GH}{2}$ or $\frac{2\times 2}{2}$, so 2.
Therefore, the area of the shaded region is $\pi - 2$.

ANSWER: (A)

23. *Solution 1*

In the first race, when Azarah crossed the finish line, Charlize was 20 m behind or Charlize had run 80 m.
Since Azarah and Charlize each travelled these respective distances in the same amount of time, then the ratio of their speeds is equal to the ratio of their distances travelled, or 100 : 80.
Similarly in the second race, when Charlize crossed the finish line, Greg was 10 m behind or Greg had run 90 m.
Since Charlize and Greg each travelled these respective distances in the same amount of time, then the ratio of their speeds is equal to the ratio of their distances travelled, or 100 : 90.
Let A, C and G represent Azarah's, Charlize's and Greg's speeds, respectively.
Then, $A : C = 100 : 80 = 25 : 20$ and $C : G = 100 : 90 = 20 : 18$.
Therefore, $A : C : G = 25 : 20 : 18$ and $A : G = 25 : 18 = 100 : 72$.
Over equal times, the ratio of their speeds is equal to the ratio of their distances travelled.
Therefore, when Azarah travels 100 m, Greg travels 72 m.
When Azarah crossed the finish line, Greg was $100 - 72 = 28$ m behind.

Solution 2

In the first race, when Azarah crossed the finish line, Charlize was 20 m behind or Charlize had run 80 m.
Since Azarah and Charlize each travelled these respective distances in the same amount of time, then the ratio of their speeds is equal to the ratio of their distances travelled, or 100 : 80.
That is, Charlize's speed is 80% of Azarah's speed.
Similarly, Greg's speed is 90% of Charlize's speed.
Therefore, Greg's speed is 90% of Charlize's speed which is 80% of Azarah's speed, or Greg's speed is 90% of 80% of Azarah's speed.
Since 90% of 80% is equivalent to $0.90 \times 0.80 = 0.72$ or 72%, then Greg's speed is 72% of Azarah's speed.
When Azarah ran 100 m (crossed the finish line), Greg ran 72% of 100 m or 72 m in the same amount of time.
When Azarah crossed the finish line, Greg was $100 - 72 = 28$ m behind.

Answer: (C)

24. The length of the longest side, z, is less than half of the perimeter 57.
Thus, $z < \frac{57}{2}$ or $z < 28\frac{1}{2}$. Since z is an integer then $z \leq 28$.
When $z = 28$, $x + y = 57 - 28 = 29$.
We list all possible values for x and y in the table below given that $z = 28$ and $x < y < z$.

y	27	26	25	24	23	22	21	20	19	18	17	16	15
x	2	3	4	5	6	7	8	9	10	11	12	13	14

Systematically, we continue decreasing the value of z and listing all possible values for x and y.
When $z = 27$, $x + y = 57 - 27 = 30$.

y	26	25	24	23	22	21	20	19	18	17	16
x	4	5	6	7	8	9	10	11	12	13	14

When $z = 26$, $x + y = 57 - 26 = 31$.

y	25	24	23	22	21	20	19	18	17	16
x	6	7	8	9	10	11	12	13	14	15

When $z = 25$, $x + y = 57 - 25 = 32$.

y	24	23	22	21	20	19	18	17
x	8	9	10	11	12	13	14	15

When $z = 24$, $x + y = 57 - 24 = 33$.

y	23	22	21	20	19	18	17
x	10	11	12	13	14	15	16

When $z = 23$, $x + y = 57 - 23 = 34$.

y	22	21	20	19	18
x	12	13	14	15	16

When $z = 22$, $x + y = 57 - 22 = 35$.

y	21	20	19	18
x	14	15	16	17

When $z = 21$, $x + y = 57 - 21 = 36$.

y	20	19
x	16	17

When $z = 20$, $x + y = 57 - 20 = 37$.

y	19
x	18

The next smallest value for z is 19 and in this case $x + y = 57 - 19 = 38$.
However, if $x + y = 38$ then at least one of x or y must be 19 or larger.
This is not possible since $z = 19$ and $x < y < z$.
Therefore, 20 is the smallest possible value for z and we have listed the side lengths of all possible triangles above.
Counting, we see that there are $13+11+10+8+7+5+4+2+1 = 61$ possible triangles that satisfy the given conditions.

ANSWER: (B)

25. *Solution 1*

Let b represent the number of boys initially registered in the class.
Let g represent the number of girls initially registered in the class.
When 11 boys transferred into the class, the number of boys in the class was $b + 11$.
When 13 girls transferred out of the class, the number of girls in the class was $g - 13$.
The ratio of boys to girls in the class at this point was $1 : 1$.
That is, the number of boys in the class was equal to the number of girls in the class, or $b + 11 = g - 13$ and so $g = b + 24$.
Since there were at least 66 students initially registered in the class, then $b + g \geq 66$.
Substituting $g = b + 24$, $b + g$ becomes $b + (b + 24) = 2b + 24$, and so $2b + 24 \geq 66$ or $2b \geq 42$, so $b \geq 21$.
At this point, we may use the conditions that $g = b + 24$ and $b \geq 21$ to determine which of the 5 answers given is not possible.
Each of the 5 answers represents a possible ratio of $b : g$ or $b : (b + 24)$ (since $g = b + 24$).
We check whether $b : (b + 24)$ can equal each of the given ratios while satisfying the condition that $b \geq 21$.
In (A), we have $b : (b + 24) = 4 : 7$ or $\frac{b}{b+24} = \frac{4}{7}$ or $7b = 4b + 96$ and then $3b = 96$, so $b = 32$.
In (B), we have $b : (b + 24) = 1 : 2$ or $\frac{b}{b+24} = \frac{1}{2}$ or $2b = b + 24$, so $b = 24$.
In (C), we have $b : (b + 24) = 9 : 13$ or $\frac{b}{b+24} = \frac{9}{13}$ or $13b = 9b + 216$ and then $4b = 216$, so $b = 54$.
In (D), we have $b : (b + 24) = 5 : 11$ or $\frac{b}{b+24} = \frac{5}{11}$ or $11b = 5b + 120$ and then $6b = 120$, so $b = 20$.

In (E), we have $b : (b + 24) = 3 : 5$ or $\frac{b}{b+24} = \frac{3}{5}$ or $5b = 3b + 72$ and then $2b = 72$, so $b = 36$.
Thus, the only ratio that does not satisfy the condition that $b \geq 21$ is $5 : 11$ (since $b = 20$).

Solution 2

As in Solution 1, we establish the conditions that $g = b + 24$ and $b \geq 21$.
Again, the ratio $b : g$ or $\frac{b}{g}$ then becomes $\frac{b}{b+24}$ (since $g = b + 24$).
Since $b \geq 21$, then b can equal $21, 22, 23, 24, \ldots$, but the smallest possible value for b is 21.
When $b = 21$, $\frac{b}{b+24}$ becomes $\frac{21}{21+24} = \frac{21}{45} = 0.4\overline{6}$.
When $b = 22$, $\frac{b}{b+24}$ becomes $\frac{22}{22+24} = \frac{22}{46} \approx 0.478$.
When $b = 23$, $\frac{b}{b+24}$ becomes $\frac{23}{23+24} = \frac{23}{47} \approx 0.489$.
When $b = 24$, $\frac{b}{b+24}$ becomes $\frac{24}{24+24} = \frac{24}{48} = 0.5$.
As b continues to increase, the value of the ratio $\frac{b}{b+24}$ or $\frac{b}{g}$ continues to increase.
Can you verify this for yourself?
Thus, the smallest possible value of the ratio $\frac{b}{g}$ is $\frac{21}{45}$ or $0.4\overline{6}$.
Comparing this ratio to the 5 answers given, we determine that $\frac{5}{11} = 0.\overline{45} < 0.4\overline{6} = \frac{21}{45}$, which is not possible.
(You may also confirm that each of the other 4 answers given are all greater than $\frac{21}{45}$ and are obtainable, as in Solution 1.)
Therefore, the only ratio of boys to girls which is not possible is (D).

ANSWER: (D)

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Grade 8 solutions follow the Grade 7 solutions

2012 Gauss Contests

(Grades 7 and 8)

Wednesday, May 16, 2012
(in North America and South America)

Thursday, May 17, 2012
(outside of North America and South America)

Solutions

Centre for Education in Mathematics and Computing Faculty and Staff

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Lloyd Auckland
Terry Bae
Steve Brown
Ersal Cahit
Karen Cole
Jennifer Couture
Serge D'Alessio
Frank DeMaio
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Grade 7

1. Evaluating, $202 - 101 + 9 = 101 + 9 = 110$.

Answer: (B)

2. Written numerically, the number 33 million is 33 000 000.

Answer: (D)

3. Each of the numbers $1, 2, 3, 4, 5$, and 6 is equally likely to appear when the die is rolled.
Since there are six numbers, then each has a one in six chance of being rolled.
The probability of rolling a 5 is $\frac{1}{6}$.

Answer: (B)

4. A positive fraction increases in value as its numerator increases and also increases in value as its denominator decreases.
Since the numerators of all five fractions are equal, then the largest of these is the fraction with the smallest denominator.
The largest fraction in the set $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{10}\right\}$ is $\frac{1}{2}$.

Answer: (A)

5. *Solution 1*
The angle marked $\square$ is vertically opposite the angle measuring 60°.
Since vertically opposite angles are equal, then the measure of the angle marked $\square$ is also 60°.

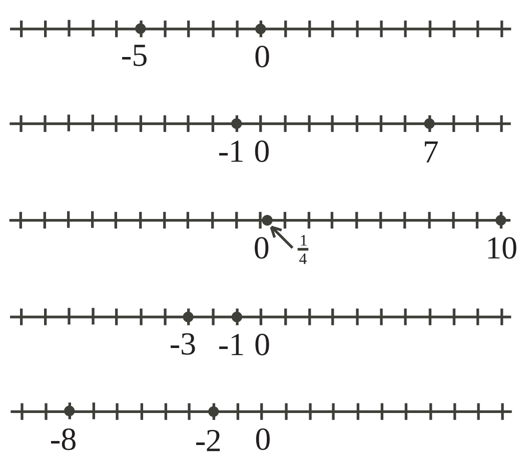
Solution 2
The measure of a straight angle is 180°.
Together, the angle measuring 120° and the angle marked $\square$ make up a straight angle.
That is, $120^\circ + \square = 180^\circ$.
Therefore, the angle marked $\square$ is 60°.

Answer: (A)

6. Since 15 times the number is 300, then the number equals 300 divided by 15, or 20.

Answer: (A)

7. Consider placing each of the two numbers from each answer on a number line.
Numbers decrease in value as we move to the left along a number line.
Since the first number listed in the pair must be less than the second, we want the first number to be to the left of the second number on the number line.
The five possible answers are each given on the number lines shown.
The last number line is the only one for which the first number listed (-8) is positioned to the left of (is less than) the second number (-2).



Answer: (E)

8. Since Bailey scores on 6 of her 8 shots, then she misses on $8 - 6 = 2$ shots.
If she misses on $\frac{2}{8} = \frac{1}{4}$ of her shots, then the percentage of shots that she does not score is $\frac{1}{4} \times 100\% = 25\%$.

ANSWER: (E)

9. The number of visits to Ben's website from Monday to Friday can be read from the graph.
These are: $300, 400, 300, 200, 200$.
The mean number of visits per day is found by adding these five totals and then dividing the sum by 5.
Thus, the mean is $(300 + 400 + 300 + 200 + 200) \div 5 = 1400 \div 5$ or 280.
The mean number of visits per day to Ben's website over the 5 days is between 200 and 300.

ANSWER: (C)

10. The graph shows that the vehicle travels at a constant speed of 20 m/s.
Travelling at 20 m/s, it will take the vehicle $100 \div 20 = 5$ seconds to travel 100 metres.

ANSWER: (E)

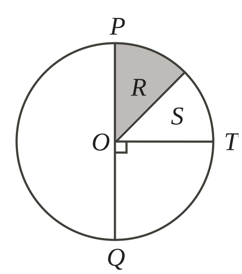
11. Since the four sides of a square are equal in length and the perimeter is 36 cm, then each side has length $\frac{36}{4} = 9$ cm.
The area of the square is the product of the length and width, which are each equal to 9 cm.
Therefore, the area of the square, in cm^2, is $9 \times 9 = 81$.

ANSWER: (B)

12. Since $\frac{14+1}{3+1} = \frac{15}{4}$ and $\frac{21}{4} - \frac{5}{4} - \frac{1}{4} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$, then answers (B) and (E) both simplify to $\frac{15}{4}$.
Written as a mixed fraction, $\frac{15}{4}$ is equal to $3\frac{3}{4}$.
Since $3.75 = 3\frac{3}{4} = 3 + \frac{3}{4}$, then answers (A) and (C) both simplify to $3\frac{3}{4}$ and thus are equivalent to $\frac{15}{4}$.
Simplifying answer (D), $\frac{5}{4} \times \frac{3}{4} = \frac{5\times3}{4\times4} = \frac{15}{16}$.
Thus, $\frac{5}{4} \times \frac{3}{4}$ is not equal to $\frac{15}{4}$.

ANSWER: (D)

13. Since PQ passes through centre O, then it is a diameter of the circle.
Since $\angle QOT = 90°$, then $\angle POT = 180° - 90° = 90°$.
Thus, the area of sector POT is $\frac{90°}{360°} = \frac{1}{4}$ or 25% of the area of the circle.
Since the areas labelled R and S are equal, then each is $25\% \div 2 = 12.5\%$ of the area of the circle.
Therefore, a spin will stop on the shaded region 12.5% of the time.



ANSWER: (E)

14. To make the difference as large as possible, we make one number as large as possible and the other number as small as possible.
The tens digit of a number contributes more to its value than its units digit.
Thus, we construct the largest possible number by choosing 8 (the largest digit) to be its tens digit, and by choosing 6 (the second largest digit) to be the ones digit.
Similarly, we construct the smallest possible number by choosing 2 (the smallest digit) to be its tens digit, and 4 (the second smallest digit) to be its ones digit.
The largest possible difference is $86 - 24 = 62$.

ANSWER: (B)

15. Since 1 mm of snow falls every 6 minutes, then 10 mm will fall every $6 \times 10 = 60$ minutes.
Since 10 mm is 1 cm and 60 minutes is 1 hour, then 1 cm of snow will fall every 1 hour.
Since 1 cm of snow falls every 1 hour, then 100 cm will fall every $1 \times 100 = 100$ hours.

ANSWER: (E)

16. Both 1 and 2012 are obvious positive integer factors of 2012.
Since 2012 is an even number and $2012 \div 2 = 1006$, then both 2 and 1006 are factors of 2012.
Since 1006 is also even then 2012 is divisible by 4.
Since $2012 \div 4 = 503$, then both 4 and 503 are factors of 2012.
We are given that 503 is a prime number; thus there are no additional factors of 503 and hence there are no additional factors of 2012.
The factors of 2012 are 1 and 2012, 2 and 1006, 4 and 503.
Therefore, there are 6 positive integers that are factors of 2012.

ANSWER: (D)

17. *Solution 1*
Since the ratio of boys to girls is $8 : 5$, then for every 5 girls there are 8 boys.
That is, the number of girls at Gauss Public School is $\frac{5}{8}$ of the number of boys.
Since the number of boys at the school is 128, the number of girls is $\frac{5}{8} \times 128 = \frac{640}{8} = 80$.
The number of students at the school is the number of boys added to the number of girls or $128 + 80 = 208$.

Solution 2
Since the ratio of boys to girls is $8 : 5$, then for every 8 boys there are $8 + 5 = 13$ students.
That is, the number of students at Gauss Public School is $\frac{13}{8}$ of the number of boys.
Since the number of boys at the school is 128, the number of students is $\frac{13}{8} \times 128 = \frac{1664}{8} = 208$.

ANSWER: (C)

18. In turn, we may use each of the three known scales to find a way to balance a circle, a diamond and a triangle.
Since many answers are possible, we must then check our solution to see if it exists among the five answers given.
First consider the scale at the top right.
A diamond and a circle are balanced by a triangle.
If we were to add a triangle to both sides of this scale, then it would remain balanced and the right side of this scale would contain what we are trying to balance, a circle, a diamond and a triangle.
That is, a circle, a diamond and a triangle are balanced by two triangles.
However, two triangles is not one of the five answers given.
Next, consider the scale at the top left.
A triangle and a circle are balanced by a square.
If we were to add a diamond to both sides of this scale, then it would remain balanced and the left side of this scale would contain what we are trying to balance, a circle, a diamond and a triangle.
That is, a circle, a diamond and a triangle are balanced by a square and a diamond.
This answer is given as one of the five answers.

In the context of a multiple choice contest, we do not expect that students will verify that the other four answers do not balance a circle, a diamond and a triangle. However, it is worth noting that it can be shown that they do not.

ANSWER: (D)

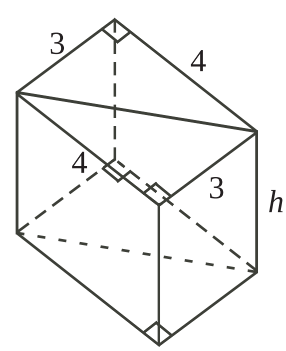
19. In an ordered list of five integers, the median is the number in the middle or third position.
Thus if we let the set of integers be a, b, c, d, e, ordered from smallest to largest, then $c = 18$.
Since the average is fixed (at 20), e (the largest number in the set) is largest when a, b and d are as small as possible.
Since the numbers in the set are different positive integers, the smallest that a can be is 1 and the smallest that b can be is 2.
Our set of integers is now $1, 2, 18, d, e$.
Again, we want d to be as small as possible, but it must be larger than the median 18.
Therefore, $d = 19$.
Since the average of the 5 integers is 20, then the sum of the five integers is $5 \times 20 = 100$.
Thus, $1 + 2 + 18 + 19 + e = 100$ or $40 + e = 100$, and so $e = 60$.
The greatest possible integer in the set is 60.

ANSWER: (A)

20. If either Chris or Mark says, "Tomorrow, I will lie." on a day that he tells a lie, then it actually means that tomorrow he will tell the truth (since he is lying).
This can only occur when he lies and then tells the truth on consecutive days.
For Chris, this only happens on Sunday, since he lies on Sunday but tells the truth on Monday.
For Mark, this only happens on Thursday, since he lies on Thursday but tells the truth on Friday.
Similarly, if either Chris or Mark says, "Tomorrow, I will lie." on a day that they tell the truth, then it means that tomorrow they will lie (since they are telling the truth).
This can only occur when they tell the truth and then lie on consecutive days.
For Chris, this only happens on Thursday, since he tells the truth on Thursday but lies on Friday.
For Mark, this only happens on Monday, since he tells the truth on Monday but lies on Tuesday.
Therefore, the only day of the week that they would both say, "Tomorrow, I will lie.", is Thursday.

ANSWER: (B)

21. The triangular prism given can be created by slicing the 3 cm by 4 cm base of a rectangular prism with equal height across its diagonal.
That is, the volume of the triangular prism in question is one half of the volume of the rectangular prism shown.
Since the volume of the triangular prism is 120 cm^3, then the volume of this rectangular prism is $2 \times 120 = 240$ cm^3.
The volume of the rectangular prism equals the area of its base 3×4 times its height, h.
Since the volume is 240, then $3 \times 4 \times h = 240$ or $12h = 240$, so $h = \frac{240}{12} = 20$.
Since the height of this rectangular prism is equal to the height of the triangular prism in question, then the required height is 20 cm.



ANSWER: (B)

22. Without changing the overall class mean, we may consider that the class has 100 students.
That is, 20 students got 0 questions correct, 5 students got 1 question correct, 40 students got 2 questions correct, and 35 students got 3 questions correct.
The combined number of marks achieved by all 100 students in the class is then,

$$(20 \times 0) + (5 \times 1) + (40 \times 2) + (35 \times 3) = 0 + 5 + 80 + 105 = 190.$$

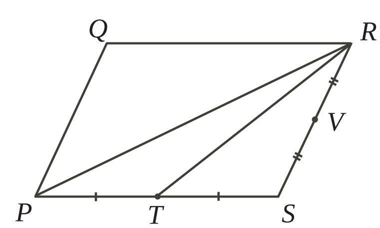
Since the 100 students earned a total of 190 marks, then the overall class average was $\frac{190}{100} = 1.9$.

Answer: (B)

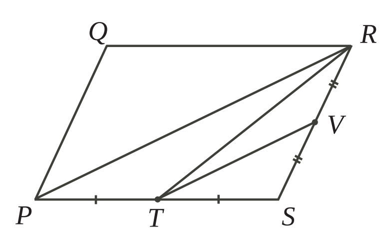
23. The units digit of any product is given by the units digit of the product of the units digits of the numbers being multiplied.
For example, the units digit of the product 12×53 is given by the product 2×3, so it is 6.
Thus to determine the units digit of N, we need only consider the product of the units digits of the numbers being multiplied to give N.
The units digits of the numbers in the product N are $1, 3, 7, 9, 1, 3, 7, 9, \ldots$, and so on.
That is, the units digits $1, 3, 7, 9$ are repeated in each group of four numbers in the product.
There are ten groups of these four numbers, $1, 3, 7, 9$, in the product.
We first determine the units digit of the product $1 \times 3 \times 7 \times 9$.
The units digit of 1×3 is 3.
The units digit of the product 3×7 is 1 (since $3 \times 7 = 21$).
The units digit of 1×9 is 9.
Therefore, the units digit of the product $1 \times 3 \times 7 \times 9$ is 9.
(We could have calculated the product $1 \times 3 \times 7 \times 9 = 189$ to determine the units digit.)
This digit 9 is the units digits of the product of each group of four successive numbers in N.
Thus, to determine the units digit of N we must determine the units digit of
$9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9$.
This product is equal to $81 \times 81 \times 81 \times 81 \times 81$.
Since we are multiplying numbers with units digit 1, then the units digit of the product is 1.

Answer: (A)

24. Diagonal PR divides parallelogram $PQRS$ into two equal areas.
That is, the area of $\triangle PRS$ is one half of the area of parallelogram $PQRS$, or 20.
In $\triangle PRS$, we construct *median* RT.
(A *median* is a line segment that joins a vertex of a triangle to the midpoint of its opposite side.)
Median RT divides $\triangle PRS$ into two equal areas since its base, PS, is halved while the height remains the same.
That is, the area of $\triangle TRS$ is one half of the area of $\triangle PRS$, or 10.



Similarly, we construct median TV in $\triangle TRS$, as shown.
Median TV divides $\triangle TRS$ into two equal areas since its base, SR, is halved while the height remains the same.
That is, the area of $\triangle TVS$ is one half of the area of $\triangle TRS$, or 5.
The area of $PRVT$ is equal to the area of $\triangle TVS$ subtracted from the area of $\triangle PRS$, or $20 - 5 = 15$.



Answer: (C)

25. A very useful and well-known formula allows us to determine the sum of the first n positive integers, $1+2+3+4+\cdots+(n-1)+n$.

The formula says that this sum, $1+2+3+4+\cdots+(n-1)+n$, is equal to $\frac{n(n+1)}{2}$ (justification of this formula is included at the end of the solution).

For example, if $n=6$ then $1+2+3+4+5+6=\frac{6(6+1)}{2}=\frac{6\times 7}{2}=\frac{42}{2}=21$.

You can check that this formula gives the correct sum, 21, by mentally adding the positive integers from 1 to 6.

In the table given, there is 1 number in Row 1, there are 2 numbers in Row 2, 3 numbers in Row 3, and so on, with n numbers in Row n.

The numbers in the rows list the positive integers in order beginning at 1 in Row 1, with each new row containing one more integer than the previous row.

Thus, the last number in each row is equal to the sum of the number of numbers in the table up to that row.

For example, the last number in Row 4 is 10, which is equal to the sum of the number of numbers in rows $1, 2, 3$, and 4.

But the number of numbers in each row is equal to the row number.

So 10 is equal to the sum $1+2+3+4$.

That is, the last number in Row n is equal to the sum $1+2+3+4+\cdots+(n-1)+n$, which is equal to $\frac{n(n+1)}{2}$.

We may now use this formula to determine in what row the number 2000 appears.

Using trial and error, we find that since $\frac{62(63)}{2}=1953$, then the last number in Row 62 is 1953.

Similarly, since $\frac{63(64)}{2}=2016$, then the last number in Row 63 is 2016.

Since 2000 is between 1953 and 2016, then 2000 must appear somewhere in Row 63.

To find how many integers less than 2000 are in the column that contains the number 2000, we must determine in which column the number 2000 appears.

Further, we must determine how many numbers there are in that column above the 2000 (since all numbers in that column in rows below the 63^{rd} are larger than 2000).

We know that 2016 is the last number in Row 63 and since it is the last number, it will have no numbers in the column above it.

Moving backward (to the left) from 2016, the number 2015 will have 1 number in the column above it, 2014 will have 2 numbers in the column above it, and so on.

That is, if we move k numbers to the left of 2016, that table entry will have k numbers in the column above it.

In other words, if the number $2016-k$ appears in Row 63, then there are k integers less than it in the column that contains it.

Since we know that 2000 appears in this 63^{rd} row, then $2016-k=2000$ means that $k=16$.

Thus, there are 16 integers less than 2000 in the column that contains the number 2000.

<u>Verification of the Formula:</u> $1+2+3+4+\cdots+(n-1)+n=\frac{n(n+1)}{2}$

If we let the sum of the first n positive integers be S, then $S=1+2+3+4+\cdots+(n-1)+n$.

If this same sum is written in the reverse order, then $S=n+(n-1)+(n-2)+(n-3)+\cdots+2+1$.

Adding the right sides of these two equations,

$$\begin{array}{ccccccccccccccc}
 & 1 & + & 2 & + & 3 & + & 4 & + & \ldots & + & (n-1) & + & n \\
+ & n & + & (n-1) & + & (n-2) & + & (n-3) & + & \ldots & + & 2 & + & 1 \\
\hline
= & (n+1) & + & (n+1) & + & (n+1) & + & (n+1) & + & \ldots & + & (n+1) & + & (n+1)
\end{array}$$

In this sum there are n occurrences of $(n+1)$, hence the sum is $n(n+1)$.
However, this sum represents $S+S$ or $2S$, so if $2S = n(n+1)$ then $S = \dfrac{n(n+1)}{2}$.

ANSWER: (D)

Grade 8

1. Using the correct order of operations, $3 \times (3+3) \div 3 = 3 \times 6 \div 3 = 18 \div 3 = 6$.

   Answer: (A)

2. Each of the numbers $1, 2, 3, 4, 5$, and 6 are equally likely to appear when the die is rolled.
   Since there are 6 numbers, then each has a one in six chance of being rolled.
   The probability of rolling a five is $\frac{1}{6}$.

   Answer: (B)

3. Written first as a fraction, fifty-six hundredths is $\frac{56}{100}$.
   The decimal equivalent of this fraction can be determined by dividing, $\frac{56}{100} = 56 \div 100 = 0.56$.

   Answer: (D)

4. Since P, Q, R lie in a straight line, $\angle PQR = 180^\circ$.
   Therefore, $42^\circ + x^\circ + x^\circ = 180^\circ$ or $42 + 2x = 180$ or $2x = 180 - 42 = 138$, and so $x = 69$.

   Answer: (A)

5. The number of 5¢ coins needed to make one dollar (100¢) is $\frac{100}{5} = 20$.
   The number of 10¢ coins needed to make one dollar (100¢) is $\frac{100}{10} = 10$.
   Therefore, it takes $20 - 10 = 10$ more 5¢ coins than it takes 10¢ coins to make one dollar.

   Answer: (B)

6. Once each of 12 equal parts is cut into 2 equal pieces, there are $12 \times 2 = 24$ equal pieces of pizza.
   Ronald eats 3 of these 24 equal pieces.
   Therefore, Ronald eats $\frac{3}{24}$ or $\frac{1}{8}$ of the pizza.

   Answer: (E)

7. Since the rectangular sheet of paper measures 25 cm by 9 cm, its area is 25×9 or 225 cm^2.
   A square sheet of paper has equal length and width.
   If the length and width of the square is s, then the area of the square is $s \times s$ or s^2.
   Therefore $s^2 = 225$, or $s = \sqrt{225} = 15$ (since s is a positive length).
   The dimensions of the square sheet of paper having the same area are 15 cm by 15 cm.

   Answer: (A)

8. Since the number in question (0.2012) is in decimal form, it is easiest to determine into which of the 5 given ranges it falls by converting the ranges into decimal form also.
   Converting, $\frac{1}{10}$ is 0.1, $\frac{1}{5}$ is 0.2, $\frac{1}{4}$ is 0.25, $\frac{1}{3}$ is $0.\overline{3}$, and $\frac{1}{2}$ is 0.5.
   Since 0.2012 is greater than 0.2 but less than 0.25, it is between $\frac{1}{5}$ and $\frac{1}{4}$.

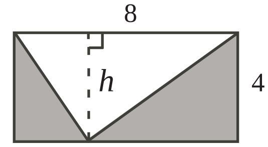
   Answer: (C)

9. Substituting $x = 2$, we get

$$3^x - x^3 = 3^2 - 2^3 = (3 \times 3) - (2 \times 2 \times 2) = 9 - 8 = 1.$$

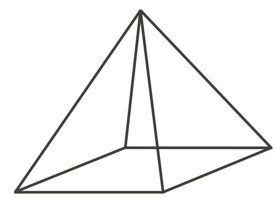
   Answer: (D)

10. The area of the rectangle is 8×4, or 32.
The unshaded portion of the rectangle is a triangle with base of length 8 and height h, as shown.
Since the dotted line (the height) with length h is parallel to the vertical side of the rectangle, then $h = 4$.
Thus, the area of the unshaded triangle is $\frac{1}{2} \times 8 \times 4 = 4 \times 4 = 16$.
The area of the shaded region is the area of the rectangle minus the area of the unshaded triangle.
Thus, the area of the shaded region is $32 - 16 = 16$.



Answer: (B)

11. Since the pyramid has a square base, the base of the pyramid has 4 edges (one for each side of the square).
An edge joins each of the 4 vertices of the square to the apex of the pyramid, as shown.
In total, a pyramid with a square base has 8 edges.



Answer: (A)

12. Since 1 mm of snow falls every 6 minutes, then 10 mm will fall every $6 \times 10 = 60$ minutes.
Since 10 mm is 1 cm and 60 minutes is 1 hour, then 1 cm of snow will fall every 1 hour.
Since 1 cm of snow falls every 1 hour, then 100 cm will fall every $1 \times 100 = 100$ hours.

Answer: (E)

13. The mode is the number that occurs most frequently in a set of numbers.
The mode of the three numbers is 9.
Thus, at least two of the three numbers must equal 9 otherwise there would be three different numbers and so three modes.
If all three of the numbers were equal to 9, then the average of the three numbers could not be 7.
Therefore, two of the three numbers are equal to 9.
If the third number is x, then the three numbers are $x, 9$ and 9.
Since the average of the three numbers is 7, then $\frac{x+9+9}{3} = 7$ or $x + 18 = 21$, so $x = 3$.
The smallest of the three numbers is 3.

Answer: (C)

14. *Solution 1*
Since half the square root of the number is one, then the square root of the number must be 2.
Since the square root of 4 is equal to 2, then the unknown number is 4.

Solution 2
Let the unknown number be x.
Since one half the square root of x is one, then $\frac{1}{2}\sqrt{x} = 1$.
Multiplying both sides of the equation by 2 gives, $2 \times \frac{1}{2}\sqrt{x} = 1 \times 2$ or $\sqrt{x} = 2$.
Squaring both sides of the equation gives, $(\sqrt{x})^2 = 2^2$ or $x = 4$.

Answer: (B)

15. To determine which combination will not be said, we list the letters and numbers recited by Yelena and Zeno in the table below.

Yelena	P	Q	R	S	T	U	P	Q	R	S	T	U
Zeno	1	2	3	4	1	2	3	4	1	2	3	4

Recognize that at this point Yelena has just said "U" and thus will begin from the beginning, P, again. Similarly, Zeno has just said "4" and thus will begin from the beginning, 1, again. That is, the sequence of letter and number combinations listed above will continue to repeat. To determine which combination will not be said we need only compare the 5 answers with the 12 possibilities given in the table.
The only combination that does not appear in the table and thus that will not be said is $R2$.

ANSWER: (D)

16. *Solution 1*
Since the lot has 25% more cars than trucks, and since we are asked to determine the ratio of cars to trucks, we may begin by choosing a convenient number of trucks in the lot.
Assume that there are 100 trucks in the lot.
Since there are 25% more cars than trucks, and since 25% of 100 is 25, then there are 125 cars in the parking lot.
Therefore, the ratio of cars to trucks in the parking lot is $125 : 100$ or $5 : 4$.

Solution 2
Assume that the number of trucks in the parking lot is x.
Since there are 25% more cars than trucks in the lot, then the number of cars is $1.25x$.
The ratio of cars to trucks is $1.25x : x$ or $1.25 : 1$ or $(1.25 \times 4) : (1 \times 4) = 5 : 4$.

ANSWER: (D)

17. The tens digit of a number contributes more to its value than its units digit.
Thus in order to make the difference between the numbers as small as possible, we begin by making the difference between the two numbers' tens digits as small as possible.
The smallest possible difference between the two tens digits is 2 and this is achieved in three ways only.
That is, we may allow the tens digits to be 2 and 4, or 4 and 6, or 6 and 8, as shown below.

$$\begin{array}{r} 4\square \\ -2\square \\ \hline \end{array} \qquad \begin{array}{r} 6\square \\ -4\square \\ \hline \end{array} \qquad \begin{array}{r} 8\square \\ -6\square \\ \hline \end{array}$$

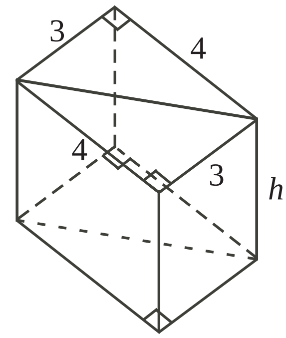
Next, we complete the numbers by using the two digits that remain as the ones digits.
We continue to make the difference as small as possible by using the larger of the two remaining digits to complete the smaller number, and using the smaller of the two remaining digits to complete the larger number.
The completion of the three possible cases as well as their respective differences are shown below.

$$\begin{array}{r} 46 \\ -28 \\ \hline 18 \end{array} \qquad \begin{array}{r} 62 \\ -48 \\ \hline 14 \end{array} \qquad \begin{array}{r} 82 \\ -64 \\ \hline 18 \end{array}$$

The smallest possible difference is $62 - 48 = 14$.

ANSWER: (B)

18. The triangular prism given can be created by slicing the 3 cm by 4 cm base of a rectangular prism with equal height across its diagonal. That is, the volume of the triangular prism in question is one half of the volume of the rectangular prism shown.
Since the volume of the triangular prism is 120 cm^3, then the volume of this rectangular prism is $2 \times 120 = 240$ cm^3.
The volume of the rectangular prism equals the area of its base 3×4 times its height, h.
Since the volume is 240, then $3 \times 4 \times h = 240$ or $12h = 240$, so $h = \frac{240}{12} = 20$.
Since the height of this rectangular prism is equal to the height of the triangular prism in question, then the required height is 20 cm.



Answer: (B)

19. Since there are 480 student participants and each student is participating in 4 events, then across all events the total number of (non-unique) participants is $480 \times 4 = 1920$.
Each event has 20 students participating.
Thus, the number of different events is $\frac{1920}{20} = 96$.
Each event is supervised by 1 adult coach, and there are 16 adult coaches each supervising the same number of events.
Therefore, the number of events supervised by each coach is $\frac{96}{16} = 6$.

Answer: (C)

20. *Solution 1*
Since the probability that Luke randomly chooses a blue marble is $\frac{2}{5}$, then for every 2 blue marbles in the bag there are 5 marbles in total.
Since there are only red and blue marbles in the bag, then for every 2 blue marbles, there are $5 - 2$ or 3 red marbles.
Thus, if there are $2k$ blue marbles in the bag then there are $3k$ red marbles in the bag (where k is some positive integer).
If Luke adds 5 blue marbles to the bag, then the number of blue marbles in the bag is $2k + 5$.
If in addition to adding 5 blue marbles to the bag Luke removes 5 red marbles from the bag, then the total number of marbles in the bag, $2k + 3k$, remains the same.
The probability that Luke randomly chooses a blue marble from the bag is determined by dividing the number of blue marbles in the bag, $2k + 5$, by the total number of marbles in the bag, $5k$.
Thus, $\dfrac{2k+5}{5k} = \dfrac{3}{5}$ or $5(2k + 5) = 3(5k)$.
Dividing both sides of this equation by 5 we have, $2k + 5 = 3k$ and so $k = 5$.
Therefore, the total number of marbles in the bag is $5k = 5(5) = 25$.

Solution 2
Let the initial number of marbles in the bag be N.
Since the probability that Luke randomly chooses a blue marble from the bag is $\frac{2}{5}$, then there are $\frac{2}{5}N$ blue marbles initially.
When 5 blue marbles are added to the bag and 5 red marbles are removed from the bag, the number of marbles in the bag is still N.
Since the probability of choosing a blue marble at the end is $\frac{3}{5}$, then there are $\frac{3}{5}N$ blue marbles in the bag at the end.

The difference between the number of blue marbles initially and the number of blue marbles at the end is 5, since 5 blue marbles were added.
Thus, $\frac{3}{5}N - \frac{2}{5}N = 5$ or $\frac{1}{5}N = 5$ or $N = 25$.
Therefore, the total number of marbles in the bag is 25.

Answer: (E)

21. From the first scale, 1 circle balances 2 triangles.
If we double what is on both sides of this scale, then 2 circles balance 4 triangles.
From the second scale, 2 circles also balance 1 triangle and 1 square.
So 4 triangles must balance 1 triangle and 1 square, or 3 triangles must balance 1 square.
If 3 triangles balance 1 square, then 6 triangles balance 2 squares.
(We note at this point that although we have found a way to balance 2 squares, 6 triangles is not one of the five answers given and so we continue on.)
Six triangles is equivalent to 4 triangles plus 2 triangles and we know from our doubling of the first scale that 4 triangles is balanced by 2 circles.
So 2 squares is balanced by 6 triangles, which is 4 triangles plus 2 triangles, which is balanced by 2 circles and 2 triangles.
Therefore, a possible replacement for the ? is 2 circles and 2 triangles.

Answer: (D)

22. Since only one of the given statements is true, then the other two statements are both false.
Assume that the second statement is true (then the other two are false).
If the second statement is true, then The Euclid is the tallest building.
Since the third statement is false, then The Galileo is the tallest building.
However, The Euclid and The Galileo cannot both be the tallest building so we have a contradiction. Therefore the second statement cannot be true.

Assume that the third statement is true.
Then The Galileo is not the tallest building.
Since the second statement must be false, then The Euclid is not the tallest building.
If both The Galileo and The Euclid are not the tallest building, then The Newton must be the tallest.
However, since the first statement must also be false, then The Newton is the shortest building and we have again reached a contradiction. Therefore, the third statement cannot be true.
Since both the second and third statements cannot be true, then the first statement must be true.

Since the first statement is true, then The Newton is either the second tallest building or the tallest building.
Since the third statement is false, then The Galileo is the tallest building which means that The Newton is the second tallest.
Since the second statement is false, then The Euclid is not the tallest and therefore must be the shortest (since The Newton is second tallest).
Ordered from shortest to tallest, the buildings are The Euclid (E), The Newton (N), and The Galileo (G).

Answer: (C)

23. Care is needed in systematically counting the different patterns.
Dividing the patterns into groups having some like attribute is one way to help this process.
We will count patterns by grouping them according to the number of "corner" triangles $(3, 2, 1,$ or $0)$ that each has shaded.

<u>3 shaded corners</u>
There is only one pattern having all 3 corners shaded, as shown in Figure 1.

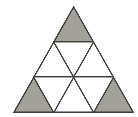


Figure 1

<u>2 shaded corners</u>
To start, we fix the 2 corners that are shaded (the top and bottom right) since rotations will give the other two possibilities that have 2 shaded corners.
For the purpose of identifying the smaller triangles, they have been numbered 1 to 6 as shown in Figure A.
We need only shade one of these 6 numbered triangles to complete a pattern.

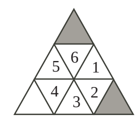


Figure A

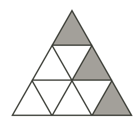


Figure 2

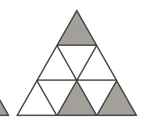


Figure 3

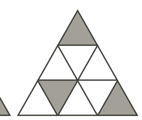


Figure 4

Since triangles 2 and 6 each share a side with a shaded corner, they cannot be shaded.
Shading triangle 1 gives our first pattern in this group, as shown in Figure 2.
Triangles 3 and 4 are then shaded to give Figures 3 and 4, respectively.
Note that shading triangle 5 gives a reflection of Figure 3 and so is not a different pattern.
These 3 patterns shown cannot be matched by rotations or reflections.
Thus, there are 3 different patterns having 2 shaded corners.

<u>1 shaded corner</u>
Again we start by fixing the corner that is shaded (the top) since rotations will give the other two possibilities that have 1 shaded corner.
We identify the smaller triangles by number in Figure B.
We need to shade 2 of these 6 numbered triangles to complete a pattern.
Since triangle 6 shares a side with the shaded corner, it cannot be shaded.

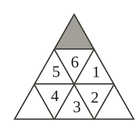


Figure B

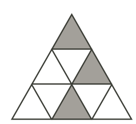


Figure 5

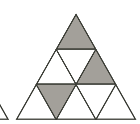


Figure 6

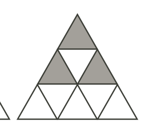


Figure 7

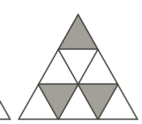


Figure 8

Also, no two adjacent triangles can be shaded since they share a side.
Shading triangles 1 and 3 gives our first pattern in this group, as shown in Figure 5.
Triangles 1 and 4 are then shaded to give Figure 6.
Figure 7 has triangles 1 and 5 shaded, while Figure 8 has triangles 2 and 4 shaded.
Shading triangles 2 and 5 gives a reflection of Figure 6 and so is not a different pattern.
Shading triangles 3 and 5 gives a reflection of Figure 5 and so is not a different pattern.
These 4 patterns shown cannot be matched by rotations or reflections.
There are no other combinations of 2 triangles that can be shaded.
Thus, there are 4 different patterns having 1 shaded corner.

No shaded corners

We identify the smaller triangles by number in Figure C.

We need to shade 3 of these 6 numbered triangles to complete a pattern.

Since no two adjacent triangles can be shaded there are only two possible patterns.

Triangles 1, 3 and 5 can be shaded (Figure 9) or triangles 2, 4 and 6 can be shaded (Figure 10).

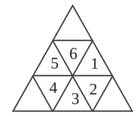


Figure C

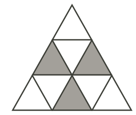


Figure 9

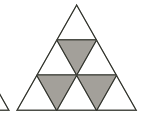


Figure 10

These 2 patterns shown cannot be matched by rotations or reflections.

There are no other combinations of 3 triangles that can be shaded.

Thus, there are 2 different patterns having no shaded corners.

Since each of the 4 groups of patterns above has a different number of corner triangles shaded, there is no pattern in any one group that can be matched to a pattern from another group by rotations or reflections.

Therefore, the total number of patterns that can be created is $1 + 3 + 4 + 2 = 10$.

Answer: (C)

24. We must find all possible groups of stones that can be selected so that the group sum is 11.
To do this, we first consider groups of size two.
There are 5 groups of size two whose numbers sum to 11.
These groups are: $\{1, 10\}$, $\{2, 9\}$, $\{3, 8\}$, $\{4, 7\}$, and $\{5, 6\}$.
Next, we consider groups of size three.
There are 5 groups of size three whose numbers sum to 11.
These are: $\{1, 2, 8\}$, $\{1, 3, 7\}$, $\{1, 4, 6\}$, $\{2, 3, 6\}$, and $\{2, 4, 5\}$.
Can you verify that there are no other groups of size three?
Although a group of size four is possible, ($\{1, 2, 3, 5\}$), it is not possible to form two more groups whose numbers sum to 11 using the remaining stones ($\{4, 6, 7, 8, 9, 10\}$).
Therefore, only groups of size two and size three need to be considered.
Next, we must count the number of ways that we can select three of the ten groups listed above such that no number is repeated between any of the three groups.
We may consider four cases or ways that these 3 groups can be chosen.
They are: all 3 groups are of size two, 2 groups are of size two and 1 group is of size three, 1 group is of size two and 2 groups are of size three, and finally all 3 groups are of size three.

Case 1: all 3 groups are of size two

Since there is no repetition of numbers between any of the 5 groups having size two, choosing any 3 groups will produce a solution.

There are 10 possible solutions using groups of size two only.

These are:

$$\begin{array}{cc}
\{1,10\}, \{2,9\}, \{3,8\} & \{1,10\}, \{4,7\}, \{5,6\} \\
\{1,10\}, \{2,9\}, \{4,7\} & \{2,9\}, \{3,8\}, \{4,7\} \\
\{1,10\}, \{2,9\}, \{5,6\} & \{2,9\}, \{3,8\}, \{5,6\} \\
\{1,10\}, \{3,8\}, \{4,7\} & \{2,9\}, \{4,7\}, \{5,6\} \\
\{1,10\}, \{3,8\}, \{5,6\} & \{3,8\}, \{4,7\}, \{5,6\}
\end{array}$$

Case 2: 2 groups are of size two and 1 group is of size three

Since there is repetition of numbers between some of the groups of size two and some of the groups of size three, we need to take care when making our choices.

To systematically work through all possible combinations, we will first choose a group of size three and then consider all pairings of groups of size two such that there is no repetition of numbers in either group of size two with the numbers in the group of size three.
This work is summarized in the table below.

1 group of size three	2 groups of size two
$\{1,2,8\}$	$\{4,7\},\{5,6\}$
$\{1,3,7\}$	$\{2,9\},\{5,6\}$
$\{1,4,6\}$	$\{2,9\},\{3,8\}$
$\{2,3,6\}$	$\{1,10\},\{4,7\}$
$\{2,4,5\}$	$\{1,10\},\{3,8\}$

There are 5 possible solutions using 2 groups of size two and 1 group of size three.

Case 3: 1 group is of size two and 2 groups are of size three
There is considerable repetition of numbers between the 5 groups of size three, $\{1,2,8\}$, $\{1,3,7\}$, $\{1,4,6\}$, $\{2,3,6\}$, and $\{2,4,5\}$.
In fact, there are only two groups whose numbers have no repetition.
These are $\{1,3,7\}$ and $\{2,4,5\}$. Can you verify that this is the only possibility?
At this point, we are unable to choose a group of size two such there is no repetition with the groups $\{1,3,7\}$ and $\{2,4,5\}$.
Therefore, there are no solutions using 1 group of size two and 2 groups of size three.

Case 4: all 3 groups are of size three
In Case 3, we found that there were only 2 groups of size three that had no repetition of numbers.
Therefore it is not possible to find 3 groups of size three without repeating numbers.
That is, Case 4 produces no solutions.

Thus, the total number of possible arrangements of the stones into three groups, each having a sum of 11, is $10+5=15$.

Answer: (E)

25. Since $WXYZ$ is a rectangle, then $\angle XWZ = \angle WZY = 90^\circ$.
Also, $WX = ZY = 15$ and $WZ = XY = 9$.
Thus, $\triangle PWS$ and $\triangle SZR$ are both right-angled triangles.
In $\triangle PWS$, $PS^2 = 3^2 + 4^2$, by the Pythagorean Theorem.
Therefore, $PS^2 = 9 + 16 = 25$, and so $PS = \sqrt{25} = 5$ (since $PS > 0$).
Similarly in $\triangle SZR$, $SR^2 = 5^2 + 12^2$, by the Pythagorean Theorem.
Therefore, $SR^2 = 25 + 144 = 169$, and so $SR = \sqrt{169} = 13$ (since $SR > 0$).
In triangles PWS and RYQ, $PW = RY = 3$, $WS = YQ = 4$, and $\angle PWS = \angle RYQ = 90^\circ$.
Therefore, the area of $\triangle PWS$ is equal to $\frac{1}{2} \times 3 \times 4 = 6$, as is the area of $\triangle RYQ$ equal to 6.
In triangles SZR and QXP, $SZ = QX = 5$, $ZR = XP = 12$, and $\angle SZR = \angle QXP = 90^\circ$.
Therefore, the area of $\triangle SZR$ is equal to $\frac{1}{2} \times 12 \times 5 = 30$, as is the area of $\triangle QXP$ equal to 30.
The area of parallelogram $PQRS$ is determined by subtracting the areas of triangles PWS, RYQ, SZR, and QXP from the area of rectangle $WXYZ$.
The area of rectangle $WXYZ$ is $WX \times XY$ or $15 \times 9 = 135$.
Therefore, the area of parallelogram $PQRS$ is $135 - (2 \times 6) - (2 \times 30) = 63$.
The area of parallelogram $PQRS$ is determined by multiplying the length of its base by its perpendicular height.

Let the base of $PQRS$ be SR and thus the perpendicular height is PT.
That is, $SR \times PT = 63$, or $13 \times PT = 63$, so $PT = \frac{63}{13}$.
Since $\angle STP = \angle PTR = 90^\circ$, then $\triangle PTS$ is a right-angled triangle.
In $\triangle PTS$, $ST^2 = PS^2 - PT^2 = 5^2 - \left(\frac{63}{13}\right)^2$, by the Pythagorean Theorem.
Therefore, $ST^2 = 25 - \frac{3969}{169} = \frac{4225-3969}{169} = \frac{256}{169}$, and so $ST = \sqrt{\frac{256}{169}} = \frac{16}{13}$ (since $ST > 0$).

ANSWER: (D)

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Grade 8 solutions follow the Grade 7 solutions

2011 Gauss Contests

(Grades 7 and 8)

Wednesday, May 11, 2011

Solutions

Centre for Education in Mathematics and Computing Faculty and Staff

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Jennifer Couture
Serge D'Alessio
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Grade 7

1. Evaluating, $5 + 4 - 3 + 2 - 1 = 9 - 3 + 2 - 1 = 6 + 2 - 1 = 8 - 1 = 7$.

   Answer: (E)

2. We must first add 9 and 16. Thus, $\sqrt{9 + 16} = \sqrt{25} = 5$.

   Answer: (E)

3. Reading from the bar graph, only 1 student chose spring.
   Since 10 students were surveyed, then the percentage of students that chose spring was $\frac{1}{10} \times 100\%$ or 10%.

   Answer: (B)

4. Since ground beef sells for \$5.00 per kg, then the cost of 12 kg is $\$5.00 \times 12 = \$60.00$.

   Answer: (C)

5. Since each of the numbers is between 1 and 2, we consider the tenths digits.
   The numbers 1.0101, 1.0011 and 1.0110 are between 1 and 1.1, while 1.1001 and 1.1100 are both greater than 1.1.
   We may eliminate answers (D) and (E).
   Next, we consider the hundredths digits.
   While 1.0101 and 1.0110 each have a 1 as their hundreths digit, 1.0011 has a 0 and is therefore the smallest number in the list.
   The ordered list from smallest to largest is $\{1.0011, 1.0101, 1.0110, 1.1001, 1.1100\}$

   Answer: (B)

6. Since you *randomly* choose one of the five answers, then each has an equally likely chance of being selected.
   Thus, the probability that you select the one correct answer from the five is $\frac{1}{5}$.

   Answer: (A)

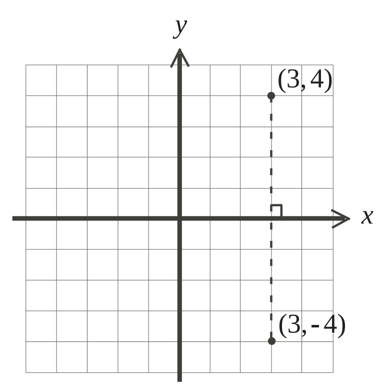
7. Since we are adding $\frac{1}{3}$ seven times, then the result is equal to $7 \times \frac{1}{3}$.

   Answer: (E)

8. Keegan paddled 12 km of his 36 km trip before lunch.
   Therefore, Keegan has $36 - 12 = 24$ km left to be completed after lunch.
   The fraction of his trip remaining to be completed is $\frac{24}{36} = \frac{2}{3}$.

   Answer: (D)

9. After reflecting the point $(3, 4)$ in the x-axis, the x-coordinate of the image will be the same as the x-coordinate of the original point, $x = 3$.
   The original point is a distance of 4 from the x-axis.
   The image will be the same distance from the x-axis, but below the x-axis.
   Thus, the image has y-coordinate -4.
   The coordinates of the image point are $(3, -4)$.



   Answer: (D)

10. Anika said that the plant was a *red* rose.
Cathy said that the plant was a *red* dahlia.
If red was not the correct colour of the plant, then Anika was incorrect about the colour and therefore must be correct about the type; in other words, the plant is a rose.
Similarly, if red was not the correct colour of the plant, then Cathy was incorrect about the colour and therefore must be correct about the type; in other words, the plant is a dahlia.
But the plant cannot be both a rose and a dahlia.
Therefore, Cathy and Anika must have been correct about the colour being red.
Bill said that the plant was a purple daisy.
Since we know that the colour of the plant is red, then Bill was incorrect about it being purple.
Therefore, Bill must have been correct about it being a daisy.
Thus, the plant is a red daisy.

ANSWER: (E)

11. The angles $2x^\circ$ and $3x^\circ$ shown are complementary and thus add to 90°.
Therefore, $2x + 3x = 90$ or $5x = 90$ and so $x = \frac{90}{5} = 18$.

ANSWER: (D)

12. Since the four sides of a square are equal in length and the perimeter is 28, then each side has length $\frac{28}{4} = 7$.
The area of the square is the product of the length and width, which are each equal to 7.
Therefore, the area of the square in cm^2 is $7 \times 7 = 49$.

ANSWER: (D)

13. Since Kayla ate less than Max and Chris ate more than Max, then Kayla ate less than Max who ate less than Chris.
Brandon and Tanya both ate less than Kayla.
Therefore, Max ate the second most.

ANSWER: (D)

14. The smallest three digit palindrome is 101.
The largest three digit palindrome is 999.
The difference between the smallest three digit palindrome and the largest three digit palindrome is $999 - 101 = 898$.

ANSWER: (B)

15. Since 10 minutes is equivalent to $\frac{10}{60} = \frac{1}{6}$ of an hour, the skier travels $12 \div 6 = 2$ km.

ANSWER: (C)

16. Any number of 2 cm rods add to give a rod having an even length.
Since we need an odd length, 51 cm, then we must combine an odd length from the 5 cm rods with the even length from the 2 cm rods to achieve this.
An odd length using 5 cm rods can only be obtained by taking an odd number of them.
All possible combinations are shown in the table below.

Number of 5 cm rods	Length in 5 cm rods	Length in 2 cm rods	Number of 2 cm rods
1	5	$51 - 5 = 46$	$46 \div 2 = 23$
3	15	$51 - 15 = 36$	$36 \div 2 = 18$
5	25	$51 - 25 = 26$	$26 \div 2 = 13$
7	35	$51 - 35 = 16$	$16 \div 2 = 8$
9	45	$51 - 45 = 6$	$6 \div 2 = 3$

Note that attempting to use 11 (or more) 5 cm rods gives more than the 51 cm length required.
Thus, there are exactly 5 possible combinations that add to 51 cm using 5 cm rods first followed by 2 cm rods.

ANSWER: (A)

17. *Solution 1*
Choosing one meat and one fruit, the possible lunches are beef and apple, beef and pear, beef and banana, chicken and apple, chicken and pear, or chicken and banana.
Of these 6 lunches, 2 of them include a banana.
Thus when randomly given a lunch, the probability that it will include a banana is $\frac{2}{6}$ or $\frac{1}{3}$.

Solution 2
Each of the possible lunches that Braydon may receive contain exactly one fruit.
The meat chosen for each lunch does not affect what fruit is chosen.
Thus, the probability that the lunch includes a banana is independent of the meat that is served with it.
Since there are 3 fruits to choose from, then the probability that the lunch includes a banana is $\frac{1}{3}$.

ANSWER: (A)

18. *Solution 1*
In kilograms, let the weights of the 3 pumpkins in increasing order be A, B and C.
The lightest combined weight, 12 kg, must come from weighing the two lightest pumpkins.
That is, $A + B = 12$.
The heaviest combined weight, 15 kg, must come from weighing the two heaviest pumpkins.
That is, $B + C = 15$.
Then the third given weight, 13 kg, is the combined weight of the lightest and heaviest pumpkins.
That is, $A + C = 13$.
Systematically, we may try each of the 5 possible answers.
If the lightest pumpkin weighs 4 kg (answer (A)), then $A + B = 12$ gives $4 + B = 12$, or $B = 8$.
If $B = 8$, then $8 + C = 15$ or $C = 7$.
Since C represents the weight of the heaviest pumpkin, C cannot be less than B and therefore the lightest pumpkin cannot weigh 4 kg.
If the lightest pumpkin weighs 5 kg (answer (B)), then $A + B = 12$ gives $5 + B = 12$, or $B = 7$.
If $B = 7$, then $7 + C = 15$ or $C = 8$.
If $A = 5$ and $C = 8$, then the third and final equation $A + C = 13$ is also true.
Therefore, the weight of the smallest pumpkin must be 5 kg.

Solution 2
In kilograms, let the weights of the 3 pumpkins in increasing order be A, B and C.
The lightest combined weight, 12 kg, must come from weighing the two lightest pumpkins.
That is, $A + B = 12$.
The heaviest combined weight, 15 kg, must come from weighing the two heaviest pumpkins.
That is, $B + C = 15$.
Then the third given weight, 13 kg, is the combined weight of the lightest and heaviest pumpkins.
That is, $A + C = 13$.
Since $A + B = 12$ and $A + C = 13$, then C is one more than B.
Since $B + C = 15$ and C is one more than B, then $B = 7$ and $C = 8$.

Since $A+B=12$, then $A+7=12$ or $A=5$.
When $A=5$, $B=7$ and $C=8$, the weights of the pairs of pumpkins are 12 kg, 13 kg and 15 kg as was given.
Therefore, the weight of the lightest pumpkin is 5 kg.

ANSWER: (B)

19. If each of the four numbers is increased by 1, then the increase in their sum is 4.
That is, these four new numbers when added together have a sum that is 4 more than their previous sum T, or $T+4$.
This new sum $T+4$ is now tripled.
The result is $3\times(T+4)=(T+4)+(T+4)+(T+4)$ or $3T+12$.

ANSWER: (C)

20. The volume of the rectangular prism equals the area of its base 6×4 times its height 2.
That is, the rectangular prism has a volume of $6\times 4\times 2=48$ cm^3.
The volume of the triangular prism is found by multiplying the area of one of its triangular faces by its length.
The triangular face has a base of length $6-3=3$ cm.
This same triangular face has a perpendicular height of $5-2=3$ cm, since the height of the rectangular prism is 2 cm.
Thus, the triangular face has area $\frac{3\times 3}{2}=\frac{9}{2}$ cm^2.
Since the length of the triangular prism is 4 cm, then its volume is $\frac{9}{2}\times 4=\frac{36}{2}=18$ cm^3.
The volume of the combined structure is equal to the sum of the volumes of the two prisms, or $48+18=66$ cm^3.

ANSWER: (E)

21. Steve counts forward by 3 beginning at 7.
That is, the numbers that Steve counts are each 7 more than some multiple of 3.
We can check the given answers to see if they satisfy this requirement by subtracting 7 from each of them and then determining if the resulting number is divisible by 3.
We summarize the results in the table below.

Answers	Result after subtracting 7	Divisible by 3?
1009	1002	Yes
1006	999	Yes
1003	996	Yes
1001	994	No
1011	1004	No

Of the possible answers, Steve only counted 1009, 1006 and 1003.
Dave counts backward by 5 beginning at 2011.
That is, the numbers that Dave counts are each some multiple of 5 less than 2011.
We can check the given answers to see if they satisfy this requirement by subtracting each of them from 2011 and then determining if the resulting number is divisible by 5.
We summarize the results in the table below.

Answers	Result after being subtracted from 2011	Divisible by 5?
1009	1002	No
1006	1005	Yes
1003	1008	No
1001	1010	Yes
1011	1000	Yes

Of the possible answers, Dave only counted 1006, 1001 and 1011.
Thus while counting, the only answer that both Steve and Dave will list is 1006.

ANSWER: (B)

22. In the first 20 minutes, Sheila fills the pool at a rate of 20 L/min and thus adds $20 \times 20 = 400$ L of water to the pool.
At this time, the pool needs $4000 - 400 = 3600$ L of water to be full.
After filling for 20 minutes, water begins to leak out of the pool at a rate of 2 L/min.
Since water is still entering the pool at a rate of 20 L/min, then the net result is that the pool is filling at a rate of $20 - 2 = 18$ L/min.
Since the pool needs 3600 L of water to be full and is filling at a rate of 18 L/min, then it will take an additional $3600 \div 18 = 200$ minutes before the pool is full of water.
Thus, the total time needed to fill the pool is $20+200 = 220$ minutes or 3 hours and 40 minutes.

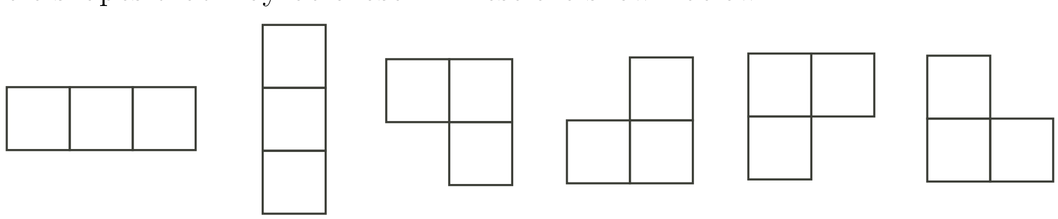
ANSWER: (B)

23. The sum of the units column is $E + E + E = 3E$.
Since E is a single digit, and $3E$ ends in a 1, then the only possibility is $E = 7$.
Then $3E = 3 \times 7 = 21$, and thus 2 is carried to the tens column.
The sum of the tens column becomes $2 + B + C + D$.
The sum of the hundreds column is $A + A + A = 3A$ plus any carry from the tens column.
Thus, $3A$ plus the carry from the tens column is equal to 20.
If there is no carry from the tens column, then $3A = 20$.
This is not possible since A is a single digit positive integer.
If the carry from the tens column is 1, then $3A + 1 = 20$ or $3A = 19$.
Again, this is not possible since A is a single digit positive integer.
If the carry from the tens column is 2, then $3A + 2 = 20$ or $3A = 18$ and $A = 6$.
Since B, C, and D are single digits (ie. they are each less than or equal to 9), then it is not possible for the carry from the tens column to be greater than 2.
Therefore, $A = 6$ is the only possibility.
Since the carry from the tens column is 2, then the sum of the tens column, $2 + B + C + D$, must equal 21.
Thus, $2 + B + C + D = 21$ or $B + C + D = 19$.
Since $A = 6$ and $E = 7$, then the sum $A + B + C + D + E = 6 + 19 + 7 = 32$.
Note that although we don't know B, C and D, it is only necessary that their sum be 19.
Although there are many possibilities, the example with $B = 2$, $C = 8$ and $D = 9$ is shown.

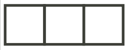
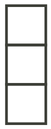
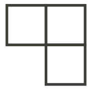
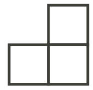
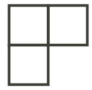
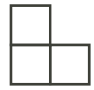
$$\begin{array}{cccc} & 6 & 2 & 7 \\ & 6 & 8 & 7 \\ + & 6 & 9 & 7 \\ \hline 2 & 0 & 1 & 1 \end{array}$$

ANSWER: (C)

24. First we recognize that given the conditions for the three selected squares, there are only 6 possible shapes that may be chosen. These are shown below.



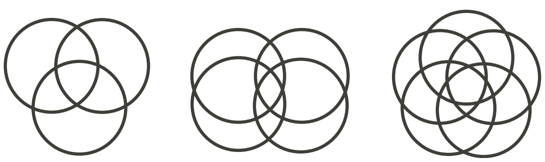
To determine the number of ways that three squares can be selected, we count the number of ways in which each of these 6 shapes can be chosen from the given figure.
The results are summarized in the table below.

Shape						
Number of Each	3	2	4	3	3	4

Thus, three of the nine squares can be selected as described in $3+2+4+3+3+4=19$ ways.

Answer: (A)

25. We first note that each circle can intersect any other circle a maximum of two times.
To begin, the first circle is drawn.
The second circle is then drawn overlapping the first, and two points of intersection are created.
Since each pair of circles overlap (but are not exactly on top of one another), then the third circle drawn can intersect the first circle twice and the second circle twice.
We continue in this manner with each new circle drawn intersecting each of the previously drawn circles exactly twice.
That is, the third circle drawn intersects each of the two previous circles twice, the fourth circle intersects each of the three previous circles twice, and so on.
Diagrams showing possible arrangements for 3, 4 and 5 circles, each giving the maximum number of intersections, are shown below.



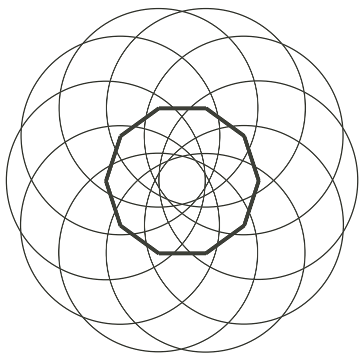
The resulting numbers of intersections are summarized in the table below.

Circle number drawn	Number of new intersections	Total number of intersections
1	0	0
2	2	2
3	$2\times 2=4$	$2+4$
4	$3\times 2=6$	$2+4+6$
5	$4\times 2=8$	$2+4+6+8$
6	$5\times 2=10$	$2+4+6+8+10$
7	$6\times 2=12$	$2+4+6+8+10+12$
8	$7\times 2=14$	$2+4+6+8+10+12+14$
9	$8\times 2=16$	$2+4+6+8+10+12+14+16$
10	$9\times 2=18$	$2+4+6+8+10+12+14+16+18$

Thus, the greatest possible total number of intersection points using ten circles is

$$2+4+6+8+10+12+14+16+18=90.$$

To be complete, we technically need to show that this number is possible, though we don't expect students to do this to answer the question.
The diagram below demonstrates a possible positioning of the ten circles that achieves the maximum 90 points of intersection.
That is, every pair of circles intersects exactly twice and all points of intersection are distinct from one another.
It is interesting to note that this diagram is constructed by positioning each of the ten circles' centres at one of the ten vertices of a suitably sized regular decagon, as shown.



Answer: (D)

Grade 8

1. The fractions $\frac{8}{12}$ and $\frac{\square}{3}$ are equivalent fractions.
To reduce the first to the second, the denominator 12 has been divided by a factor of 4.
Therefore, we divide the numerator 8 by the same factor, 4.
Thus, $\frac{8}{12} = \frac{2}{3}$ and the value represented by $\square$ is 2.

Answer: (D)

2. Since ground beef sells for \$5.00 per kg, then the cost of 12 kg is $\$5.00 \times 12 = \$60.00$.

Answer: (C)

3. When the unknown angle y° is added to the 90° angle, the result is a complete rotation, or 360°.
Thus, $y^\circ + 90^\circ = 360^\circ$ or $y = 360 - 90 = 270$.

Answer: (E)

4. *Solution 1*
To compare the five fractions we rewrite them each with a common denominator of 100.
Then the list equivalent to the given list $\left\{\frac{3}{10}, \frac{9}{20}, \frac{12}{25}, \frac{27}{50}, \frac{49}{100}\right\}$ is $\left\{\frac{30}{100}, \frac{45}{100}, \frac{48}{100}, \frac{54}{100}, \frac{49}{100}\right\}$.
The largest number in this new list is $\frac{54}{100}$, and therefore $\frac{27}{50}$ is the largest number in the given list.
Solution 2
We notice that with the exception of $\frac{27}{50}$, each numerator in the given list is less than one half of its corresponding denominator.
Thus, each fraction except $\frac{27}{50}$ has a value that is less than one half.
The fraction $\frac{27}{50}$ is larger than one half.
Therefore $\frac{27}{50}$ is the largest number in the list.

Answer: (D)

5. Since 3 of the 15 balls are red, then the probability that Alex randomly selects a red ball is $\frac{3}{15}$ or $\frac{1}{5}$.

Answer: (A)

6. *Solution 1*
Since double the original number plus 3 is 23, then double the original number must equal 20 (that is, $23 - 3$).
Therefore, the original number is 20 divided by 2, or 10.

Solution 2
Let the original number be represented by the variable x.
Then doubling the original number and adding 3 gives $2x + 3$.
Thus, $2x + 3 = 23$ or $2x = 23 - 3 = 20$, so $x = \frac{20}{2} = 10$.

Answer: (B)

7. *Solution 1*
To make half of the recipe, only half of the $4\frac{1}{2}$ cups of flour are needed.
Since half of 4 is 2 and half of $\frac{1}{2}$ is $\frac{1}{4}$, then $2\frac{1}{4}$ cups of flour are needed.

Solution 2
To make half of the recipe, only half of the $4\frac{1}{2}$ cups of flour are needed.
To determine half of $4\frac{1}{2}$, we divide $4\frac{1}{2}$ by 2, or multiply by $\frac{1}{2}$.
Thus, $4\frac{1}{2} \times \frac{1}{2} = \frac{9}{2} \times \frac{1}{2} = \frac{9}{4} = 2\frac{1}{4}$ cups of flour are needed to make half of the recipe.

Answer: (B)

8. Since $\angle PQR = \angle PRQ$, then $\triangle PQR$ is an isosceles triangle and $PQ = PR = 7$.
Therefore, the perimeter of $\triangle PQR$ is $PQ + QR + PR = 7 + 5 + 7 = 19$.

ANSWER: (E)

9. If 15 of 27 students in the class are girls, then the remaining $27 - 15 = 12$ students are boys.
The ratio of boys to girls in the class is $12 : 15 = 4 : 5$.

ANSWER: (A)

10. Since Kayla ate less than Max and Chris ate more than Max, then Kayla ate less than Max who ate less than Chris.
Brandon and Tanya both ate less than Kayla.
Therefore, Max ate the second most.

ANSWER: (D)

11. Evaluating each of the expressions,
(A): $(2 \times 3)^2 = 6^2 = 36$
(B): $3 + 2^2 = 3 + 4 = 7$
(C): $2^3 - 1 = 8 - 1 = 7$
(D): $3^2 - 2^2 = 9 - 4 = 5$
(E): $(3 + 2)^2 = 5^2 = 25$,
we see that only expression (D) is equal to 5.

ANSWER: (D)

12. Nick charges \$10 per hour of babysitting.
If Nick babysits for y hours, then his charge just for babysitting is $10y$ dollars.
In addition, Nick charges a one-time fee of \$7 for travel costs.
Thus, the expression that represents the total number of dollars that Nick charges for y hours of babysitting is $10y + 7$.

ANSWER: (A)

13. Measured in cm^2, the area of Kalob's window is 50×80.
Measured in cm^2, twice the area of Kalob's window is $50 \times 80 \times 2$ which is equal to 50×160.
Thus, a window with dimensions 50 cm $\times$ 160 cm is a window with area double the area of Kalob's window.

ANSWER: (C)

14. First recognize that the day and the month must be equal.
Next, since $3^2 = 9$ and $10^2 = 100$, both the day and the month must be larger than 3 but less than 10 so that the year lies between 2012 and 2099.
We list all possible square root days in the table below.

Day and Month	Last Two Digits of the Year	Date
4	$4^2 = 16$	4/4/2016
5	$5^2 = 25$	5/5/2025
6	$6^2 = 36$	6/6/2036
7	$7^2 = 49$	7/7/2049
8	$8^2 = 64$	8/8/2064
9	$9^2 = 81$	9/9/2081

Since these are all actual dates between January 1, 2012 and December 31, 2099, then the number of square root days is 6.

ANSWER: (E)

15. In $\triangle CDE$, $CE = 5$, $DE = 3$, and $\angle CDE = 90^\circ$.
By the Pythagorean Theorem, $CE^2 = CD^2 + DE^2$ or $CD^2 = CE^2 - DE^2$ or $CD^2 = 5^2 - 3^2 = 25 - 9 = 16$, so $CD = 4$ (since $CD > 0$).
In $\triangle ABC$, $AB = 9$, $BC = BD - CD = 16 - 4 = 12$, and $\angle ABC = 90^\circ$.
By the Pythagorean Theorem, $AC^2 = AB^2 + BC^2$ or $AC^2 = 9^2 + 12^2 = 81 + 144 = 225$, so $AC = 15$ (since $AC > 0$).

Answer: (C)

16. *Solution 1*
Beatrix is twice the height of Violet who is $\frac{2}{3}$ the height of Georgia.
Therefore, Beatrix is 2 times $\frac{2}{3}$ the height of Georgia, or $\frac{4}{3}$ the height of Georgia.

Solution 2
Let the heights of Beatrix, Violet and Georgia be represented by B, V and G respectively.
Since Beatrix is twice the height of Violet, then $B = 2V$.
Since Violet is $\frac{2}{3}$ the height of Georgia, then $V = \frac{2}{3}G$.
Substituting $\frac{2}{3}G$ for V in the first equation, we get $B = 2V = 2(\frac{2}{3}G) = \frac{4}{3}G$.
Thus, Beatrix's height is $\frac{4}{3}$ of Georgia's height.

Answer: (C)

17. Since x can be any value in between 0 and 1, we choose a specific value for x in this range.
For example, we allow x to be $\frac{1}{4}$ and then evaluate each of the five expressions.
We get, $x = \frac{1}{4}$; $x^2 = (\frac{1}{4})^2 = \frac{1}{16}$; $2x = 2\left(\frac{1}{4}\right) = \frac{2}{4} = \frac{1}{2}$; $\sqrt{x} = \sqrt{\frac{1}{4}} = \frac{1}{2}$; $\frac{1}{x} = \frac{1}{\frac{1}{4}} = 4$.
Since $\frac{1}{16}$ is the smallest value, then for any value of x between 0 and 1, x^2 will produce the smallest value of the five given expressions.
In fact, no matter what x between 0 and 1 is chosen, x^2 is always smallest.

Answer: (B)

18. Assume that each square has side length 2, and thus has area 4.
In square $ABCD$, diagonal AC divides the square into 2 equal areas.
Thus, the area of $\triangle ACD$ is one half of the area of square $ABCD$ or 2.
Since AC is the diagonal of square $ABCD$, $\angle ACD = \angle ACB = 90^\circ \div 2 = 45^\circ$.
Also, $\angle DCH = \angle BCH - \angle DCB = 180^\circ - 90^\circ = 90^\circ$.
Since ACJ is a straight line segment, $\angle ACJ = \angle ACD + \angle DCH + \angle HCJ = 180^\circ$.
Thus, $\angle HCJ = 180^\circ - \angle ACD - \angle DCH = 180^\circ - 45^\circ - 90^\circ = 45^\circ$.
In $\triangle CHJ$, $\angle HCJ = 45^\circ$ and $\angle CHJ = 90^\circ$, so $\angle HJC = 180^\circ - 90^\circ - 45 = 45^\circ$.
Therefore $\triangle CHJ$ is isosceles, so $CH = HJ = 1$, since J is the midpoint of GH.
Therefore $\triangle CHJ$ has area $\frac{1}{2} \times 1 \times 1$ or $\frac{1}{2}$.
Since $\triangle ACD$ has area 2, and $\triangle CHJ$ has area $\frac{1}{2}$, then the combined areas of the two shaded regions is $2 + \frac{1}{2}$ or $\frac{5}{2}$.
Since the total area of the two large squares is 8, the fraction of the two squares that is shaded is $\frac{5}{2} \div 8 = \frac{5}{2} \times \frac{1}{8} = \frac{5}{16}$.

Answer: (D)

19. We must consider that the integers created could be one-digit, two-digit or three-digit integers.
First, consider one-digit integers.
Since 1, 2 and 3 are the only digits that may be used, then there are only 3 one-digit positive integers less than 400, namely 1, 2 and 3.
Next, consider the number of two-digit integers that can be created.
Since digits may be repeated, the integers 11, 12, 13, 21, 22, 23, 31, 32, 33 are the only possibilities.

Thus, there are 9 two-digit positive integers that can be created.
Instead of listing these 9 integers, another way to count how many there are is to consider that there are 3 choices for the first digit (either 1, 2 or 3 may be used) and also 3 choices for the second digit (since repetition of digits is allowed), so $3 \times 3 = 9$ possibilities.
Finally, we count the number of three-digit positive integers.
There are 3 choices for the first digit, 3 choices for the second digit and 3 choices for the third digit.
Thus, there are $3 \times 3 \times 3 = 27$ possible three-digit positive integers that can be created. (Note that all of these are less than 400, since the largest of them is 333.)
In total, the number of positive integers less than 400 that can be created using only the digits 1, 2 or 3 (with repetition allowed), is $3 + 9 + 27 = 39$.

ANSWER: (D)

20. Since the average height of all 22 students is 103 cm, then the sum of the heights of all students in the class is $22 \times 103 = 2266$ cm.
Since the average height of the 12 boys in the class is 108 cm, then the sum of the heights of all boys in the class is $12 \times 108 = 1296$ cm.
The sum of the heights of all the girls in class is the combined height of all students in the class less the height of all the boys in the class, or $2266 - 1296 = 970$ cm.
Since there are 10 girls in the class, their average height is $970 \div 10 = 97$ cm.

ANSWER: (B)

21. We first consider the minimum number of coins needed to create 99¢, the greatest amount that we are required to create.
We can create 75¢ with a minimum number of coins by placing three quarters in the collection.
Once at 75¢, we need $99 - 75 = 24$¢ more to reach 99¢.
To create 24¢ with a minimum number of coins we use two dimes and 4 pennies.
Therefore, the minimum number of coins required to create 99¢ is 3 quarters, 2 dimes and 4 pennies, or 9 total coins in the collection.
In fact, this is the only group of 9 coins that gives exactly 99¢.
To see this, consider that we must have the 4 pennies and the 3 quarters, but these 7 coins only give 79¢.
We now need exactly 20¢ and have only two coins left to include in our collection.
Therefore the two remaining coins must both be dimes, and thus the only group of 9 coins that gives exactly 99¢ is 3 quarters, 2 dimes and 4 pennies.
We now attempt to check if all other amounts of money less than one dollar can be created using only these 9 coins.
Using the 4 pennies, we can create each of the amounts 1¢, 2¢, 3¢, and 4¢.
However, we don't have a way to create 5¢ using 3 quarters, 2 dimes and 4 pennies.
Thus, at least one more coin is needed.
It is clear that if the 10th coin that we add to the collection is a penny, then we will have 5 pennies and thus be able to create 5¢.
However, with 3 quarters, 2 dimes and 5 pennies, we won't be able to create 9¢.
If instead of a penny we add a nickel as our 10th coin in the collection, then we will have 3 quarters, 2 dimes, 1 nickel and 4 pennies.
Obviously then we can create 5¢ using the 1 nickel.
In fact, we are now able to create the following sums of money:

- any amount from 1¢ to 4¢ using 4 pennies;

– any amount from 5¢ to 9¢ using 1 nickel and 4 pennies;
– any amount from 10¢ to 14¢ using 1 dime and 4 pennies;
– any amount from 15¢ to 19¢ using 1 dime, 1 nickel and 4 pennies;
– any amount from 20¢ to 24¢ using 2 dimes and 4 pennies.

If in addition to the amounts from 1¢ to 24¢ we use a quarter, we will be able to create any amount from 25¢ to 49¢.
Similarly, using 2 quarters we can create any amount from 50¢ to 74¢.
Finally, using 3 quarters we can create any amount from 75¢ to 99¢.
Therefore, the smallest number of coins needed to create any amount of money less than one dollar is 10.

Can you verify that there is a different combination of 10 coins with which we can create any amount of money less than one dollar?

Answer: (A)

22. Consider the possible ways that the numbers from 1 to 9 can be used three at a time to sum to 18.
Since we may not repeat any digit in the sum, the possibilities are:
$1+8+9, 2+7+9, 3+6+9, 3+7+8, 4+5+9, 4+6+8, 5+6+7$.
Next, consider the row $1+d+f$ in Fig.1.
Since the sum of every row is 18, $d+f=17$.
This gives that either $d=8$ and $f=9$ or $d=9$ and $f=8$.
Next, consider the 3 rows in which x appears.
The sums of these 3 rows are $a+x+d$, $b+x+f$, and $c+x+6$.
That is, x appears in exactly 3 distinct sums.
Searching our list of possible sums above, we observe that only the numbers $6, 7$ and 8 appear in 3 distinct sums.
That is, x must equal either $6, 7$ or 8.
However, the number 6 already appears in the table.
Thus, x is not 6.
Similarly, we already concluded that either d or f must equal 8.
Thus, x is not 8. Therefore, $x=7$ is the only possibility.
Fig.2 shows the completed table and verifies that the number represented by x is indeed 7.

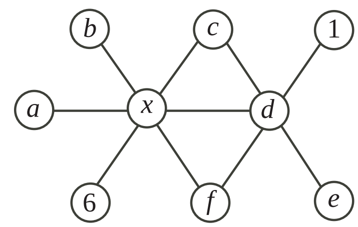


Fig.1

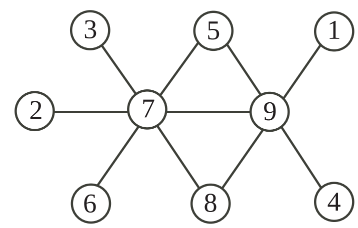
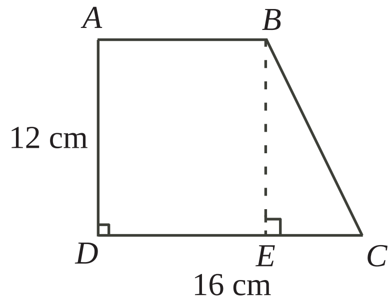


Fig.2

Answer: (C)

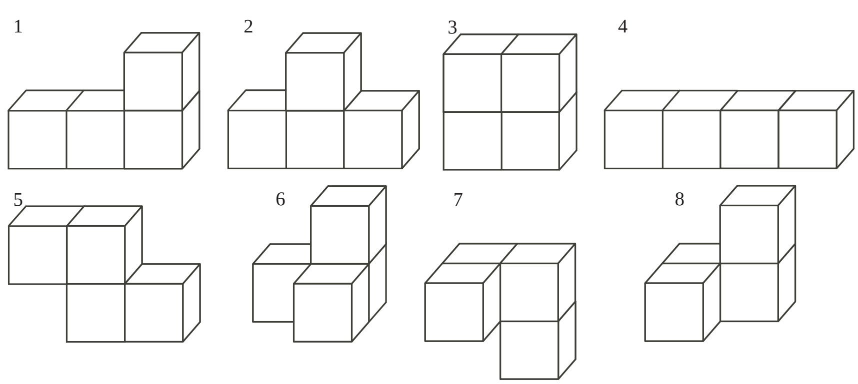
23. We first label the trapezoid $ABCD$ as shown in the diagram below.
Since AD is the perpendicular height of the trapezoid, then AB and DC are parallel.
The area of the trapezoid is $\frac{AD}{2} \times (AB + DC)$ or $\frac{12}{2} \times (AB + 16)$ or $6 \times (AB + 16)$.
Since the area of the trapezoid is 162, then $6 \times (AB + 16) = 162$ and $AB + 16 = \frac{162}{6}$ or $AB + 16 = 27$, so $AB = 11$.
Construct a perpendicular from B to E on DC.
Since AB is parallel to DE and both AD and BE are perpendicular to DE, then $ABED$ is a rectangle.
Thus, $DE = AB = 11$, $BE = AD = 12$, and $EC = DC - DE = 16 - 11 = 5$.
Since $\angle BEC = 90°$, then $\triangle BEC$ is a right-angled triangle.
Thus by the Pythagorean Theorem, $BC^2 = BE^2 + EC^2$ or $BC^2 = 12^2 + 5^2$ or $BC^2 = 169$ so

$BC = 13$ (since $BC > 0$).
The perimeter of the trapezoid is $AB + BC + CD + DA = 11 + 13 + 16 + 12 = 52$ cm.



Answer: (B)

24. When Ada glues cube faces together, they must coincide.
Also, each of the 4 cubes must have a face that coincides with a face of at least one of the other 3 cubes.
We this in mind, we begin by repositioning the 4 identical cubes, attempting to construct figures different from those already constructed.
Figure 1 shown below was given in the question.
The next 4 figures (numbered 2 to 5) have the same thickness as figure 1.
That is, each of the first 5 figures is 1 cube thick from front to back.
These are the only 5 figures that can be constructed having this 1 cube thickness and these 5 figures are all unique.
Can you verify this for yourself? Attempt to construct a different figure with a 1 cube thickness. Now rotate this figure to see if it will match one of the first 5 figures shown.
Next, we consider constructing figures with a front to back depth of 2 cubes.
The only 3 unique figures that can be constructed in this way are labeled 6 to 8 below.
While figures 7 and 8 look to be the same, there is no way to rotate one of them so that it is identical to the other.
Thus, not only are these 3 figures the only figures that can be constructed with a 2 cube thickness, but they are all different from one another.
Since no rotation of any of these 3 figures will result in it having a 1 cube thickness from front to back, the figures 6 to 8 are all different than any of the previous 5 figures.
With some rotation, we can verify that the only figures having a 3 cube thickness have already been constructed (figures 1, 2 and 5).
Similarly, the only figure that can possibly be built with a 4 cube thickness has already been constructed (figure 4).
The 8 figures shown below are the only figures that Ada can construct.



Answer: (D)

25. The given sequence allows for many different patterns to be discovered depending on how terms in the sequence are grouped and then added.
One possibility is to add groups of four consecutive terms in the sequence.
That is, consider finding the sum of the sequence, S, in the manner shown below.

$$S = (1 + (-4) + (-9) + 16) + (25 + (-36) + (-49) + 64) + (81 + (-100) + (-121) + 144) + \ldots$$

The pattern that appears when grouping terms in this way is that each consecutive group of 4 terms, beginning at the first term, adds to 4.
That is, $1+(-4)+(-9)+16 = 4$, $25+(-36)+(-49)+64 = 4$, $81+(-100)+(-121)+144 = 4$, and so on.
For now, we will assume that this pattern of four consecutive terms adding to 4 continues and wait to verify this at the end of the solution.
Since each consecutive group of four terms adds to 4, the first eight terms add to 8, the first twelve terms add to 12 and the first n terms add to n provided that n is a multiple of 4.
Thus, the sum of the first 2012 terms is 2012, since 2012 is a multiple of 4.
Since we are required to find the sum of the first 2011 terms, we must subtract the value of the 2012^{th} term from our total of 2012.
We know that the n^{th} term in the sequence is either n^2 or it is $-n^2$.
Therefore, we must determine if the 2012^{th} term is positive or negative.
By the alternating pattern of the signs, the first and fourth terms in each of the consecutive groupings will be positive, while the second and third terms are negative.
Since the 2012^{th} term is fourth in its group of four, its sign is positive.
Thus, the 2012^{th} term is 2012^2.
Therefore, the sum of the first 2011 terms is the sum of the first 2012 terms, which is 2012, less the 2012^{th} term, which is 2012^2.
Thus, $S = 2012 - 2012^2 = 2012 - 4048144 = -4046132$.

Verifying the Pattern

While we do not expect that students will verify this pattern in the context of a multiple choice contest, it is always important to verify patterns.
One way to verify that the sum of each group of four consecutive terms (beginning with the first term) adds to 4, is to use algebra.
If the first of four consecutive integers is n, then the next three integers in order are $n + 1$, $n + 2$, and $n + 3$.
Since the terms in our sequence are the squares of consecutive integers, we let n^2 represent the first term in the group of four.
The square of the next integer larger than n is $(n + 1)^2$, and thus the remaining two terms in the group are $(n + 2)^2$ and $(n + 3)^2$.
Since the first and fourth terms are positive, while the second and third terms are negative, the sum of the four terms is $n^2 - (n + 1)^2 - (n + 2)^2 + (n + 3)^2$.
(For example, if $n = 5$ then we have $5^2 - 6^2 - 7^2 + 8^2$.)
To simplify this expression, we must first understand how to simplify its individual parts such as $(n + 1)^2$.
The expression $(n + 1)^2$ means $(n + 1) \times (n + 1)$.
To simplify this product, we multiply the n in the first set of brackets by each term in the second set of brackets and do the same for the 1 appearing in the first set of brackets.
The operation between each product remains as it appears in the expression, as an addition.

That is,

$$\begin{aligned}(n+1)^2 &= (n+1)\times(n+1)\\ &= n\times n + n\times 1 + 1\times n + 1\times 1\\ &= n^2+n+n+1\\ &= n^2+2n+1\end{aligned}$$

Applying this process again,
$(n+2)^2 = (n+2)\times(n+2) = n\times n + n\times 2 + 2\times n + 2\times 2 = n^2+2n+2n+4 = n^2+4n+4$,
and
$(n+3)^2 = (n+3)\times(n+3) = n\times n + n\times 3 + 3\times n + 3\times 3 = n^2+3n+3n+9 = n^2+6n+9$.
Therefore, the sum of each group of four consecutive terms (beginning with the first term) is,

$$\begin{aligned}n^2-(n+1)^2-(n+2)^2+(n+3)^2 &= n^2-(n^2+2n+1)-(n^2+4n+4)+(n^2+6n+9)\\ &= n^2-n^2-2n-1-n^2-4n-4+n^2+6n+9\\ &= n^2-n^2-n^2+n^2-2n-4n+6n+9-1-4\\ &= 4\end{aligned}$$

Answer: (E)

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Grade 8 solutions follow the Grade 7 solutions

2010 Gauss Contests

(Grades 7 and 8)

Wednesday, May 12, 2010

Solutions

Centre for Education in Mathematics and Computing Faculty and Staff

Ed Anderson
Lloyd Auckland
Terry Bae
Janet Baker
Steve Brown
Ersal Cahit
Karen Cole
Jennifer Couture
Frank DeMaio
Fiona Dunbar
Jeff Dunnett
Mike Eden
Barry Ferguson
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Angie Hildebrand
Judith Koeller
Joanne Kursikowski
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Jen Nissen
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Linda Schmidt
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David Switzer, Sixteenth Ave. P.S., Richmond Hill, ON
Tanya Thompson, Nottawa, ON
Chris Wu, Amesbury M.S., Toronto, ON

Grade 7

1. Reading the number on the vertical axis corresponding to the pet *fish*, we find that 40 students chose fish as their favourite pet.

Answer: (D)

2. By dividing, we find the fraction $\frac{20}{25}$ is equivalent to the decimal 0.80.
We convert this to a percent by multiplying by 100%.
Thus, Tanya scored $0.80 \times 100\% = 80\%$ on her math quiz.

Answer: (C)

3. Using the correct order of operations, $4 \times 5 + 5 \times 4 = 20 + 20 = 40$.

Answer: (E)

4. To find the location of the point $(-2, -3)$, we begin at the origin, $(0, 0)$, and move left 2 units and down 3 units.
The point $(-2, -3)$ is located at D.

Answer: (D)

5. Going down 2 floors from the 11th floor brings Chaz to the 9th floor.
Going down 4 floors from the 9th floor brings Chaz to the 5th floor.
Thus, Chaz gets off the elevator on the 5th floor.

Answer: (D)

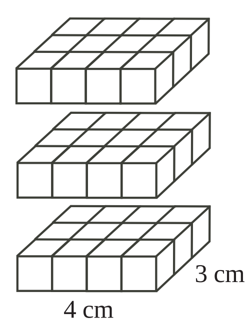
6. The answer, 10000.3, is 1000 times bigger than 10.0003. This can be determined either by dividing 10000.3 by 10.0003 or by recognizing that the decimal point in 10.0003 is moved three places to the right to obtain 10000.3. Thus, the number that should replace the $\square$ is 1000.

Answer: (B)

7. The four angles shown, 150°, 90°, x°, and 90°, form a complete rotation, a 360° angle.
Thus, $150^\circ + 90^\circ + x^\circ + 90^\circ = 360^\circ$, or $x^\circ = 360^\circ - 150^\circ - 90^\circ - 90^\circ = 30^\circ$.

Answer: (D)

8. *Solution 1*
To build the solid rectangular prism, we could first construct the 4 cm by 3 cm base using $4 \times 3 = 12$ of the 1 cm $\times$ 1 cm $\times$ 1 cm blocks.
Two more layers identical to the first layer, placed on top of the first layer, would give the prism its required 3 cm height.
This would require 12 more 1 cm $\times$ 1 cm $\times$ 1 cm blocks in layer two and 12 more in layer three, or $12 \times 3 = 36$ blocks in total.



Solution 2
Equivalently, this question is asking for the volume of the rectangular prism.
The volume of a prism is the area of the base times the height,
or $V = 4 \text{ cm} \times 3 \text{ cm} \times 3 \text{ cm} = 36 \text{ cm}^3$.
Since the volume of each of the 1 cm $\times$ 1 cm $\times$ 1 cm blocks is 1 cm^3, then 36 blocks are needed to build the solid rectangular prism.
(The prism can actually be built with 36 blocks as seen in Solution 1.)

Answer: (E)

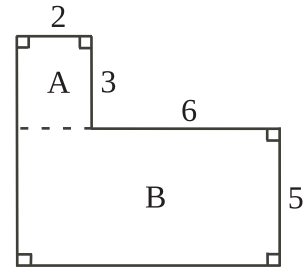
9. If the time reads 3:33, the next time that all of the digits on the clock are equal to one another is 4:44. Since the amount of time between 3:33 and 4:44 is 1 hour and 11 minutes, the shortest length of time in minutes is $60 + 11 = 71$.

Answer: (A)

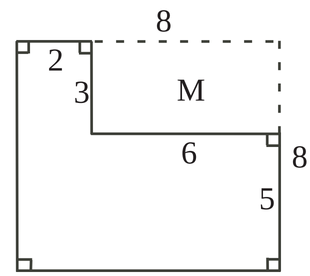
10. Since 700 is the product of 35 and y, then $35 \times y = 700$ or $y = 700 \div 35 = 20$.
Since 20 is the product of 5 and x, then $5 \times x = 20$ or $x = 20 \div 5 = 4$.

Answer: (B)

11. *Solution 1*
We divide the shape into two rectangles, A and B, by constructing the dotted line segment of length 2 units shown.
The area of rectangle A is $2 \times 3 = 6$ square units.
The length of rectangle B is 6 units plus the length of the dotted line segment, or $6 + 2 = 8$.
Thus, the area of rectangle B is $8 \times 5 = 40$ square units.
The area of the entire figure is the sum of the areas of rectangles A and B, or $6 + 40 = 46$ square units.



Solution 2
By constructing the dotted lines shown, we form a rectangle with length $2 + 6 = 8$ units and width $5 + 3 = 8$ units (in fact, this large rectangle is a square).
We find the required area by subtracting the area of rectangle M from the area of the 8 by 8 square.
Thus, the area is $(8 \times 8) - (6 \times 3) = 64 - 18 = 46$ square units.



Answer: (C)

12. If 4 schools each recycle $\frac{3}{4}$ of a tonne of paper, then combined, they recycle $4 \times \frac{3}{4} = \frac{12}{4} = 3$ tonnes of paper.
Since recycling 1 tonne of paper will save 24 trees, recycling 3 tonnes of paper will save $3 \times 24 = 72$ trees.

Answer: (B)

13. *Solution 1*
The mean of 5 consecutive integers is equal to the number in the middle.
Since the numbers have a mean of 21, if we were to distribute the quantities equally, we would have 21, 21, 21, 21, and 21.
Since the numbers are consecutive, the second number is 1 less than the 21 in the middle, while the fourth number is 1 more than the 21 in the middle.
Similarly, the first number is 2 less than the 21 in the middle, while the fifth number is 2 more than the 21 in the middle.
Thus, the numbers are $21 - 2$, $21 - 1$, 21, $21 + 1$, $21 + 2$.
The smallest of 5 consecutive integers having a mean of 21, is 19.

Solution 2
Since 21 is the mean of five consecutive integers, the smallest of these five integers must be less than 21.
Suppose that the smallest number is 20.

The mean of $20, 21, 22, 23$, and 24 is $\dfrac{20+21+22+23+24}{5} = 22$.
This mean of 22 is greater than the required mean of 21; thus, the smallest of the 5 consecutive integers must be less than 20.
Suppose that the smallest number is 19.
The mean of $19, 20, 21, 22$, and 23, is $\dfrac{19+20+21+22+23}{5} = 21$, as required.
Thus, the smallest of the 5 consecutive integers is 19.

Answer: (E)

14. *Solution 1*
Since the bag contains green mints and red mints only, the remaining $100\% - 75\% = 25\%$ of the mints must be red.
Thus, the ratio of the number of green mints to the number of red mints is $75 : 25 = 3 : 1$.

Solution 2
Since 75% of the mints are green, then $\frac{3}{4}$ of the mints are green.
Since the bag contains only green mints and red mints, then $1 - \frac{3}{4} = \frac{1}{4}$ of the mints in the bag are red.
Thus, there are 3 times as many green mints as red mints.
The ratio of the number of green mints to the number of red mints is $3 : 1$.

Answer: (B)

15. The area of square N is four times the area of square M or $4 \times 100 \text{ cm}^2 = 400 \text{ cm}^2$.
Thus, each side of square N has length $\sqrt{400} = 20$ cm.
The perimeter of square N is 4×20 cm $= 80$ cm.

Answer: (C)

16. First we must find the *magic constant*, that is, the sum of each row, column and diagonal.
From column one, we find that the magic constant is $(+1) + (-4) + (-3) = -6$.
In the diagonal extending from the top left corner to the bottom right corner, the two existing numbers $+1$ and -5 have a sum of -4.
Thus, to obtain the magic constant of -6 in this diagonal, -2 must occupy the centre square.
In the diagonal extending from the bottom left corner to the top right corner, the two numbers -3 and -2, have a sum of -5.
Thus, to obtain the magic constant of -6 in this diagonal, Y must equal -1.
The completed magic square is shown below.

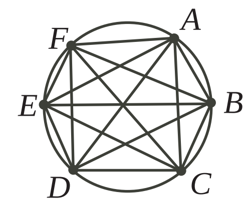
+1	−6	−1
−4	−2	0
−3	+2	−5

Answer: (A)

17. The smallest possible three-digit integer that is 17 more than a two-digit integer is 100 (100 is 17 more than 83 and 100 is in fact the smallest possible three-digit integer).
Notice that 101 is 17 more than 84, 102 is 17 more than 85, and so on. This continues until we reach 117 which is 17 more than 100, but 100 is not a two-digit integer. Thus, 116 is the largest possible three-digit integer that is 17 more than a two-digit integer (116 is 17 more than 99).
Therefore, all of the integers from 100 to 116 inclusive, or 17 three-digit integers, are exactly 17 more than a two-digit integer.

Answer: (A)

18. *Solution 1*



We label the 6 points A through F as shown and proceed to connect the points in all possible ways.
From point A, 5 line segments are drawn, 1 to each of the other points, B through F.
From point B, 4 new line segments are drawn, 1 to each of the points C through F, since the segment AB has already been drawn.
This continues, with 3 line segments drawn from point C, 2 from point D, 1 from point E, and 0 from point F since it will have already been joined to each of the other points.
In total, there are $5 + 4 + 3 + 2 + 1 = 15$ line segments.

Solution 2
Label the 6 points A through F as shown above.
From each of the 6 points, 5 line segments can be drawn leaving the point, 1 to each of the other 5 points.
Thus, the total number of line segments leaving the 6 points is $6 \times 5 = 30$.
However, this counts each of the line segments twice, since each segment will be counted as leaving both of its ends.
For example, the segment leaving point A and ending at point D is also counted as a segment leaving point D and ending at point A.
Thus, the actual number of line segments is $30 \div 2 = 15$.

Answer: (D)

19. The value of any positive fraction is increased by increasing the numerator and/or decreasing the denominator.
Thus, to obtain the largest possible sum, we choose 6 and 7 as the numerators, and 3 and 4 as the denominators.
We then calculate:

$$\frac{7}{3} + \frac{6}{4} = \frac{28}{12} + \frac{18}{12} = \frac{46}{12} = \frac{23}{6}$$

We recognize that $\frac{23}{6}$ is the largest of the 5 possible answers, and thus is the correct response.
(This means that we do not need to try $\frac{7}{4} + \frac{6}{3}$.)

Answer: (E)

20. *Solution 1*
To determine who *cannot* be sitting in the middle seat, we may eliminate the 4 people who *can* be sitting in the middle seat.
First, assume that Sally and Mike, who must be beside one another, are in seats 1 and 2, or in seats 2 and 1.
Since Andy and Jen are not beside each other, either Andy is in seat 3 (the middle seat) and Jen is in seat 5, or vice versa.
Thus, Andy and Jen can each be sitting in the middle seat and are eliminated as possible choices.
Next, assume that Sally and Mike are in seats 2 and 3, or in seats 3 and 2.
That is, either Sally is in the middle (seat 3), or Mike is.
In either case, seats 1, 4 and 5 are empty, allowing either Andy or Jen to choose seat 1 and hence, they are not next to one another.
This demonstrates that Sally and Mike can each be sitting in the middle seat.

Having eliminated Andy, Jen, Sally, and Mike, it must be Tom who cannot be sitting in the middle seat.

Solution 2
Assume that Tom is sitting in the middle (seat 3).
Since Sally and Mike are seated beside each other, they are either sitting in seats 4 and 5 or seats 1 and 2.
In either case, seats 1 and 2 remain empty or seats 4 and 5 remain empty.
However, Andy and Jen cannot sit beside each other.
Therefore, this arrangement is not possible.
Thus, Tom cannot be sitting in the middle seat.
Since the question implies that there is a unique answer, then Tom is the answer.

ANSWER: (E)

21. Traveling at a constant speed of 15 km/h, in 3 hours the bicycle will travel $15 \times 3 = 45$ km.
At the start, the bicycle was 195 km ahead of the bus.
Therefore, in order to catch up to the bicycle, the bus must travel 195 km plus the additional 45 km that the bicycle travels, or $195 + 45 = 240$ km.
To do this in 3 hours, the bus must travel at an average speed of $240 \div 3 = 80$ km/h.

ANSWER: (B)

22. When tossing a single coin, there are two possible outcomes, a head (H) or a tail (T).
When tossing 2 coins, there are $2 \times 2 = 4$ possible outcomes.
These are HH, HT, TH, and TT.
When tossing 3 coins, there are $2 \times 2 \times 2 = 8$ possible outcomes.
These are HHH, HHT, HTH, HTT, THH, THT, TTH, and TTT.
Of these 8 possible outcomes, there are 2 winning outcomes, HHH and TTT.
Thus, the probability of winning the *Coin Game* is $\frac{2}{8} = \frac{1}{4}$.

ANSWER: (B)

23. Since $\mathrm{M} \times \mathrm{O} \times \mathrm{M} = 49$, either $\mathrm{M} = 7$ and $\mathrm{O} = 1$ or $\mathrm{M} = 1$ and $\mathrm{O} = 49$.
However, since the value of the word TOTE is 18, O cannot have a value of 49 because 18 is not divisible by 49.
Thus, $\mathrm{M} = 7$ and $\mathrm{O} = 1$.
Since $\mathrm{T} \times \mathrm{O} \times \mathrm{T} \times \mathrm{E} = 18$ and $\mathrm{O} = 1$, we have $\mathrm{T} \times \mathrm{T} \times \mathrm{E} = 18$.
Therefore, either $\mathrm{T} = 3$ and $\mathrm{E} = 2$ or $\mathrm{T} = 1$ and $\mathrm{E} = 18$.
However, $\mathrm{O} = 1$, and since every letter has a different value, T cannot be equal to 1.
Thus, $\mathrm{T} = 3$ and $\mathrm{E} = 2$.
The value of the word TEAM is 168, so $\mathrm{T} \times \mathrm{E} \times \mathrm{A} \times \mathrm{M} = 168$, or $3 \times 2 \times \mathrm{A} \times 7 = 168$.
Thus, $42 \times \mathrm{A} = 168$ or $\mathrm{A} = 168 \div 42 = 4$.
The value of the word HOME is 70, so $\mathrm{H} \times \mathrm{O} \times \mathrm{M} \times \mathrm{E} = 70$, or $\mathrm{H} \times 1 \times 7 \times 2 = 70$.
Thus, $14 \times \mathrm{H} = 70$ or $\mathrm{H} = 70 \div 14 = 5$.
Finally, the value of the word MATH is $\mathrm{M} \times \mathrm{A} \times \mathrm{T} \times \mathrm{H} = 7 \times 4 \times 3 \times 5 = 420$.

ANSWER: (C)

24. The sum of two even numbers is even. The sum of two odd numbers is even.
The sum of an odd number and an even number is odd.
Thus, for the sum $m + n$ to be even, both m and n must be even, or they must both be odd.
If $m = 2$, then n must be even and greater than 2.
Thus, if $m = 2$ then n can be $4, 6, 8, 10, 12, 14, 16, 18$, or 20.
This gives 9 different pairs (m, n) when $m = 2$.
If $m = 4$, then n must be even and greater than 4.
Thus, if $m = 4$ then n can be $6, 8, 10, 12, 14, 16, 18$, or 20.
This gives 8 different pairs (m, n) when $m = 4$.
Continuing in this manner, each time we increase m by 2, the number of choices for n, and thus for (m, n), decreases by 1.
This continues until $m = 18$, at which point there is only one choice for n, namely $n = 20$.
Therefore, the total number of different pairs (m, n) where both m and n are even is, $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$.
Similarly, if $m = 1$, then n must be odd and greater than 1.
Thus, if $m = 1$, then n can be $3, 5, 7, 9, 11, 13, 15, 17$, or 19.
This gives 9 different pairs (m, n) when $m = 1$.
If $m = 3$, then n must be odd and greater than 3.
Thus, if $m = 3$ then n can be $5, 7, 9, 11, 13, 15, 17$, or 19.
This gives 8 different pairs (m, n) when $m = 3$.
Continuing in this manner, each time we increase m by 2, the number of choices for n, and thus for (m, n), decreases by 1.
This continues until $m = 17$, at which point there is only one choice for n, namely $n = 19$.
Therefore, the total number of different pairs (m, n) where both m and n are odd is, $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$.
Thus, the total number of different pairs (m, n) using numbers from the list $\{1, 2, 3, \ldots, 20\}$ such that $m < n$ and $m + n$ is even is $45 + 45 = 90$.

Answer: (B)

25. Together, Hose A and Hose B fill the pool in 6 hours.
Thus, it must take Hose A more than 6 hours to fill the pool when used by itself.
Therefore, $a \geq 7$, since a is a positive integer.
Similarly, it must take Hose B more than 6 hours to fill the pool when used by itself.
Therefore, $b \geq 7$, since b is a positive integer.
When used by itself, the fraction of the pool that Hose A fills in 6 hours is $\frac{6}{a}$.
When used by itself, the fraction of the pool that Hose B fills in 6 hours is $\frac{6}{b}$.
When used together, Hose A and Hose B fill the pool *once* in 6 hours. Thus, $\frac{6}{a} + \frac{6}{b} = 1$.
Since $a \geq 7$, $b \geq 7$, and both a and b are integers, then we can find values for a and b that satisfy the equation $\frac{6}{a} + \frac{6}{b} = 1$ by using systematic trial and error.
For example, let $a = 7$. Then $\frac{6}{7} + \frac{6}{b} = 1$, or $\frac{6}{b} = 1 - \frac{6}{7}$, or $\frac{6}{b} = \frac{1}{7}$.
Since $\frac{6}{42} = \frac{1}{7}$, then $b = 42$ and $a = 7$ is one possible solution to the equation $\frac{6}{a} + \frac{6}{b} = 1$.
Compare this to what happens when we let $a = 11$.
We have $\frac{6}{11} + \frac{6}{b} = 1$, or $\frac{6}{b} = 1 - \frac{6}{11}$, or $\frac{6}{b} = \frac{5}{11}$, or $5b = 66$.
Since there is no integer value for b that makes $5b$ equivalent to 66, then $a = 11$ does not give a possible solution.
The possible solutions found by systematic trial and error are shown below.

a	7	8	9	10	12	15	18	24	42
b	42	24	18	15	12	10	9	8	7

Any value for a larger than 42 requires b to be smaller than 7, but we know that $b \geq 7$.
Thus, there are only 9 different possible values for a.
Note:
We can reduce the time it takes to complete this trial and error above by recognizing that in the equation $\frac{6}{a} + \frac{6}{b} = 1$, the a and b are interchangeable.
That is, interchanging a and b in the equation, gives $\frac{6}{b} + \frac{6}{a} = 1$, which is the same equation.
For example, this tells us that since $a = 7$, $b = 42$ satisfies the equation, then $a = 42$, $b = 7$ satisfies the equation as well.
Moreover, if the pair (a, b) satisfies the equation, then (b, a) satisfies the equation, and if (a, b) does not satisfy the equation, then (b, a) does not satisfy the equation.
This interchangeability of a and b is seen in the symmetry of the list of possible solutions above.
Recognizing that this symmetry must exist allows us to quickly determine the 4 remaining solutions that follow after $a = 12$, $b = 12$.

Answer: (C)

Grade 8

1. Using the correct order of operations, $2 + 3 \times 4 + 10 = 2 + 12 + 10 = 24$.

   Answer: (A)

2. The athlete who won the race is the one who had the shortest running time.
   Thus, athlete C won the race.

   Answer: (C)

3. Substituting $x = 2$ and $y = 1$ into the expression $2x - 3y$, we have $2 \times 2 - 3 \times 1$.
   Using the correct order of operations, $2 \times 2 - 3 \times 1 = 4 - 3 = 1$.

   Answer: (B)

4. *Solution 1*
   Evaluating the left side of the equation, we get $44 \times 25 = 1100$.
   Thus, the number that should replace the $\square$ is $1100 \div 100 = 11$.

   Solution 2
   The left side of the equation, 44×25, can be rewritten as $11 \times 4 \times 25$.
   Since $11 \times 4 \times 25 = 11 \times 100$, then we can write $11 \times 100 = \square \times 100$, so the number that should replace the $\square$ is 11.

   Answer: (A)

5. The area of the rectangle, 12, is found by multiplying its length by its width.
   Since the length and width must be whole numbers, the only possible dimensions are:
   12 by 1, 6 by 2, and 4 by 3.
   (We recognize that 1 by 12, 2 by 6, and 3 by 4 are also possibilities, but the perimeter of the rectangle is unchanged by reversing the dimensions.)
   The perimeter of a rectangle is found by doubling the sum of the length and width.
   The results are shown in the table below.

Length	Width	Perimeter
12	1	26
6	2	16
4	3	14

   Therefore, the smallest possible perimeter of a rectangle with the given conditions is 14.

   Answer: (D)

6. We first recognize that $\frac{1}{4}$ is a common fraction in each of the five sums, and so the relative size of the sums depends only on the other fractions.
   Since $\frac{1}{3}$ is the largest of the fractions $\frac{1}{5}, \frac{1}{6}, \frac{1}{3}, \frac{1}{8}, \frac{1}{7}$, we conclude that $\frac{1}{4} + \frac{1}{3}$ is the largest sum.

   Answer: (C)

7. Since 1 gram is the approximate weight of 15 seeds, 300 grams is the approximate weight of $300 \times 15 = 4500$ seeds.
   Therefore, there are approximately 4500 seeds in the container.

   Answer: (B)

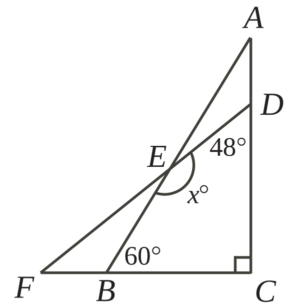
8. The first time after 10:25 at which all of the digits on the clock will be equal to one another is 11:11.
   Thus, the shortest length of time required is the elapsed time between 10:25 and 11:11,
   or 46 minutes (because the time from 10:25 to 11:00 is 35 minutes and the time from 11:00 to 11:11 is 11 minutes).

   Answer: (D)

9. Chris was given $\frac{1}{3}$ of 84 cookies, or $\frac{1}{3} \times 84 = \frac{84}{3} = 28$ cookies.
He ate $\frac{3}{4}$ of the 28 cookies he was given, or $\frac{3}{4} \times 28 = \frac{84}{4} = 21$ cookies.

ANSWER: (E)

10. *Solution 1*
In $\triangle ABC$ shown below, $\angle BAC = 180^\circ - \angle ABC - \angle ACB = 180^\circ - 60^\circ - 90^\circ = 30^\circ$.
Since $\angle ADC$ is a straight angle, $\angle ADE = 180^\circ - \angle CDE = 180^\circ - 48^\circ = 132^\circ$.
In $\triangle AED$, $\angle AED = 180^\circ - \angle ADE - \angle EAD = 180^\circ - 132^\circ - 30^\circ = 18^\circ$.
Since $\angle AEB$ is a straight angle, $\angle DEB = 180^\circ - \angle AED = 180^\circ - 18^\circ = 162^\circ$.
Thus, the value of x is 162.



Solution 2
The sum of the interior angles of a quadrilateral is 360°.
In quadrilateral $BCDE$ shown above,

$$\angle DEB = 360^\circ - \angle EDC - \angle DCB - \angle CBE = 360^\circ - 48^\circ - 90^\circ - 60^\circ = 162^\circ.$$

Thus, the value of x is 162.

ANSWER: (E)

11. *Solution 1*
The mean of 5 consecutive integers is equal to the number in the middle.
Since the numbers have a mean of 21, if we were to distribute the quantities equally, we would have 21, 21, 21, 21, and 21.
Since the numbers are consecutive, the second number is 1 less than the 21 in the middle, while the fourth number is 1 more than the 21 in the middle.
Similarly, the first number is 2 less than the 21 in the middle, while the fifth number is 2 more than the 21 in the middle.
Thus, the numbers are $21 - 2$, $21 - 1$, 21, $21 + 1$, $21 + 2$.
The smallest of 5 consecutive integers having a mean of 21, is 19.

Solution 2
Since 21 is the mean of five consecutive integers, the smallest of these five integers must be less than 21.
Suppose the smallest integer is 20.
The mean of $20, 21, 22, 23$, and 24 is $\dfrac{20 + 21 + 22 + 23 + 24}{5} = 22$.
This mean of 22 is greater than the required mean of 21; thus, the smallest of the 5 consecutive integers must be less than 20.
Suppose the smallest integer is 19.
The mean of $19, 20, 21, 22$, and 23 is $\dfrac{19 + 20 + 21 + 22 + 23}{5} = 21$, as required.
Thus, the smallest of 5 consecutive integers having a mean of 21, is 19.

ANSWER: (E)

12. For every 3 white balls in the jar, there are 2 red balls in the jar.
Since there are 9 white balls in the jar, which is 3 groups of 3 white balls, there must be 3 groups of 2 red balls in the jar.
Thus, there are $3 \times 2 = 6$ red balls in the jar.

Answer: (D)

13. Evaluating, $\left(\frac{11}{12}\right)^2 = \left(\frac{11}{12}\right) \times \left(\frac{11}{12}\right) = \frac{11\times 11}{12\times 12} = \frac{121}{144}$.
Since $\frac{121}{144} > \frac{72}{144} = \frac{1}{2}$ and $\frac{121}{144} < \frac{144}{144} = 1$, the value of $\left(\frac{11}{12}\right)^2$ is between $\frac{1}{2}$ and 1.

Answer: (B)

14. During the 5 games, Gina faced $10 + 13 + 7 + 11 + 24 = 65$ shots in total.
During the 5 games, Gina saved $7 + 9 + 6 + 9 + 21 = 52$ of the 65 shots.
Thus, the percentage of total shots saved is $\frac{52}{65} \times 100\% = 0.80 \times 100\% = 80\%$.

Answer: (C)

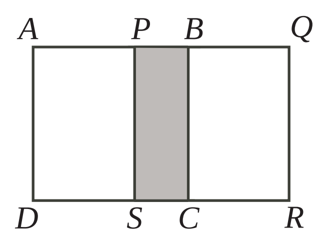
15. To find the smallest possible sum, we first choose the tens digit of each number to be as small as possible.
Therefore, we choose 5 and 6 as the two tens digits.
Next, we choose the units digits to be as small as possible.
Since 7 and 8 are each less than 9, we choose 7 and 8 as the two units digits.
Using 5 and 6 as the tens digits, 7 and 8 as the units digits, we evaluate the only two possibilities.

$$\begin{array}{rr} & 57 \\ + & 68 \\ \hline & 125 \end{array} \qquad \begin{array}{rr} & 58 \\ + & 67 \\ \hline & 125 \end{array}$$

(Can you see why these two sums should be equal?)
The smallest possible sum is 125.

Answer: (B)

16. *Solution 1*
Since $AQ = 20$ and $AB = 12$, then $BQ = AQ - AB = 20 - 12 = 8$.
Thus, $PB = PQ - BQ = 12 - 8 = 4$.
Since $PS = 12$, the area of rectangle $PBCS$ is $12 \times 4 = 48$.

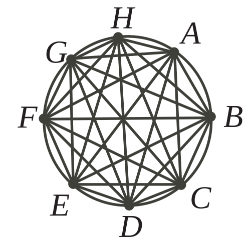


Solution 2
The sum of the areas of squares $ABCD$ and $PQRS$ is $2 \times (12 \times 12) = 2 \times 144 = 288$.
The area of rectangle $AQRD$ is $12 \times 20 = 240$.
The sum of the areas of $ABCD$ and $PQRS$ is equal to the sum of the areas of $APSD$, $PBCS$, $PBCS$, and $BQRC$.
The area of rectangle $AQRD$ is equal to the sum of the areas of $APSD$, $PBCS$, and $BQRC$.
Therefore, the sum of the areas of $ABCD$ and $PQRS$, minus the area of $AQRD$, is the area of $PBCS$.
Thus, the shaded rectangle $PBCS$ has area $288 - 240 = 48$.

Answer: (C)

17. *Solution 1*

Label the 8 points A through H as shown and proceed to connect the points in all possible ways.



From point A, 7 line segments are drawn, 1 to each of the other points, B through H.
From point B, 6 new line segments are drawn, 1 to each of the points C through H, since the segment AB has already been drawn.
This continues, with 5 line segments drawn from point C, 4 from D, 3 from E, 2 from F, 1 from G and 0 from point H since it will already be joined to each of the other points.
In total, there are $7 + 6 + 5 + 4 + 3 + 2 + 1 = 28$ line segments.

Solution 2

Label the 8 points A through H as shown above and proceed to connect the points in all possible ways.
From each of the 8 points, 7 line segments can be drawn leaving the point, 1 to each of the other 7 points.
Thus, the total number of line segments leaving the 8 points is $8 \times 7 = 56$.
However, this counts each of the line segments twice, since each segment will be counted as leaving both of its ends.
For example, the segment leaving point A and ending at point D is also counted as a segment leaving point D and ending at point A.
Thus, the actual number of line segments is half of 56 or $56 \div 2 = 28$.

Answer: (E)

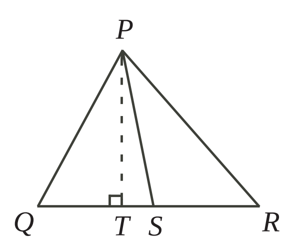
18. Traveling at a constant speed of 15 km/h, in 3 hours the bicycle will travel $15 \times 3 = 45$ km.
At the start, the bicycle was 195 km ahead of the bus.
Therefore, in order to catch up to the bicycle, the bus must travel 195 km plus the additional 45 km that the bicycle travels, or $195 + 45 = 240$ km.
To do this in 3 hours, the bus must travel at an average speed of $240 \div 3 = 80$ km/h.

Answer: (B)

19. Figure 1 is formed with 1 square.
Figure 2 is formed with $4 + 1$ squares.
Figure 3 is formed with $4 + 4 + 1 = 2 \times 4 + 1$ squares.
Figure 4 is formed with $4 + 4 + 4 + 1 = 3 \times 4 + 1$ squares.
Figure 5 is formed with $4 + 4 + 4 + 4 + 1 = 4 \times 4 + 1$ squares.
Thus, the number of groups of 4 squares needed to help form the Figure is increasing by 1.
Also, in each case the number of groups of 4 squares needed is one less than the Figure number.
For example, Figure 6 will be formed with 5 groups of 4 squares plus 1 additional square.
In general, we can say that Figure N will be formed with $N - 1$ groups of 4 squares, plus 1 additional square.
Thus, Figure 2010 will be formed with $2009 \times 4 + 1 = 8036 + 1 = 8037$ squares.

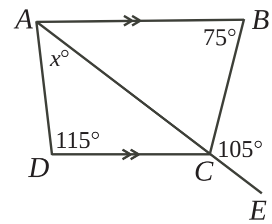
Answer: (A)

20. Position point T on QR such that PT is perpendicular to QR.
Line segment PT is the height of $\triangle PQS$ with base QS, and is also the height of $\triangle PRS$ with base SR.
Since $\triangle PQS$ and $\triangle PRS$ have equal heights and equal areas, their bases must be equal. Thus, $QS = SR$.



Answer: (D)

21. Since $\angle ACE$ is a straight angle, $\angle ACB = 180° - 105° = 75°$.
In $\triangle ABC$, $\angle BAC = 180° - \angle ABC - \angle ACB = 180° - 75° - 75° = 30°$.
Since AB is parallel to DC, $\angle ACD = \angle BAC = 30°$ (alternate angles).
In $\triangle ADC$, $\angle DAC = 180° - \angle ADC - \angle ACD = 180° - 115° - 30° = 35°$.
Thus, the value of x is 35.



Answer: (A)

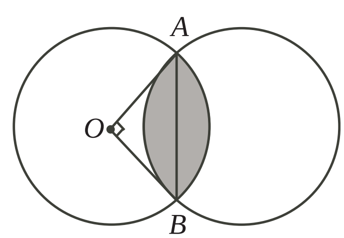
22. We first recognize that in the products, $r \times s$, $u \times r$ and $t \times r$, r is the only variable that occurs in all three.
Thus, to make $r \times s + u \times r + t \times r$ as large as possible, we choose $r = 5$, the largest value possible.
Since each of s, u and t is multiplied by r once only, and the three products are then added, it does not matter which of s, u or t we let equal 2, 3 or 4, as the result will be the same.
Therefore, let $s = 2$, $u = 3$ and $t = 4$.
Thus, the largest possible value of $r \times s + u \times r + t \times r$ is $5 \times 2 + 3 \times 5 + 4 \times 5 = 10 + 15 + 20 = 45$.

Answer: (B)

23. Since Kevin needs 12 hours to shovel all of his snow, he shovels $\frac{1}{12}$ of his snow every hour.
Since Dave needs 8 hours to shovel all of Kevin's snow, he shovels $\frac{1}{8}$ of Kevin's snow every hour.
Similarly, John shovels $\frac{1}{6}$ of Kevin's snow every hour, and Allison shovels $\frac{1}{4}$ of Kevin's snow every hour.
Together, Kevin, Dave, John, and Allison can shovel $\frac{1}{12} + \frac{1}{8} + \frac{1}{6} + \frac{1}{4} = \frac{2}{24} + \frac{3}{24} + \frac{4}{24} + \frac{6}{24} = \frac{15}{24}$ of Kevin's snow every hour.
Therefore, together they can shovel $\frac{15}{24} \div 60 = \frac{15}{24} \times \frac{1}{60} = \frac{15}{1440} = \frac{1}{96}$ of Kevin's snow every minute.
Thus, by shoveling $\frac{1}{96}$ of Kevin's snow per minute, together they will shovel all of Kevin's snow in 96 minutes.

Answer: (D)

24. Label points A and B, the points of intersection of the two circles, and point O, the centre of the left circle.
Construct line segment AB, which by symmetry divides the shaded area in half.
Construct radii OA and OB with $OA = OB = 10$ cm.
Since each circle contains 25% or $\frac{1}{4}$ of the other circle's circumference, $\angle AOB = \frac{1}{4} \times 360° = 90°$.
Thus, the area of sector AOB is $\frac{1}{4}$ of the area of the entire circle, or $\frac{1}{4}\pi r^2 = \frac{1}{4}\pi 10^2 = 25\pi$ cm^2.
The area of $\triangle AOB$ is $\dfrac{OA \times OB}{2} = \dfrac{10 \times 10}{2} = 50$ cm^2.
The area remaining after $\triangle AOB$ is subtracted from sector AOB is equal to half of the shaded area. Thus, the shaded area is $2 \times (25\pi - 50) \approx 2 \times (28.5398) = 57.0796$ cm^2.
The area of the shaded region is closest to 57.08 cm^2.



Answer: (A)

25. We are given that the first two terms of a 10 term sequence are 1 and x.
Since each term after the second is the sum of the previous two terms, then the third term is $1 + x$.
Since the fourth term is the sum of the second and third terms, then the fourth term is $x + (1 + x) = 1 + 2x$.
Continuing in this manner, we construct the 10 term sequence:

$$1, x, 1 + x, 1 + 2x, 2 + 3x, 3 + 5x, 5 + 8x, 8 + 13x, 13 + 21x, 21 + 34x.$$

Each of the second through tenth terms is dependent on the value of x, and thus, any one of these terms could potentially equal 463.
For the second term to equal 463, we need $x = 463$, which is possible since the only requirement is that x is a positive integer.
Thus, if $x = 463$ then 463 appears as the second term in the sequence.
For the third term to equal 463, we need $1 + x = 463$, or $x = 462$.
Thus, if $x = 462$ then 463 appears as the third term in the sequence.
For the fourth term to equal 463, we need $1 + 2x = 463$, or $2x = 462$ or $x = 231$.
Thus, if $x = 231$ then 463 appears as the fourth term in the sequence.
For the fifth term to equal 463, we need $2 + 3x = 463$, or $3x = 461$ or $x = \frac{461}{3}$.
However, $\frac{461}{3}$ is not an integer, and thus, 463 cannot appear as the fifth term in the sequence.
We continue in this manner and summarize all the results in the table below.

Term	Expression	Equation	Value of x	Is x an integer?
2nd	x	$x = 463$	$x = 463$	Yes
3rd	$1 + x$	$1 + x = 463$	$x = 462$	Yes
4th	$1 + 2x$	$1 + 2x = 463$	$x = 231$	Yes
5th	$2 + 3x$	$2 + 3x = 463$	$x = \frac{461}{3}$	No
6th	$3 + 5x$	$3 + 5x = 463$	$x = 92$	Yes
7th	$5 + 8x$	$5 + 8x = 463$	$x = \frac{458}{8}$	No
8th	$8 + 13x$	$8 + 13x = 463$	$x = 35$	Yes
9th	$13 + 21x$	$13 + 21x = 463$	$x = \frac{450}{21}$	No
10th	$21 + 34x$	$21 + 34x = 463$	$x = 13$	Yes

Therefore, the sum of all possible integer values of x for which 463 appears in the sequence is $463 + 462 + 231 + 92 + 35 + 13 = 1296$.

Answer: (B)

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Grade 8 solutions follow the Grade 7 solutions

2009 Gauss Contests

(Grades 7 and 8)

Wednesday, May 13, 2009

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Lloyd Auckland
Terry Bae
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Jennifer Couture
Fiona Dunbar
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Joanne Kursikowski
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Tanya Thompson, Nottawa, ON
Chris Wu, Amesbury M.S., Toronto, ON

Grade 7

1. Adding, $4.1 + 1.05 + 2.005 = 5.15 + 2.005 = 7.155$.

   Answer: (A)

2. Since the triangle is equilateral, all sides are equal in length.
   Therefore, the perimeter of the triangle is $8 + 8 + 8 = 8 \times 3 = 24$.

   Answer: (C)

3. The numbers 12, 14 and 16 are even, and therefore divisible by 2 so not prime.
   The number 15 is divisible by 5; therefore, it is also not prime.
   Each of the remaining numbers, 11, 13 and 17, has no positive divisor other than 1 and itself.
   Therefore, 3 numbers in the list are prime.

   Answer: (D)

4. *Solution 1*
   Since each number in the set is between 0 and 1, they can be ordered from smallest to largest by comparing their tenths digits first. In order from smallest to largest the list is

   $$\{0.05, 0.25, 0.37, 0.40, 0.81\}.$$

   The smallest number in the list is 0.05.

   Solution 2
   Consider the equivalent fraction for each decimal:
   $0.40 = \frac{40}{100}$, $0.25 = \frac{25}{100}$, $0.37 = \frac{37}{100}$, $0.05 = \frac{5}{100}$, and $0.81 = \frac{81}{100}$.
   Since the denominators all equal 100, we choose the fraction with the smallest numerator.
   Therefore, $0.05 = \frac{5}{100}$ is the smallest number in the set.

   Answer: (D)

5. The x-coordinate of point P lies between -2 and 0. The y-coordinate lies between 2 and 4.
   Of the possible choices, $(-1, 3)$ is the only point that satisfies both of these conditions.

   Answer: (E)

6. The temperature in Vancouver is 22°C.
   The temperature in Calgary is $22°\text{C} - 19°\text{C} = 3°\text{C}$.
   The temperature in Quebec City is $3°\text{C} - 11°\text{C} = -8°\text{C}$.

   Answer: (C)

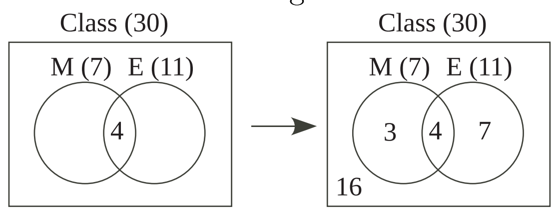
7. Since a real distance of 60 km is represented by 1 cm on the map, then a real distance of 540 km is represented by $\dfrac{540}{60}$ cm or 9 cm on the map.

   Answer: (A)

8. The sum of the three angles in any triangle is always 180°.
   In $\triangle PQR$, the sum of $\angle P$ and $\angle Q$ is $60°$, and thus $\angle R$ must measure $180° - 60° = 120°$.

   Answer: (C)

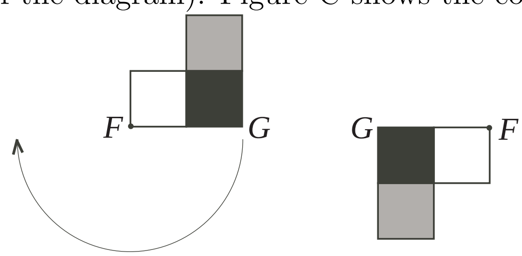
9. The first Venn diagram below shows that there are 30 students in the class, 7 students have been to Mexico, 11 students have been to England and 4 students have been to both countries.
Of the 7 students that have been to Mexico, 4 have also been to England.
Therefore, $7 - 4 = 3$ students have been to Mexico and not England.
Of the 11 students that have been to England, 4 have also been to Mexico.
Therefore, $11 - 4 = 7$ students have been to England and not Mexico.



Therefore, 3 students have been to Mexico only, 7 students have been to England only, and 4 students have been to both.
In the class of 30 students, this leaves $30 - 3 - 7 - 4 = 16$ students who have not been to Mexico or England.

Answer: (B)

10. Consider rotating the horizontal line segment FG (as shown below) 180° about point F.
A 180° rotation is half of a full rotation. Point F stays fixed, while segment FG rotates to the left of F (as does the rest of the diagram). Figure C shows the correct result.



Answer: (C)

11. Since Scott runs 4 m for every 5 m Chris runs, Scott runs $\frac{4}{5}$ of the distance that Chris runs in the same time. When Chris crosses the finish line he will have run 100 m.
When Chris has run 100 m, Scott will have run $\frac{4}{5} \times 100 = 80$ m.

Answer: (E)

12. The area of a triangle can be calculated using the formula Area $= \frac{1}{2} \times$ base $\times$ height.
The area is 27 cm^2 and the base measures 6 cm. Substituting these values into the formula, $A = \frac{1}{2} \times b \times h$ becomes $27 = \frac{1}{2} \times 6 \times h$ or $27 = 3h$. Therefore, $h = 9$ cm.

Answer: (A)

13. There are 60 seconds in a minute, 60 minutes in an hour, 24 hours in a day and 7 days in a week. Therefore, the number of seconds in one week is $60 \times 60 \times 24 \times 7$.

Answer: (D)

14. *Solution 1*
S represents a value of approximately 1.5 on the number line, while T is approximately 1.6. Then $S \div T$ is approximately equal to $1.5 \div 1.6 = 0.9375$. R is the only value on the number line that is slightly less than 1 and therefore best represents the value of $S \div T$.

Solution 2
S is slightly less than T, so $\dfrac{S}{T}$ is slightly less than 1. Thus, $\dfrac{S}{T}$ is best represented by R.

Answer: (C)

15. For the sum to be a maximum, we try to use the largest divisor possible.
Although 144 is the largest divisor, using it would require that the remaining two divisors both equal 1 (since the divisors are integers).
Since the question requires the product of three different divisors, $144 = 144 \times 1 \times 1$ is not possible and the answer cannot be $144 + 1 + 1 = 146$ or (C).
The next largest divisor of 144 is 72 and $144 = 72 \times 2 \times 1$.
Now the three factors are different and their sum is $72 + 2 + 1 = 75$.
Since 75 is the largest possible answer remaining, we have found the maximum.

Answer: (B)

16. *Solution 1*
For the square to have an area of 25, each side length must be $\sqrt{25} = 5$.
The rectangle's width is equal to that of the square and therefore must also be 5.
The length of the rectangle is double its width or $5 \times 2 = 10$.
The area of the rectangle is thus $5 \times 10 = 50$.

Solution 2
The rectangle has the same width as the square but twice the length.
Thus, the rectangle's area is twice that of the square or $2 \times 25 = 50$.

Answer: (D)

17. The six other players on the team averaged 3.5 points each.
The total of their points was $6 \times 3.5 = 21$.
Vanessa scored the remainder of the points, or $48 - 21 = 27$ points.

Answer: (E)

18. Since x and z are positive integers and $xz = 3$, the only possibilities are $x = 1$ and $z = 3$ or $x = 3$ and $z = 1$.
Assuming that $x = 1$ and $z = 3$, $yz = 6$ implies $3y = 6$ or $y = 2$.
Thus, $x = 1$ and $y = 2$ and $xy = 2$.
This contradicts the first equation $xy = 18$.
Therefore, our assumption was incorrect and it must be true that $x = 3$ and $z = 1$.
Then $yz = 6$ and $z = 1$ implies $y = 6$.
Checking, $x = 3$ and $y = 6$ also satisfies $xy = 18$, the first equation.
Therefore, the required sum is $x + y + z = 3 + 6 + 1 = 10$.

Answer: (B)

19. The value of all quarters is \$10.00.
Each quarter has a value of \$0.25.
There are thus $10 \div 0.25 = 40$ quarters in the jar.
Similarly, there are $10 \div 0.05 = 200$ nickels, and $10 \div 0.01 = 1000$ pennies in the jar.
In total, there are $40 + 200 + 1000 = 1240$ coins in the jar.
The probability that the selected coin is a quarter is $\dfrac{\text{the number of quarters}}{\text{the total number of coins}} = \dfrac{40}{1240} = \dfrac{1}{31}$.

Answer: (B)

20. Since V is the midpoint of PR, then $PV = VR$.
Since $UVRW$ is a parallelogram, then $VR = UW$.
Since W is the midpoint of US, then $UW = WS$.
Thus, $PV = VR = UW = WS$.

Similarly, $QW = WR = UV = VT$.
Also, R is the midpoint of TS and therefore, $TR = RS$.
Thus, $\triangle VTR$ is congruent to $\triangle WRS$, and so the two triangles have equal area.
Diagonal VW in parallelogram $UVRW$ divides the area of the parallelogram in half.
Therefore, $\triangle UVW$ and $\triangle RWV$ have equal areas.
In quadrilateral $VRSW$, $VR = WS$ and VR is parallel to WS.
Thus, $VRSW$ is a parallelogram and the area of $\triangle RWV$ is equal to the area of $\triangle WRS$.
Therefore, $\triangle VTR$, $\triangle WRS$, $\triangle RWV$, and $\triangle UVW$ have equal areas, and so these four triangles divide $\triangle STU$ into quarters.
Parallelogram $UVRW$ is made from two of these four quarters of $\triangle STU$, or one half of $\triangle STU$.
The area of parallelogram $UVRW$ is thus $\frac{1}{2}$ of 1, or $\frac{1}{2}$.

ANSWER: (B)

21. Together, Lara and Ryan ate $\frac{1}{4} + \frac{3}{10} = \frac{5}{20} + \frac{6}{20} = \frac{11}{20}$ of the pie.
Therefore, $1 - \frac{11}{20} = \frac{9}{20}$ of the pie remained.
The next day, Cassie ate $\frac{2}{3}$ of the pie that remained.
This implies that $1 - \frac{2}{3} = \frac{1}{3}$ of the pie that was remaining was left after Cassie finished eating.
Thus, $\frac{1}{3}$ of $\frac{9}{20}$, or $\frac{3}{20}$ of the original pie was not eaten.

ANSWER: (D)

22. The first row is missing a 4 and a 2. Since there is already a 2 in the second column (in the second row), the first row, second column must contain a 4 and the first row, fourth column must contain a 2. To complete the upper left 2×2 square, the second row, first column must contain a 3. The second row is now missing both a 4 and a 1. But the fourth column already contains a 4 (in the fourth row), therefore the second row, fourth column must contain a 1. To complete the fourth column, we place a 3 in the third row. Now the P cannot be a 3, since there is already a 3 in the third row. Also, the P cannot be a 4 or a 2, since the second column already contains these numbers. By process of elimination, the digit 1 must replace the P.

ANSWER: (A)

23. *Solution 1*
We can suppose that the jug contains 1 litre of water at the start. The following table shows the quantity of water poured in each glass and the quantity of water remaining in each glass after each pouring, stopping when the quantity of water remaining is less than 0.5 L.

Number of glasses	Number of litres poured	Number of litres remaining
1	10 % of 1 = 0.1	$1 - 0.1 = 0.9$
2	10 % of 0.9 = 0.09	$0.9 - 0.09 = 0.81$
3	10 % of 0.81 = 0.081	$0.81 - 0.081 = 0.729$
4	10 % of 0.729 = 0.0729	$0.729 - 0.0729 = 0.6561$
5	10 % of 0.6561 = 0.06561	$0.6561 - 0.06561 = 0.59049$
6	10 % of 0.59049 = 0.059049	$0.59049 - 0.059049 = 0.531441$
7	10 % of 0.531441 = 0.0531441	$0.531441 - 0.0531441 = 0.4782969$

We can see from the table that the minimum number of glasses that Kim must pour so that less than half of the water remains in the jug is 7.

Solution 2
Removing 10% of the water from the jug is equivalent to leaving 90% of the water in the jug.

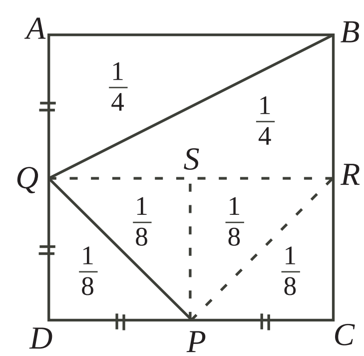
Thus, to find the total fraction remaining in the jug after a given pour, we multiply the previous total by 0.9.
We make the following table; stopping when the fraction of water remaining in the glass is first less than 0.5 (one half).

Number of glasses poured	Fraction of water remaining
1	$0.9 \times 1 = 0.9$
2	$0.9 \times 0.9 = 0.81$
3	$0.9 \times 0.81 = 0.729$
4	$0.9 \times 0.729 = 0.6561$
5	$0.9 \times 0.6561 = 0.59049$
6	$0.9 \times 0.59049 = 0.531441$
7	$0.9 \times 0.531441 = 0.4782969$

We can see from the table that the minimum number of glasses that Kim must pour so that less than half of the water remains in the jug is 7.

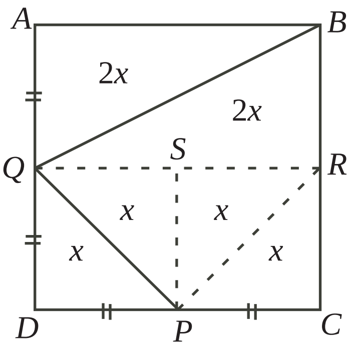
Answer: (C)

24. *Solution 1*
Draw line segment QR parallel to DC, as in the following diagram. This segment divides square $ABCD$ into two halves. Since triangles ABQ and RQB are congruent, each is half of rectangle $ABRQ$ and therefore one quarter of square $ABCD$. Draw line segment PS parallel to DA, and draw line segment PR. Triangles PDQ, PSQ, PSR and PCR are congruent. Therefore each is one quarter of rectangle $DCRQ$ and therefore one eighth of square $ABCD$.



Quadrilateral $QBCP$ therefore represents $\frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{8}$ of square $ABCD$. Its area is therefore $\frac{5}{8}$ of the area of the square.
Therefore, $\frac{5}{8}$ of the area of the square is equal to 15. Therefore, $\frac{1}{8}$ of the area of the square is equal to 3. Therefore the square has an area of 24.

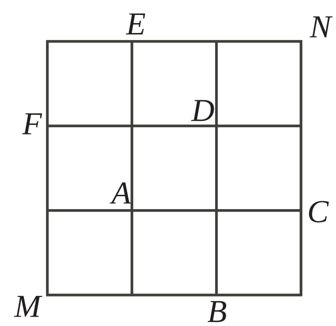
Solution 2
Draw a line segment from Q to R, the midpoint of BC.
Draw a line segment from P to S, the midpoint of QR.
Let the area of $\triangle QSP$ equal x. Thus, the area of $\triangle QDP$ is also x and $QDPS$ has area $2x$.
Square $SPCR$ is congruent to square $QDPS$ and thus has area $2x$.
Rectangle $QDCR$ has area $4x$, as does the congruent rectangle $AQRB$.
Also, $\triangle AQB$ and $\triangle BRQ$ have equal areas and thus, each area is $2x$.



Quadrilateral $QBCP$ is made up of $\triangle BRQ$, $\triangle QSP$ and square $SPCR$, and thus has area $2x + x + 2x = 5x$.
Since quadrilateral $QBCP$ has area 15, then $5x = 15$ or $x = 3$.
Therefore, the area of square $ABCD$, which is made up of quadrilateral $QBCP$, $\triangle AQB$ and $\triangle QDP$, is $5x + 2x + x = 8x = 8(3) = 24$.

ANSWER: (E)

25. Labeling the diagram as shown below, we can describe paths using the points they pass through.



The path $MADN$ is the only path of length 3 (traveling along 3 diagonals).
Since the diagram is symmetrical about MN, all other paths will have a *reflected path* in the line MN and therefore occur in pairs. This observation alone allows us to eliminate (B) and (D) as possible answers since they are even.
The following table lists all possible paths from M to N traveling along diagonals only.

Path length	Path Name	Reflected Path
3	$MADN$	same
5	$MABCDN$	$MAFEDN$
9	$MABCDEFADN$	$MAFEDCBADN$
	$MABCDAFEDN$	$MAFEDABCDN$
	$MADCBAFEDN$	$MADEFABCDN$

At this point we have listed 9 valid paths. Since paths occur in pairs (with the exception of $MADN$), the next possible answer would be 11. Since 11 is not given as an answer (and 9 is the largest possible answer given), we can be certain that we have found them all.

ANSWER: (E)

Grade 8

1. Using the correct order of operations, $1 + 3^2 = 1 + 9 = 10$.

ANSWER: (B)

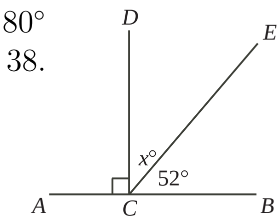
2. Calculating, $-10 + (-12) = -22$.

ANSWER: (D)

3. Using a 1 litre jug of water, Jack could fill two 0.5 (one half) litre bottles of water.
Using a 3 litre jug of water, Jack could fill $3 \times 2 = 6$ bottles of water.
(Check: $3 \div 0.5 = 3 \div \frac{1}{2} = 3 \times 2 = 6$)

ANSWER: (C)

4. Since AB is a line segment, $\angle ACD + \angle DCE + \angle ECB = 180^\circ$ or $90^\circ + x^\circ + 52^\circ = 180^\circ$ or $x^\circ = 180^\circ - 90^\circ - 52^\circ$ or $x = 38$.



ANSWER: (B)

5. Calculating, $\frac{7}{9} = 7 \div 9 = 0.7777\cdots = 0.\overline{7}$. Rounded to 2 decimal places, $\frac{7}{9}$ is 0.78.

ANSWER: (C)

6. The graph shows that vehicle X uses the least amount of fuel for a given distance.
Therefore, it is the most fuel efficient vehicle and will travel the farthest using 50 litres of fuel.

ANSWER: (D)

7. Kayla spent $\frac{1}{4} \times \$100 = \$25$ on rides and $\frac{1}{10} \times \$100 = \$10$ on food.
The total that she spent was \$35.

ANSWER: (E)

8. The polyhedron has 6 faces and 8 vertices.
The equation $F + V - E = 2$ becomes $6 + 8 - E = 2$ or $14 - E = 2$ or $E = 14 - 2 = 12$.
Therefore, the polyhedron has 12 edges.

ANSWER: (A)

9. Eliminating multiple occurrences of the same letter, the word 'PROBABILITY' uses 9 different letters of the alphabet, A, B, I, L, O, P, R, T, and Y.
Since there are 26 letters in the alphabet, the probability that Jeff picks one of the 9 different letters in 'PROBABILITY' is $\frac{9}{26}$.

ANSWER: (A)

10. *Solution 1*
Since the two numbers differ by 2 but add to 20, the smaller number must be 1 less than half of 20 while the larger number is 1 greater than half of twenty.
The smaller number is 9 and the larger is 11.

Solution 2
Since the numbers differ by 2, let the smaller number be represented by x and the larger number be represented by $x + 2$.
Since their sum is 20, then $x + x + 2 = 20$ or $2x = 18$ or $x = 9$.
The smaller number is 9 and the larger is 11.

ANSWER: (A)

11. Since $\angle ABC = \angle ACB$, then $\triangle ABC$ is isosceles and $AB = AC$.
Given that $\triangle ABC$ has a perimeter of 32, $AB + AC + 12 = 32$ or $AB + AC = 20$.
But $AB = AC$, so $2AB = 20$ or $AB = 10$.

Answer: (C)

12. Substituting $C = 10$, the equation $F = \frac{9}{5}C + 32$ becomes $F = \frac{9}{5} \times 10 + 32 = 18 + 32 = 50$.
A temperature of 10 degrees Celsius is equal to 50 degrees Fahrenheit.

Answer: (D)

13. Beginning with the positive integer 1 as a number in the first pair, we get the sum $101 = 1+100$.
From this point we can continue to increase the first number by one while decreasing the second number by one, keeping the sum equal to 101.
The list of possible sums is:

$$\begin{aligned} 101 &= 1 + 100 \\ 101 &= 2 + 99 \\ 101 &= 3 + 98 \\ &\vdots \\ 101 &= 50 + 51 \end{aligned}$$

After this point, the first number will no longer be smaller than the second if we continue to add 1 to the first number and subtract 1 from the second.
There are 50 possible sums in all.

Answer: (A)

14. The six other players on the team averaged 3.5 points each.
The total of their points was $6 \times 3.5 = 21$.
Vanessa scored the remainder of the points, or $48 - 21 = 27$ points.

Answer: (E)

15. Triangle PQR is a right-angled triangle since $\angle PQR = 90^\circ$ (because $PQRS$ is a rectangle).
In $\triangle PQR$, the Pythagorean Theorem gives,

$$\begin{aligned} PR^2 &= PQ^2 + QR^2 \\ 13^2 &= 12^2 + QR^2 \\ 169 &= 144 + QR^2 \\ 169 - 144 &= QR^2 \\ QR^2 &= 25 \end{aligned}$$

So $QR = 5$ since $QR > 0$. The area of $PQRS$ is thus $12 \times 5 = 60$.

Answer: (B)

16. When it is 3:00 p.m. in Victoria, it is 6:00 p.m. in Timmins.
Therefore, Victoria time is always 3 hours earlier than Timmins time.
When the flight arrived at 4:00 p.m. local Timmins time, the time in Victoria was 1:00 p.m.
The plane left at 6:00 a.m. Victoria time and arrived at 1:00 p.m. Victoria time.
The flight was 7 hours long.

Answer: (D)

17. The value of all quarters is \$10.00.
Each quarter has a value of \$0.25.
There are thus $10 \div 0.25 = 40$ quarters in the jar.
Similarly, there are $10 \div 0.05 = 200$ nickels, and $10 \div 0.01 = 1000$ pennies in the jar.
In total, there are $40 + 200 + 1000 = 1240$ coins in the jar.
The probability that the selected coin is a quarter is $\dfrac{\text{the number of quarters}}{\text{the total number of coins}} = \dfrac{40}{1240} = \dfrac{1}{31}$.

Answer: (B)

18. Of the 40 students, 12 did not like either dessert.
Therefore, $40 - 12 = 28$ students liked at least one of the desserts.
But 18 students said they liked apple pie, 15 said they liked chocolate cake, and $18 + 15 = 33$, so $33 - 28 = 5$ students must have liked both of the desserts.

Answer: (E)

19. In the ones column, $P + Q$ ends in 9. So $P + Q = 9$ or $P + Q = 19$.
Since P is at most 9 and Q is at most 9, then $P + Q$ is at most 18.
Therefore, $P + Q = 9$ since $P + Q$ cannot equal 19.
In the tens column, since $Q + Q$ ends in zero, $Q + Q$ equals 0 or 10.
Therefore, either $Q = 0$ or $Q = 5$.
If $Q = 0$, there would be no "carry" to the hundreds column where $P + Q$ (plus no carry) ends in a zero.
This is not possible since we already determined that $P + Q = 9$.
Therefore, $Q = 5$ and $P = 4$, giving a 1 carried from the tens column to the hundreds column.
In the hundreds column, we have $1 + 4 + 5$ which gives a 1 carried from the hundreds column to the thousands column.
Then R plus the 1 carried equals 2, so $R = 1$. Thus, $P + Q + R = 4 + 5 + 1 = 10$.

$$\begin{array}{rcccc} & & P & Q & P \\ + & R & Q & Q & Q \\ \hline & 2 & 0 & 0 & 9 \end{array} \qquad \begin{array}{rcccc} & & 4 & 5 & 4 \\ + & 1 & 5 & 5 & 5 \\ \hline & 2 & 0 & 0 & 9 \end{array}$$

Answer: (B)

20. Since the area of the square is 144, each side has length $\sqrt{144} = 12$.
The length of the string equals the perimeter of the square which is $4 \times 12 = 48$.
The largest circle that can be formed from this string has a circumference of 48 or $2\pi r = 48$.
Solving for the radius r, we get $r = \frac{48}{2\pi} \approx 7.64$.
The maximum area of a circle that can be formed using the string is $\pi(\frac{48}{2\pi})^2 \approx \pi(7.64)^2 \approx 183$.

Answer: (E)

21. For the sum to be a maximum, we must choose the three smallest divisors in an effort to make the fourth divisor as large as possible.
The smallest 3 divisors of 360 are 1, 2 and 3, making $\frac{360}{1\times2\times3} = 60$ the fourth divisor.
We note here that 1, 2 and 3 are the smallest three different divisors of 360.
Therefore, it is not possible to use a divisor greater than 60, since there aren't three divisors smaller than 1, 2 and 3.
Replacing the divisor 60 with a smaller divisor will decrease the sum of the four divisors.
To see this, we recognize that the product of 3 different positive integers is always greater than

or equal to the sum of the 3 integers. For example, $1 \times 2 \times 4 = 8 > 1 + 2 + 4 = 7$.
The next largest divisor less than 60 is 45, thus the remaining three divisors would have a product of $360 \div 45 = 8$, and therefore have a sum that is less than or equal to 8.
This gives a combined sum that is less than or equal to $45+8 = 53$, much less than the previous sum of $1 + 2 + 3 + 60 = 66$. In the same way, we obtain sums smaller than 66 if we consider the other divisors of 360 as the largest of the four integers. Therefore, the maximum possible sum is 66.

Answer: (B)

22. The first vertical line through the letter S cuts the S into 4 pieces, 2 pieces to the left of the line and 2 to the right.
Each additional vertical line adds 3 new pieces while preserving the 4 original pieces.
The chart below shows the number of pieces increasing by three with each additional line drawn.

Lines	Pieces
1	4
2	7
3	10
4	13
⋮	⋮

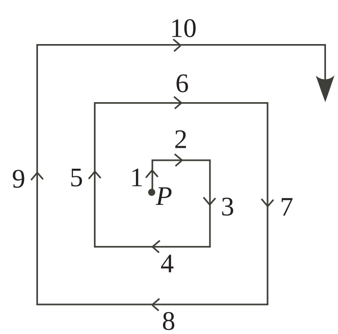
We want 154 pieces. Since the first line gives 4 pieces, we require an additional 150 pieces. Since three new pieces are created for each additional line drawn, we need to add $150 \div 3 = 50$ new lines after the first, or 51 lines in total.

Answer: (D)

23. Since we are looking for the percentage of the whole length, we may take the side length of the square to be any convenient value, as the actual length will not affect the final answer.
Let us assume that the side length of the square is 2. Then the diameter of the circle is also 2 because the width of the square and the diameter of the circle are equal.
Using the Pythagorean Theorem, $XY^2 = 2^2 + 2^2 = 4 + 4 = 8$ or $XY = \sqrt{8}$.
The portion of line segment XY lying outside the circle has length XY minus the diameter of the circle, or $\sqrt{8} - 2$.
The percentage of line segment XY lying outside the circle is $\dfrac{\sqrt{8}-2}{\sqrt{8}} \times 100\% \approx 29.3\%$.

Answer: (A)

24. *Solution 1*
Breenah travels along each of the sides in a direction that is either up, down, right or left.
The "up" sides occur every fourth segment, thus they have lengths $1, 5, 9, 13, 17, 21, \ldots$ or lengths that are one more than a multiple of 4. As we see, the segment of length 21 is an up side.

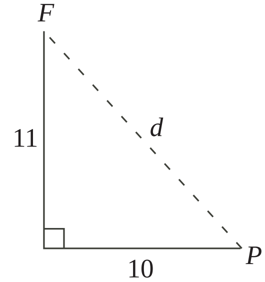


We see that the upper endpoint of each up segment is 2 units to the left and 2 units above the upper endpoint of the previous up segment. Thus, when Breenah is standing at the upper endpoint of the up segment of length 21, she is $2 + 2 + 2 + 2 + 2 = 10$ units to the left and $1 + 2 + 2 + 2 + 2 + 2 = 11$ units above P. The following paragraphs prove this in a more formal way.

We now determine the horizontal distance from point P to the up side of length 21.
Following the spiral outward from P, the first horizontal line segment moves right 2, or $+2$, where the positive sign indicates movement to the right.
The second horizontal segment moves left 4, or -4, where the negative sign indicates movement to the left.
After these two horizontal movements, we are at the line segment of length 5, an up side. To get there, we moved a horizontal distance of $(+2) + (-4) = -2$ or 2 units to the left.
The horizontal distance from P to the next up side (length 9), can be found similarly.
Beginning on the segment of length 5, we are already at -2 or 2 units left of P and we move right 6 (or $+6$), then left 8 (or -8).
Thus, to reach the up side with length 9, we have moved horizontally $(-2) + (+6) + (-8) = -4$ or 4 units left of P.
This pattern of determining the horizontal distances from P to each of the up sides is continued in the table below.
We now determine the vertical distance from point P to the upper endpoint of the up side with length 21.
From P, the first vertical segment moves up 1 or $+1$, the second moves down 3 or -3 and the third moves up 5 or $+5$.
Therefore the vertical position of the endpoint of the up side with length 5 is $(+1) + (-3) + (+5) = +3$ or 3 units above P.
Similarly, we can calculate the vertical position of each of the up side endpoints relative to P and have summarized this in the table below.

Side Length	Horizontal Distance	Vertical Distance
5	$(+2) + (-4) = -2$	$(+1) + (-3) + (+5) = +3$
9	$(-2) + (+6) + (-8) = -4$	$(+3) + (-7) + (+9) = +5$
13	$(-4) + (+10) + (-12) = -6$	$(+5) + (-11) + (+13) = +7$
17	$(-6) + (+14) + (-16) = -8$	$(+7) + (-15) + (+17) = +9$
21	$(-8) + (+18) + (-20) = -10$	$(+9) + (-19) + (+21) = +11$

We now calculate the distance, d, from P to the upper endpoint, F, of the up side of length 21.



Since the calculated distances are horizontal and vertical, we have created a right angle and may find the required distance using the Pythagorean Theorem.
Then $d^2 = 10^2 + 11^2 = 100 + 121 = 221$ or $d = \sqrt{221} \approx 14.866$.

Solution 2

If we place the spiral on an xy-plane with point P at the origin, the coordinates of the key points reveal a pattern.

Side Length	Endpoint Coordinates
1	$(0, 1)$
2	$(2, 1)$
3	$(2, -2)$
4	$(-2, -2)$
5	$(-2, 3)$
6	$(4, 3)$
7	$(4, -4)$
8	$(-4, -4)$
$\vdots$	$\vdots$

From the table we can see that after finishing a side having length that is a multiple of 4, say $4k$, we are at the point $(-2k, -2k)$ (the basis for this argument is shown in solution 1).
Therefore, after completing the side of length 20, we are at the point $(-10, -10)$.
All sides of length $4k$ travel toward the left.
We must now move vertically upward 21 units from this point $(-10, -10)$.
Moving upward, this last side of length 21 will end at the point $F(-10, 11)$.
This point is left 10 units and up 11 units from $P(0, 0)$.
Using the Pythagorean Theorem, $PF^2 = 10^2 + 11^2 = 100 + 121 = 221$ or $PF = \sqrt{221} \approx 14.866$.

ANSWER: (B)

25. Consider all possible sums of pairs of numbers that include p: $p+q$, $p+r$, $p+s$, $p+t$ and $p+u$.
We see that p is included in 5 different sums (once with each of the other 5 numbers in the list).
Similarly, each of the numbers will be included in 5 sums.
If the sums of all 15 pairs are added together, each of p, q, r, s, t, and u will have been included 5 times.
Therefore, $5p + 5q + 5r + 5s + 5t + 5u = 25 + 30 + 38 + 41 + \ldots + 103 + 117 = 980$ or $5(p + q + r + s + t + u) = 980$.
Dividing this equation by 5, we obtain $p + q + r + s + t + u = \frac{980}{5} = 196$.
Since p and q are the smallest two integers, their sum must be the smallest of all of the pairs; thus, $p + q = 25$.
Similarly, t and u are the largest two integers and their sum must be the largest of all of the pairs; thus, $t + u = 117$.
Now, $(p+q)+r+s+(t+u) = 196$ becomes $25+r+s+117 = 196$ or $r+s = 196-142 = 54$.

ANSWER: (B)

Canadian
Mathematics
Competition
An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Grade 8 solutions follow the Grade 7 solutions

2008 Gauss Contests

(Grades 7 and 8)

Wednesday, May 14, 2008

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Lloyd Auckland
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Fiona Dunbar
Jeff Dunnett
Barry Ferguson
Judy Fox
Sandy Graham
Judith Koeller
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Chris Wu, Elia M.S., Toronto, ON

Grade 7

Grade 8 solutions follow the Grade 7 solutions

1. Calculating, $6 \times 2 - 3 = 12 - 3 = 9$.

   Answer: (A)

2. Calculating, $1 + 0.01 + 0.0001 = 1.01 + 0.0001 = 1.0100 + 0.0001 = 1.0101$.

   Answer: (E)

3. Using a common denominator of 8, we have $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{4}{8} + \frac{2}{8} + \frac{1}{8} = \frac{7}{8}$.

   Answer: (D)

4. Since the polygon has perimeter 108 cm and each side has length 12 cm, then the polygon has $108 \div 12 = 9$ sides.

   Answer: (D)

5. In the set, three of the numbers are greater than or equal to 3, and two of the numbers are less than 3.
   The smallest number must be one of the numbers that is less than 3, that is, 2.3 or 2.23.
   Of these two numbers, 2.23 is the smallest, so is the smallest number in the set.

   Answer: (D)

6. Since PQ is a straight line, then $x^\circ + x^\circ + x^\circ + x^\circ + x^\circ = 180^\circ$ or $5x = 180$ or $x = 36$.

   Answer: (A)

7. 20 is not a prime number, since it is divisible by 2.
   21 is not a prime number, since it is divisible by 3.
   25 is not a prime number, since it is divisible by 5.
   27 is not a prime number, since it is divisible by 3.
   23 is a prime number, since its only positive divisors are 1 and 23.

   Answer: (C)

8. Kayla walked 8 km on Monday.
   On Tuesday, she walked $8 \div 2 = 4$ km.
   On Wednesday, she walked $4 \div 2 = 2$ km.
   On Thursday, she walked $2 \div 2 = 1$ km.
   On Friday, she walked $1 \div 2 = 0.5$ km.

   Answer: (E)

9. Since 50% selected chocolate and 10% selected strawberry as their favourite flavour, then overall $50\% + 10\% = 60\%$ chose chocolate or strawberry as their favourite flavour.
   Now $60\% = \frac{60}{100} = \frac{3}{5}$, so $\frac{3}{5}$ of the people surveyed selected chocolate or strawberry as their favourite flavour.

   Answer: (A)

10. Since Max sold 41 glasses of lemonade on Saturday and 53 on Sunday, he sold $41 + 53 = 94$ glasses in total.
    Since he charged 25 cents for each glass, then his total sales were $94 \times \$0.25 = \$23.50$.

    Answer: (A)

11. Since Chris spent \$68 in total and \$25 on the helmet, then he spent \$68 − \$25 = \$43 on the two hockey sticks.
Since the two sticks each cost the same amount, then this cost was \$43 ÷ 2 = \$21.50.
Answer: (C)

12. The number below and between 17 and 6 is $17 - 6 = 11$.
The number below and between 8 and 11 is $11 - 8 = 3$.
The number below and between 11 and 2 is $11 - 2 = 9$.
The number below and between 7 and 3 is $7 - 3 = 4$.
The number below and between 3 and 9 is $9 - 3 = 6$.

$$\begin{array}{ccccccccc} 8 & & 9 & & 17 & & 6 & & 4 \\ & 1 & & 8 & & 11 & & 2 & \\ & & 7 & & 3 & & 9 & & \\ & & & 4 & & 6 & & & \\ & & & & x & & & & \end{array}$$

Therefore, $x = 6 - 4 = 2$.
Answer: (B)

13. Since $PQ = PR$, then $\angle PQR = \angle PRQ$.
Since the angles in a triangle add up to 180°, then $40^\circ + \angle PQR + \angle PRQ = 180^\circ$, so $\angle PQR + \angle PRQ = 140^\circ$.
Since $\angle PQR = \angle PRQ$, then $\angle PQR = \angle PRQ = 70^\circ$.
Since the angle labelled as x° is opposite $\angle PRQ$, then $x^\circ = \angle PRQ = 70^\circ$, so $x = 70$.
Answer: (B)

14. The sum of Wesley's and Breenah's ages is 22.
After each year, each of their ages increases by 1, so the sum of their ages increases by 2.
For the sum to increase from 22 to $2 \times 22 = 44$, the sum must increase by 22, which will take $22 \div 2 = 11$ years.
Answer: (E)

15. The first transformation is a 180° rotation of the letter, which gives G $\rightarrow$ ⅁.
The second transformation is a reflection across a vertical axis, which gives G $\rightarrow$ ⅁ $\rightarrow$ Ͼ.
Answer: (D)

16. In the diagram, the length of one side of the large square is equal to eight side lengths of the smaller squares, so the large square consists of $8 \times 8 = 64$ small squares.
Of these 64 small squares, 48 are shaded. (We can obtain this number by counting the 48 shaded squares or by counting the 16 *unshaded* squares.)
As a percentage, this fraction equals $\frac{48}{64} \times 100\% = \frac{3}{4} \times 100\% = 75\%$.
Answer: (D)

17. *Solution 1*
Since the perimeter of a rectangle equals twice the length plus twice the width, then the length plus the width equals $120 \div 2 = 60$.
Since the length equals twice the width plus 6, then twice the width plus the width equals $60 - 6 = 54$.
In other words, three times the width equals 54, so the width equals $54 \div 3 = 18$.

Solution 2
Let the width of the rectangle be w.
Then the length of the rectangle is $2w+6$.
Since the perimeter is 120, then

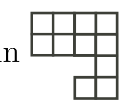
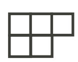
$$\begin{aligned} 2w+2(2w+6) &= 120 \\ 2w+4w+12 &= 120 \\ 6w &= 108 \\ w &= 18 \end{aligned}$$

so the width is 18.

Answer: (B)

18. The sum of Rishi's marks so far is $71+77+80+87=315$.
Since Rishi's mark on his next test is between 0 and 100, the sum of his marks will be between $315+0=315$ and $315+100=415$ after his next test.
Since his average equals the sum of his marks divided by the number of marks, then his average will be between $\frac{315}{5}=63$ and $\frac{415}{5}=83$.
Of the given choices, the only one in this range is 82.

Answer: (C)

19. After some experimentation, the only way in which the two given pieces can be put together to stay within a 4×4 grid and so that one of the given choices can fit together with them is to rotate the second piece by 90° clockwise, and combine to obtain  .
Therefore, the missing piece is  .

Answer: (C)

20. The possible ways of writing 72 as the product of three different positive integers are: $1\times2\times36$; $1\times3\times24$; $1\times4\times18$; $1\times6\times12$; $1\times8\times9$; $2\times3\times12$; $2\times4\times9$; $3\times4\times6$.
(We can find all of these possibilities systematically by starting with the smallest possible first number and working through the possible second numbers, then go to the next possible smallest first number and continue.)
The sums of these sets of three numbers are 39, 28, 23, 19, 18, 17, 15, 13, so the smallest possible sum is 13.

Answer: (A)

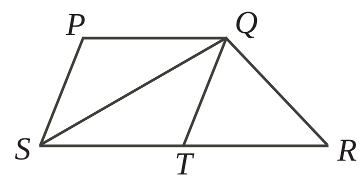
21. Since Andrea has completed $\frac{3}{7}$ of the total 168 km, then she has completed $\frac{3}{7}\times 168$ km or $3\times24=72$ km.
This means that she has $168-72=96$ km remaining.
To complete the 96 km in her 3 remaining days, she must average $\frac{96}{3}=32$ km per day.

Answer: (D)

22. *Solution 1*
Since PQ is parallel to SR, then the height of $\triangle PQS$ (considering PQ as the base) and the height of $\triangle SRQ$ (considering SR as the base) are the same (that is, the vertical distance between PQ and SR).

Since SR is twice the length of PQ and the heights are the same, then the area of $\triangle SRQ$ is twice the area of $\triangle PQS$.
In other words, the area of $\triangle PQS$ is $\frac{1}{3}$ of the total area of the trapezoid, or $\frac{1}{3} \times 12 = 4$.

Solution 2
Draw a line from Q to T, the midpoint of SR.



Since $SR = 2(PQ)$ and T is the midpoint of SR, then $PQ = ST = TR$.
We consider PQ, ST and TR as the bases of $\triangle PQS$, $\triangle STQ$ and $\triangle TRQ$, respectively.
Using these three segments as the bases, each of these triangles has the same height, since PQ is parallel to SR.
Since $PQ = ST = TR$ and these triangles have the same height, then the three triangles each have the same area.
The trapezoid is thus cut into three triangles of equal area.
Therefore, the area of $\triangle PQS$ is one-third of the area of entire trapezoid, or $\frac{1}{3} \times 12 = 4$.

ANSWER: (B)

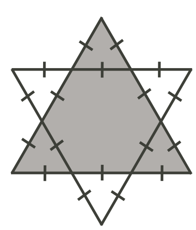
23. Since Ethan does not sit next to Dianne, the four must arrange themselves in one of the configurations:

D __ E __	D __ __ E	__ D __ E
E __ D __	E __ __ D	__ E __ D

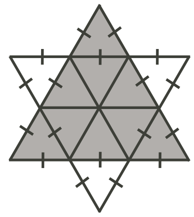
For each of these six configurations, there are two ways for Beverly and Jamaal to sit (either with Beverly on the left or with Jamaal on the left).
Therefore, there are $6 \times 2 = 12$ possible ways that the four can sit. (Try listing them out!)

ANSWER: (B)

24. Since the two large triangles are equilateral, then each of their three angles equals 60°.
Therefore, each of 6 small triangles in the star has an angle of 60° between the two equal sides.
But each of these 6 small triangles is isosceles so each of the remaining two angles must equal $\frac{1}{2}(180^\circ - 60^\circ)$ or 60°.
Therefore, each of the small triangles is equilateral.



This shows us that the inner hexagon has all sides equal, and also that each angle is $180^\circ - 60^\circ$ or 120°, so the hexagon is regular.
Next, we draw the three diagonals of the hexagon that pass through its centre (this is possible because of the symmetry of the hexagon).



Also, because of symmetry, each of the angles of the hexagon is split in half, to get $120° \div 2 = 60°$. Therefore, each of the 6 new small triangles has two $60°$ angles, and so must have its third angle equal to $60°$ as well. Thus, each of the 6 new small triangles is equilateral.
So all 12 small triangles are equilateral. Since each has one side length marked by a single slash, then these 12 small triangles are all identical.
Since the total area of the star is 36, then the area of each small triangle is $36 \div 12 = 3$.
Since the shaded area is made up of 9 of these small triangles, its area is $9 \times 3 = 27$.

Answer: (C)

25. First we look at the integers from 2000 to 2008.
Since we can ignore the 0s when adding up the digits, the sum of all of the digits of these integers is

$$2 + (2 + 1) + (2 + 2) + (2 + 3) + (2 + 4) + (2 + 5) + (2 + 6) + (2 + 7) + (2 + 8) = 54$$

Next, we look at the integers from 1 to 1999.
Again, since we can ignore digits of 0, we consider these numbers as 0001 to 1999, and in fact as the integers from 0000 to 1999, including 0000 to make 2000 integers in total.
Of these 2000 integers, 200 have a units digit of 0, 200 have a units digit of 1, and so on.
(One integer out of every 10 has a units digit of 0, and so on.)
Therefore, the sum of the units digits of these integers is

$$200(0)+200(1)+\cdots+200(8)+200(9) = 200+400+600+800+1000+1200+1400+1600+1800 = 9000$$

Of these 2000 integers, 200 have a tens digit of 0, 200 have a tens digit of 1, and so on.
(Ten integers out of every 100 have a tens digit of 0, and so on.)
Therefore, the sum of the tens digits of these integers is

$$200(0) + 200(1) + \cdots + 200(8) + 200(9) = 9000$$

Of these 2000 integers, 200 have a hundreds digit of 0 (that is, 0000 to 0099 and 1000 to 1099), 200 have a hundreds digit of 1, and so on.
(One hundred integers out of every 1000 have a hundreds digit of 0, and so on.)
Therefore, the sum of the hundreds digits of these integers is

$$200(0) + 200(1) + \cdots + 200(8) + 200(9) = 9000$$

Of these 2000 integers, 1000 have a thousands digit of 0 and 1000 have a thousands digits of 1.
Therefore, the sum of the thousands digits of these integers is

$$1000(0) + 1000(1) = 1000$$

Overall, the sum of all of the digits of these integers is $54 + 9000 + 9000 + 9000 + 1000 = 28\,054$.

Answer: (E)

Grade 8

1. Using the correct order of operations, $8 \times (6-4)+2 = 8 \times 2 + 2 = 16 + 2 = 18$.

   Answer: (C)

2. Since the polygon has perimeter 108 cm and each side has length 12 cm, then the polygon has $108 \div 12 = 9$ sides.

   Answer: (D)

3. Since $\angle PQR = 90^\circ$, then $2x^\circ + x^\circ = 90^\circ$ or $3x = 90$ or $x = 30$.

   Answer: (A)

4. Calculating, $(1+2)^2 - (1^2+2^2) = 3^2 - (1+4) = 9 - 5 = 4$.

   Answer: (B)

5. When these four numbers are listed in increasing order, the two negative numbers come first, followed by the two positive numbers.
   Of the two positive numbers, 0.28 and 2.8, the number 0.28 is the smallest.
   Of the two negative numbers, -0.2 and -8.2, the number -8.2 is the smallest.
   Therefore, the correct order is -8.2, -0.2, 0.28, 2.8.

   Answer: (A)

6. From the given formula, the number that should be placed in the box is $5^3+5-1 = 125+4 = 129$.

   Answer: (E)

7. Since 50% selected chocolate and 10% selected strawberry as their favourite flavour, then overall $50\% + 10\% = 60\%$ chose chocolate or strawberry as their favourite flavour.
   Now $60\% = \frac{60}{100} = \frac{3}{5}$, so $\frac{3}{5}$ of the people surveyed selected chocolate or strawberry as their favourite flavour.

   Answer: (A)

8. *Solution 1*
   Since 5 times the number minus 9 equals 51, then 5 times the number must equal 60 (that is, $51 + 9$).
   Therefore, the original number is 60 divided by 5, or 12.

   Solution 2
   Let the original number be x.
   Then $5x - 9 = 51$, so $5x = 51 + 9 = 60$, so $x = \dfrac{60}{5} = 12$.

   Answer: (D)

9. *Solution 1*
   Since Danny weighs 40 kg, then 20% of his weight is $\frac{20}{100} \times 40 = \frac{1}{5} \times 40 = 8$ kg.
   Since Steven weighs 20% more than Danny, his weight is $40 + 8 = 48$ kg.

   Solution 2
   Since Steven weighs 20% more than Danny, then Steven's weight is 120% of Danny's weight.
   Since Danny's weight is 40 kg, then Steven's weight is $\frac{120}{100} \times 40 = \frac{6}{5} \times 40 = 48$ kg.

   Answer: (C)

10. Of the given 11 numbers, the numbers 3, 5, 7, 11 and 13 are prime. (4, 6, 8, 10 and 12 are not prime, since they are divisible by 2, and 9 is not prime since it is divisible by 3.)
Therefore, 5 of the 11 numbers are prime.
Thus, if a card is chosen at random and flipped over, the probability that the number on this card is a prime number is $\frac{5}{11}$.

Answer: (E)

11. In centimetres, the dimensions of the box are 20 cm, 50 cm, and 100 cm (since 1 m equals 100 cm).
Therefore, the volume of the box is

$$(20 \text{ cm}) \times (50 \text{ cm}) \times (100 \text{ cm}) = 100\,000 \text{ cm}^3$$

Answer: (D)

12. *Solution 1*
Since each pizza consists of 8 slices and each slice is sold for \$1, then each pizza is sold for \$8 in total.
Since 55 pizzas are sold, the total revenue is $55 \times \$8 = \$440$.
Since 55 pizzas were bought initially, the total cost was $55 \times \$6.85 = \$376.75$.
Therefore, the total profit was $\$440 - \$376.75 = \$63.25$.

Solution 2
Since each pizza consists of 8 slices and each slice is sold for \$1, then each pizza is sold for \$8 in total.
Since each pizza was bought for \$6.85 initially, then the school makes a profit of $\$8.00 - \$6.85$ or \$1.15 per pizza.
Since the school completely sold 55 pizzas, then its total profit was $55 \times \$1.15 = \$63.25$.

Answer: (D)

13. Since RSP is a straight line, then $\angle RSQ + \angle QSP = 180^\circ$, so $\angle RSQ = 180^\circ - 80^\circ = 100^\circ$.
Since $\triangle RSQ$ is isosceles with $RS = SQ$, then

$$\angle RQS = \tfrac{1}{2}(180^\circ - \angle RSQ) = \tfrac{1}{2}(180^\circ - 100^\circ) = 40^\circ$$

Similarly, since $\triangle PSQ$ is isosceles with $PS = SQ$, then

$$\angle PQS = \tfrac{1}{2}(180^\circ - \angle PSQ) = \tfrac{1}{2}(180^\circ - 80^\circ) = 50^\circ$$

Therefore, $\angle PQR = \angle PQS + \angle RQS = 50^\circ + 40^\circ = 90^\circ$.

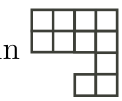
Answer: (B)

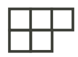
14. On Monday, Amos read 40 pages.
On Tuesday, Amos read 60 pages, for a total of $40 + 60 = 100$ pages so far.
On Wednesday, Amos read 80 pages, for a total of $100 + 80 = 180$ pages so far.
On Thursday, Amos read 100 pages, for a total of $180 + 100 = 280$ pages so far.
On Friday, Amos read 120 pages, for a total of $280 + 120 = 400$ pages so far.
Therefore, Amos finishes the 400 page book on Friday.

Answer: (A)

15. If Abby had 23 nickels, the total value would be $23 \times \$0.05 = \$1.15$.
But the total value of Abby's coins is \$4.55, which is \$3.40 more.
Since a quarter is worth 20 cents more than a nickel, then every time a nickel is replaced by a quarter, the total value of the coins increase by 20 cents.
For the total value to increase by \$3.40, we must replace $\$3.40 \div \$0.20 = 17$ nickels with quarters.
Therefore, Abby has 17 quarters.
(To check, if Abby has 17 quarters and 6 nickels, the total value of the coins that she has is $17 \times \$0.25 + 6 \times \$0.05 = \$4.25 + \$0.30 = \$4.55$.)

Answer: (B)

16. After some experimentation, the only way in which the two given pieces can be put together to stay within a 4×4 grid and so that one of the given choices can fit together with them is to rotate the second piece by 90° clockwise, and combine to obtain  .

Therefore, the missing piece is  .

Answer: (C)

17. The digits after the decimal point occur in repeating blocks of 6 digits.
Since $2008 \div 6 = 334.666...$, then the 2008th digit after the decimal point occurs after 334 blocks of digits have been used.
In 334 blocks of 6 digits, there are $334 \times 6 = 2004$ digits in total. Therefore, the 2008th digit is 4 digits into the 335th block, so must be 8.

Answer: (A)

18. Since Andrea has completed $\frac{3}{7}$ of the total 168 km, then she has completed $\frac{3}{7} \times 168$ km or $3 \times 24 = 72$ km.
This means that she has $168 - 72 = 96$ km remaining.
To complete the 96 km in her 3 remaining days, she must average $\frac{96}{3} = 32$ km per day.

Answer: (D)

19. *Solution 1*
After some trial and error, you might discover that $x = 0$ and $y = 7$ works, since
$307 + 703 = 1010$.
Therefore, since we are asked for the unique value of $y - x$, it must be $7 - 0 = 7$.

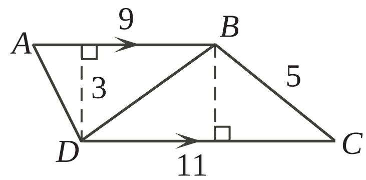
Solution 2
When performing this addition, in the units column either $y + 3 = x$ or $y + 3 = x$ with a carry of 1, meaning that $y + 3 = 10 + x$.
Therefore, either $y - x = -3$ or $y - x = 10 - 3 = 7$.
If there was no carry, then adding up the tens digits, we would get $x + x$ ending in a 1, which is impossible as $x + x = 2x$ which is even.
Therefore, the addition $y + 3$ must have a carry of 1.
Therefore, $y - x = 7$.

Answer: (C)

20. *Solution 1*
The area of a trapezoid equals one-half times the sum of the bases times the height. Therefore, the area of this trapezoid is

$$\tfrac{1}{2} \times (9 + 11) \times 3 = \tfrac{1}{2} \times 20 \times 3 = 10 \times 3 = 30$$

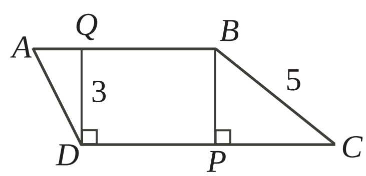
Solution 2
We draw diagonal BD.



$\triangle ABD$ has a base of 9 and a height of 3.
$\triangle BCD$ has a base of 11 and a height of 3.
The area of the trapezoid is equal to the sum of the areas of these two triangles, or

$$\tfrac{1}{2} \times 9 \times 3 + \tfrac{1}{2} \times 11 \times 3 = \tfrac{27}{2} + \tfrac{33}{2} = \tfrac{60}{2} = 30$$

Solution 3
Draw perpendicular lines from B to CD and from D to AB, as shown.



By the Pythagorean Theorem, $BP^2 + PC^2 = BC^2$ or $3^2 + PC^2 = 5^2$ or $PC^2 = 25 - 9 = 16$, so $PC = 4$.
Since $DC = 11$, then $DP = 11 - 4 = 7$.
Since $QBPD$ is a rectangle, then $QB = DP = 7$, so $AQ = 9 - 7 = 2$.
The area of trapezoid $ABCD$ equals the sum of the area of $\triangle AQD$, rectangle $QBPD$ and $\triangle BPC$, or

$$\tfrac{1}{2} \times 2 \times 3 + 7 \times 3 + \tfrac{1}{2} \times 4 \times 3 = 3 + 21 + 6 = 30$$

Answer: (D)

21. The object has 7 "front" faces, each of which is 1×1. Therefore, the surface area of the front is $7 \times 1 \times 1 = 7$.
Similarly, the surface area of the "back" is 7.
Now consider the faces on the left, top, right and bottom. Each of these faces is 1×2, so each has an area of 2.
How many of these faces are there?
If we start at the bottom left and travel clockwise around the figure, we have 2 left faces, 2 top faces, 2 left faces, 1 top face, 3 right faces, 1 bottom face, 1 right face, and 2 bottom faces, or 14 faces in total.
Therefore, the surface area accounted for by these faces is $14 \times 2 = 28$.
Therefore, the total surface area of the object is $7 + 7 + 28 = 42$.

Answer: (A)

22. There are 6 possibilities for the first row of the grid:

$$1,2,3 \quad 1,3,2 \quad 2,1,3 \quad 2,3,1 \quad 3,1,2 \quad 3,2,1$$

Consider the first row of 1, 2, 3:

1	2	3

.

The first column could be 1, 2, 3 or 1, 3, 2:

1	2	3
2		
3		

or

1	2	3
3		
2		

.

Each of these grids can be finished with the given rules, but can only be finished in one way.
(In the first grid, the middle number in the bottom row cannot be 2 or 3, so is 1, so the middle number in the middle row is 3, so the right column is 3, 1, 2.
Similarly, in the second grid, the middle number in the middle row must be 1. Try completing this grid!)
Therefore, a first row of 1, 2, 3 gives two possible grids.
Similarly, each of the other 5 possible first rows will give two other grids.
(We can see this by trying each of these possibilities or by for example switching all of the 2s and 3s to get the grids with a first row of 1, 3, 2.)
Therefore, the total number of different ways of filling the grid is $6 \times 2 = 12$.

ANSWER: (B)

23. Since the area of the larger circle is 64π and each circle is divided into two equal areas, then the larger shaded area is $\frac{1}{2}$ of 64π, or 32π.
Let r be the radius of the larger circle.
Since the area of the larger circle is 64π, then $\pi r^2 = 64\pi$ or $r^2 = 64$ or $r = \sqrt{64} = 8$, since $r > 0$.
Since the smaller circle passes through the centre of the larger circle and just touches the outer circle, then by symmetry, its diameter must equal the radius of the larger circle. (In other words, if we join the centre of the larger circle to the point where the two circles just touch, this line will be a radius of the larger circle and a diameter of the smaller circle.)
Therefore, the diameter of the smaller circle is 8, so its radius is 4.
Therefore, the area of the smaller circle is $\pi(4^2) = 16\pi$, so the smaller shaded area is $\frac{1}{2} \times 16\pi$ or 8π.
Therefore, the total of the shaded areas is $32\pi + 8\pi = 40\pi$.

ANSWER: (D)

24. First we look at the integers from 2000 to 2008.
Since we can ignore the 0s when adding up the digits, the sum of all of the digits of these integers is

$$2 + (2+1) + (2+2) + (2+3) + (2+4) + (2+5) + (2+6) + (2+7) + (2+8) = 54$$

Next, we look at the integers from 1 to 1999.
Again, since we can ignore digits of 0, we consider these numbers as 0001 to 1999, and in fact as the integers from 0000 to 1999.
Of these 2000 integers, 200 have a units digit of 0, 200 have a units digit of 1, and so on.
(One integer out of every 10 has a units digit of 0, and so on.)
Therefore, the sum of the units digits of these integers is

$$200(0)+200(1)+\cdots+200(8)+200(9) = 200(0+1+2+3+4+5+6+7+8+9) = 200(45) = 9000$$

Of these 2000 integers, 200 have a tens digit of 0, 200 have a tens digit of 1, and so on.
(Ten integers out of every 100 have a tens digit of 0, and so on.)
Therefore, the sum of the tens digits of these integers is

$$200(0) + 200(1) + \cdots + 200(8) + 200(9) = 9000$$

Of these 2000 integers, 200 have a hundreds digit of 0 (that is, 0000 to 0099 and 1000 to 1099), 200 have a hundreds digit of 1, and so on.
(One hundred integers out of every 1000 have a hundreds digit of 0, and so on.)
Therefore, the sum of the hundreds digits of these integers is

$$200(0) + 200(1) + \cdots + 200(8) + 200(9) = 9000$$

Of these 2000 integers, 1000 have a thousands digit of 0 and 1000 have a thousands digits of 1.
Therefore, the sum of the thousands digits of these integers is

$$1000(0) + 1000(1) = 1000$$

Overall, the sum of all of the digits of these integers is $54 + 9000 + 9000 + 9000 + 1000 = 28\,054$.

Answer: (E)

25. Since the length of the candles was equal at 9 p.m., the longer one burned out at 10 p.m., and the shorter one burned out at midnight, then it took 1 hour for the longer candle and 3 hours for the shorter candle to burn this equal length.
Therefore, the longer candle burned 3 times as quickly as the shorter candle.
Suppose that the shorter candle burned x cm per hour.
Then the longer candle burned $3x$ cm per hour.
From its lighting at 3 p.m. to 9 p.m., the longer candles burned for 6 hours, so burned $6 \times 3x$ or $18x$ cm.
From its lighting at 7 p.m. to 9 p.m., the shorter candle burned for 2 hours, so burns $2 \times x = 2x$ cm.
But, up to 9 p.m., the longer candle burned 32 cm more than the shorter candle, since it began 32 cm longer.
Therefore, $18x - 2x = 32$ or $16x = 32$ or $x = 2$.
In summary, the shorter candle burned for 5 hours at 2 cm per hour, so its initial length was 10 cm.
Also, the longer candle burned for 7 hours at 6 cm per hour, so its initial length was 42 cm.
Thus, the sum of the original lengths is $42 + 10 = 52$ cm.

Answer: (E)

Canadian
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Grade 8 solutions
follow the
Grade 7 solutions

2007 Gauss Contests

(Grades 7 and 8)

Wednesday, May 16, 2007

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Grade 7

Grade 8 solutions follow the Grade 7 solutions

1. Calculating, $(4-3)\times 2 = 1\times 2 = 2$.

   Answer: (B)

2. Since one thousand is 1000, then ten thousand is 10 000.

   Answer: (C)

3. When we subtract 5 from the missing number, the answer is 2, so to find the missing number, we add 5 to 2 and obtain 7. (Check: $7-5=2$.)

   Answer: (A)

4. *Solution 1*
   As a fraction, 80% is $\frac{80}{100}$ or $\frac{4}{5}$.
   Therefore, Mukesh got $\frac{4}{5}$ of the possible 50 marks, or $\frac{4}{5}\times 50 = 40$ marks.

   Solution 2
   Since Mukesh got 80% of the 50 marks, he got $\frac{80}{100}\times 50 = \frac{80}{2} = 40$ marks.

   Answer: (A)

5. *Solution 1*

$$\begin{aligned}\frac{7}{10}+\frac{3}{100}+\frac{9}{1000} &= \frac{700}{1000}+\frac{30}{1000}+\frac{9}{1000} \qquad \text{(using a common denominator)}\\ &= \frac{739}{1000}\\ &= 0.739\end{aligned}$$

   Solution 2

$$\begin{aligned}\frac{7}{10}+\frac{3}{100}+\frac{9}{1000} &= 0.7+0.03+0.009 \qquad \text{(converting each fraction to a decimal)}\\ &= 0.739\end{aligned}$$

   Answer: (D)

6. *Solution 1*
   Mark has $\frac{3}{4}$ of a dollar, or 75 cents.
   Carolyn has $\frac{3}{10}$ of a dollar, or 30 cents.
   Together, they have $75+30=105$ cents, or \$1.05.

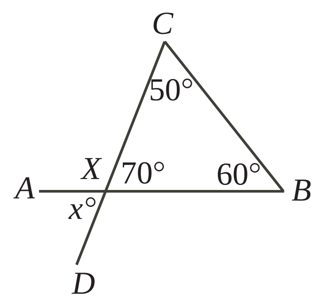
   Solution 2
   Since Mark has $\frac{3}{4}$ of a dollar and Carolyn has $\frac{3}{10}$ of a dollar, then together they have $\frac{3}{4}+\frac{3}{10}=\frac{15}{20}+\frac{6}{20}=\frac{21}{20}$ of a dollar.
   Since $\frac{21}{20}$ is equivalent to $\frac{105}{100}$, they have \$1.05.

   Answer: (E)

7. From the graph, the student who ate the most apples ate 6 apples, so Lorenzo ate 6 apples. Also from the graph, the student who ate the fewest apples ate 1 apple, so Jo ate 1 apple. Therefore, Lorenzo ate $6-1=5$ more apples than Jo.

   Answer: (B)

8. Since the angles in a triangle add to 180°, then the missing angle in the triangle is $180^\circ - 50^\circ - 60^\circ = 70^\circ$.
We then have:



Since $\angle BXC = 70^\circ$, then $\angle AXC = 180^\circ - \angle BXC = 110^\circ$.
Since $\angle AXC = 110^\circ$, then $\angle DXA = 180^\circ - \angle AXC = 70^\circ$.
Therefore, $x = 70$.
(Alternatively, we could note that when two lines intersect, the vertically opposite angles are equal so $\angle DXA = \angle BXC = 70^\circ$.)

Answer: (E)

9. When the word BANK is viewed from the inside of the window, the letters appear in the reverse order and the letters themselves are all backwards, so the word appears as ꓘИAB.

Answer: (D)

10. Since a large box costs \$3 more than a small box and a large box and a small box together cost \$15, then replacing the large box with a small box would save \$3.
This tells us that two small boxes together cost \$12.
Therefore, one small box costs \$6.

Answer: (D)

11. Since each number in the Fibonacci sequence, beginning with the 2, is the sum of the two previous numbers, then the sequence continues as $1, 1, 2, 3, 5, 8, 13, 21$.
Thus, 21 appears in the sequence.

Answer: (B)

12. The probability that Mary wins the lottery is equal to the number of tickets that Mary bought divided by the total number of tickets in the lottery.
We are told that the probability that Mary wins is $\frac{1}{15}$.
Since there were 120 tickets in total sold, we would like to write $\frac{1}{15}$ as a fraction with 120 in the denominator.
Since $120 \div 15 = 8$, then we need to multiply the numerator and denominator of $\frac{1}{15}$ each by 8 to obtain a denominator of 120.
Therefore, the probability that Mary wins is $\dfrac{1 \times 8}{15 \times 8} = \dfrac{8}{120}$. Since there were 120 tickets sold, then Mary must have bought 8 tickets.

Answer: (D)

13. *Solution 1*
We look at each of the choices and try to make them using only 3 cent and 5 cent stamps:

(A): 7 cannot be made, since no more than one 5 cent and two 3 cent stamps could be used (try playing with the possibilities!)
(B): $13 = 5 + 5 + 3$
(C): 4 cannot be the answer since a larger number (7) already cannot be made
(D): $8 = 5 + 3$
(E): $9 = 3 + 3 + 3$

Therefore, the answer must be 7.
(We have not really justified that 7 is the largest number that cannot be made using only 3s and 5s; we have, though, determined that 7 must be the answer to this question, since it is the only possible answer from the given possibilities! See Solution 2 for a justification that 7 is indeed the answer.)

Solution 2
We make a table to determine which small positive integers can be made using 3s and 5s:

Integer	Combination
1	Cannot be made
2	Cannot be made
3	3
4	Cannot be made
5	5
6	$3 + 3$
7	Cannot be made
8	$5 + 3$
9	$3 + 3 + 3$
10	$5 + 5$
11	$5 + 3 + 3$

Every integer larger than 11 can also be made because the last three integers in our table can be made and we can add a 3 to our combinations for 9, 10 and 11 to get combinations for 12, 13 and 14, and so on.
From the table, the largest amount of postage that cannot be made is 7.

Answer: (A)

14. We list all of the possible orders of finish, using H, R and N to stand for Harry, Ron and Neville.
The possible orders are HNR, HRN, NHR, NRH, RHN, RNH.
(It is easiest to list the orders in alphabetical order to better keep track of them.)
There are 6 possible orders.

Answer: (B)

15. *Solution 1*
The positive whole numbers that divide exactly into 40 are 1, 2, 4, 5, 8, 10, 20, 40.
The positive whole numbers that divide exactly into 72 are 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.
The numbers that occur in both lists are 1, 2, 4, 8, or four numbers in total.

Solution 2
The greatest common divisor of 40 and 72 is 8.
Any common divisor of 40 and 72 is a divisor of the greatest common divisor (namely 8) and vice-versa.
Since the positive divisors of 8 are 1, 2, 4, and 8, there are four such common positive divisors.

Answer: (C)

16. The first scale tells us that a square and a circle together have a mass of 8.
The second scale tells us that a square and two circles together have a mass of 11.
We can replace the square and one circle on the second scale with an "8", so 8 plus the mass of a circle gives a mass of 11. This tells us that the mass of a circle is 3.
From the third scale, since the mass of a circle and two triangles is 15, then the mass of the two triangles only is $15 - 3 = 12$.
Therefore, the mass of one triangle is $12 \div 2 = 6$.

Answer: (D)

17. The total cost to use the kayak for 3 hours is $3 \times \$5 = \$15$. Since the total rental cost for 3 hours is \$30, then the fixed fee to use the paddle is $\$30 - \$15 = \$15$.
For a six hour rental, the total cost is thus $\$15 + (6 \times \$5) = \$15 + \$30 = \$45$.

Answer: (C)

18. *Solution 1*
Julie's birthday was $37 + 67 = 104$ days before Fred's birthday.
When we divide 104 by 7 (the number of days in one week), we obtain a quotient of 14 and a remainder of 6.
In 14 weeks, there are $14 \times 7 = 98$ days, so 98 days before Fred's birthday was also a Monday.
Since Julie's birthday was 104 days before Fred's, this was 6 days still before the Monday 98 days before Fred's birthday. The 6th day before a Monday is a Tuesday.
Therefore, Julie's birthday was a Tuesday.

Solution 2
37 days is 5 weeks plus 2 days. Since Fred's birthday was on a Monday and Pat's birthday was 37 days before Fred's, then Pat's birthday was on a Saturday.
67 days is 9 weeks plus 4 days. Since Pat's birthday was on a Saturday and Julie's birthday was 67 days before Pat's, then Julie's birthday was on a Tuesday.

Answer: (D)

19. The positive whole numbers less than 1000 that end with 77 are 77, 177, 277, 377, 477, 577, 677, 777, 877, 977.
The positive whole numbers less than 1000 which begin with 77 are 77, 770, 771, 772, 773, 774, 775, 776, 777, 778, 779.
There is no other way for a positive whole number less than 1000 to contain at least two 7's side-by-side.
There are 10 numbers in the first list and 11 numbers in the second list. Since 2 numbers appear in both lists, the total number of whole numbers in the two lists is $10 + 11 - 2 = 19$.

Answer: (E)

20. Since the perimeter of the square is 48, its side length is $48 \div 4 = 12$.
Since the side length of the square is 12, its area is $12 \times 12 = 144$.
The area of the triangle is $\frac{1}{2} \times 48 \times x = 24x$.
Since the area of the triangle equals the area of the square, then $24x = 144$ or $x = 6$.

Answer: (C)

21. *Solution 1*
Starting at the "K" there are two possible paths that can be taken. At each "A", there are again two possible paths that can be taken. Similarly, at each "R" there are two possible paths that can be taken.
Therefore, the total number of paths is $2 \times 2 \times 2 = 8$.
(We can check this by actually tracing out the paths.)

Solution 2
Each path from the K at the top to one of the L's at the bottom has to spell KARL.
There is 1 path that ends at the first L from the left. This path passes through the first A and the first R.
There are 3 paths that end at the second L. The first of these passes through the first A and the first R. The second of these passes through the first A and the second R. The third of these passes through the second A and the second R.
There are 3 paths that end at the third L. The first of these passes through the first A and the second R. The second of these passes through the second A and the second R. The third of these passes through the second A and the third R.
There is 1 path that ends at the last L. This path passes through the last A and the last R.
So the total number of paths to get to the bottom row is $1+3+3+1 = 8$, which is the number of paths that can spell KARL.

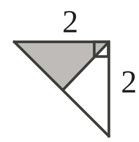
Answer: (D)

22. Since the average of four numbers is 4, their sum is $4 \times 4 = 16$.
For the difference between the largest and smallest of these numbers to be as large as possible, we would like one of the numbers to be as small as possible (so equal to 1) and the other (call it B for big) to be as large as possible.
Since one of the numbers is 1, the sum of the other three numbers is $16 - 1 = 15$.
For the B to be as large as possible, we must make the remaining two numbers (which must be different and not equal to 1) as small as possible. So these other two numbers must be equal to 2 and 3, which would make B equal to $15 - 2 - 3 = 10$.
So the average of these other two numbers is $\dfrac{2+3}{2} = \dfrac{5}{2}$ or $2\frac{1}{2}$.

Answer: (B)

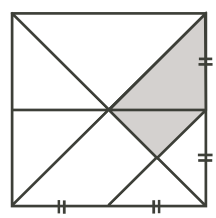
23. *Solution 1*
Since we are dealing with fractions of the whole area, we may make the side of the square any convenient value.
Let us assume that the side length of the square is 4.
Therefore, the area of the whole square is $4 \times 4 = 16$.
The two diagonals of the square divide it into four pieces of equal area (so each piece has area $16 \div 4 = 4$).
The shaded area is made up from the "right" quarter of the square with a small triangle removed, and so has area equal to 4 minus the area of this small triangle.

This small triangle is half of a larger triangle.



This larger triangle has its base and height each equal to half of the side length of the square (so equal to 2) and has a right angle. So the area of this larger triangle is $\frac{1}{2} \times 2 \times 2 = 2$.
So the area of the small triangle is $\frac{1}{2} \times 2 = 1$, and so the area of the shaded region is $4 - 1 = 3$.
Therefore, the shaded area is $\frac{3}{16}$ of the area of the whole square.

Solution 2
Draw a horizontal line from the centre of the square through the shaded region.



The two diagonals divide the square into four pieces of equal area. The new horizontal line divides one of these pieces into two parts of equal area. Therefore, the shaded region above the new horizontal line is $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$ of the total area of the square.
The shaded piece below this new horizontal line is half of the bottom right part of this right-hand piece of the square. (It is half of this part because the shaded triangle and unshaded triangle making up this part have the same shape.) So this remaining shaded piece is $\frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$ of the total area of the square.
In total, the shaded region is $\frac{1}{8} + \frac{1}{16} = \frac{3}{16}$ of the total area of the square.

Answer: (C)

24. First, we try to figure out what digit Q is.
Since the product is not equal to 0, Q cannot be 0. Since the product has four digits and the top number has three digits, then Q (which is multiplying the top number) must be bigger than 1.
Looking at the units digits in the product, we see that $Q \times Q$ has a units digit of Q.
Since $Q > 1$, then Q must equal 5 or 6 (no other digit gives itself as a units digit when multiplied by itself).
But Q cannot be equal to 5, since if it was, the product $RQ5Q$ would end "55" and each of the two parts (PPQ and Q) of the product would end with a 5. This would mean that each of the parts of the product was divisible by 5, so the product should be divisible by $5 \times 5 = 25$. But a number ending in 55 is not divisible by 25.
Therefore, $Q = 6$.
So the product now looks like

$$\begin{array}{r} PP6 \\ \times \quad 6 \\ \hline R656 \end{array}$$

Now when we start the long multiplication, 6×6 gives 36, so we write down 6 and carry a 3.
When we multiply $P \times 6$ and add the carry of 3, we get a units digit of 5, so the units digit of

$P \times 6$ should be 2.
For this to be the case, $P = 2$ or $P = 7$.
We can now try these possibilities: $226 \times 6 = 1356$ and $776 \times 6 = 4656$. Only the second ends "656" like the product should.
So $P = 7$ and $R = 4$, and so $P + Q + R = 7 + 6 + 4 = 17$.

Answer: (E)

25. The easiest way to keep track of the letters here is to make a table of what letters arrive at each time, what letters are removed, and what letters stay in the pile.

Time	Letters Arrived	Letters Removed	Remaining Pile (bottom to top)
12:00	1, 2, 3	3, 2	1
12:05	4, 5, 6	6, 5	1, 4
12:10	7, 8, 9	9, 8	1, 4, 7
12:15	10, 11, 12	12, 11	1, 4, 7, 10
12:20	13, 14, 15	15, 14	1, 4, 7, 10, 13
12:25	16, 17, 18	18, 17	1, 4, 7, 10, 13, 16
12:30	19, 20, 21	21, 20	1, 4, 7, 10, 13, 16, 19
12:35	22, 23, 24	24, 23	1, 4, 7, 10, 13, 16, 19, 22
12:40	25, 26, 27	27, 26	1, 4, 7, 10, 13, 16, 19, 22, 25
12:45	28, 29, 30	30, 29	1, 4, 7, 10, 13, 16, 19, 22, 25, 28
12:50	31, 32, 33	33, 32	1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31
12:55	34, 35, 36	36, 35	1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34
1:00	None	34, 31	1, 4, 7, 10, 13, 16, 19, 22, 25, 28
1:05	None	28, 25	1, 4, 7, 10, 13, 16, 19, 22
1:10	None	22, 19	1, 4, 7, 10, 13, 16
1:15	None	16, 13	1, 4, 7, 10

(At 12:55, all 36 letters have been delivered, so starting at 1:00 letters are only removed and no longer added.)
Letter #13 is removed at 1:15.

Answer: (A)

Grade 8

1. Calculating, $(4 \times 12) - (4 + 12) = 48 - 16 = 32$.

Answer: (E)

2. Converting to decimals, $\frac{3}{10} + \frac{3}{1000} = 0.3 + 0.003 = 0.303$.

Answer: (B)

3. We use the graph to determine the difference between the high and low temperature each day.

Day	High	Low	Difference
Monday	20°	10°	10°
Tuesday	20°	15°	5°
Wednesday	25°	25°	0°
Thursday	30°	10°	20°
Friday	25°	20°	5°

Therefore, the difference was the greatest on Thursday.

Answer: (D)

4. When the cube is tossed, the total number of possibilities is 6 and the number of desired outcomes is 2.
So the probability of tossing a 5 or 6 is $\frac{2}{6}$ or $\frac{1}{3}$.

Answer: (C)

5. Since the side length of the cube is x cm, its volume is x^3 cm^3.
Since the volume is known to be 8 cm^3, then $x^3 = 8$ so $x = 2$ (the cube root of 8).

Answer: (A)

6. Since a 3 minute phone call costs \$0.18, then the rate is $\$0.18 \div 3 = \$0.06$ per minute.
For a 10 minute call, the cost would be $10 \times \$0.06 = \$0.60$.

Answer: (B)

7. Since there are 1000 metres in a kilometre, then 200 metres is equivalent to $\dfrac{200}{1000}$ km or 0.2 km.

Answer: (A)

8. The children in the Gauss family have ages 7, 7, 7, 14, 15.
The mean of their ages is thus $\dfrac{7+7+7+14+15}{5} = \dfrac{50}{5} = 10$.

Answer: (E)

9. Since $x = 5$ and $y = x + 3$, then $y = 5 + 3 = 8$.
Since $y = 8$ and $z = 3y + 1$, then $z = 3(8) + 1 = 24 + 1 = 25$.

Answer: (B)

10. The possible three-digit numbers that can be formed using the digits 5, 1 and 9 are: 519, 591, 951, 915, 195, 159.
The largest of these numbers is 951 and the smallest is 159.
The difference between these numbers is $951 - 159 = 792$.

Answer: (C)

11. *Solution 1*
Since Lily is 90 cm tall, Anika is $\frac{4}{3}$ as tall as Lily, and Sadaf is $\frac{5}{4}$ as tall as Anika, then Sadaf is $\frac{5}{4} \times \frac{4}{3} \times 90 = \frac{5}{3} \times 90 = \frac{450}{3} = 150$ cm tall.

Solution 2
Since Lily is 90 cm tall and Anika is $\frac{4}{3}$ of her height, then Anika is $\frac{4}{3} \times 90 = \frac{360}{3} = 120$ cm tall. Since Anika is 120 cm tall and Sadaf is $\frac{5}{4}$ of her height, then Sadaf is $\frac{5}{4} \times 120 = \frac{600}{4} = 150$ cm tall.

Answer: (E)

12. *Solution 1*
Since $\angle BCA = 40^\circ$ and $\triangle ADC$ is isosceles with $AD = DC$, then $\angle DAC = \angle ACD = 40^\circ$.
Since the sum of the angles in a triangle is 180°, then
$\angle ADC = 180^\circ - \angle DAC - \angle ACD = 180^\circ - 40^\circ - 40^\circ = 100^\circ$.
Since $\angle ADB$ and $\angle ADC$ are supplementary, then $\angle ADB = 180^\circ - \angle ADC = 180^\circ - 100^\circ = 80^\circ$.
Since $\triangle ADB$ is isosceles with $AD = DB$, then $\angle BAD = \angle ABD$.
Thus, $\angle BAD = \frac{1}{2}(180^\circ - \angle ADB) = \frac{1}{2}(180^\circ - 80^\circ) = \frac{1}{2}(100^\circ) = 50^\circ$.
Therefore, $\angle BAC = \angle BAD + \angle DAC = 50^\circ + 40^\circ = 90^\circ$.

Solution 2
Since $\triangle ABD$ and $\triangle ACD$ are isosceles, then $\angle BAD = \angle ABD$ and $\angle DAC = \angle ACD$.
These four angles together make up all of the angles of $\triangle ABC$, so their sum is 180°.
Since $\angle BAC$ is half of the the sum of these angles (as it incorporates one angle of each pair), then $\angle BAC = \frac{1}{2}(180^\circ) = 90^\circ$.

Answer: (D)

13. For each of the 2 art choices, Cayli can chose 1 of 3 sports choices and 1 of 4 music choices. So for each of the 2 art choices, Cayli has $3 \times 4 = 12$ possible combinations of sports and music.
Since Cayli has 2 art choices, her total number of choices is $2 \times 12 = 24$.

Answer: (B)

14. At the 2007 Math Olympics, Canada won 17 of 100 possible medals, or 0.17 of the possible medals.
We convert each of the possible answers to a decimal and see which is closest to 0.17:

(A) $\frac{1}{4} = 0.25$ (B) $\frac{1}{5} = 0.2$ (C) $\frac{1}{6} = 0.166666...$ (D) $\frac{1}{7} = 0.142857...$ (E) $\frac{1}{8} = 0.125$

The choice that is closest to 0.17 is $\frac{1}{6}$, or (C).

Answer: (C)

15. Let us try the integers 5, 6, 7, 8.
When 5 is divided by 4, the quotient is 1 and the remainder is 1.
When 6 is divided by 4, the quotient is 1 and the remainder is 2.
When 7 is divided by 4, the quotient is 1 and the remainder is 3.
When 8 is divided by 4, the quotient is 2 and the remainder is 0.
The sum of these remainders is $1 + 2 + 3 + 0 = 6$.
(When any four consecutive integers are chosen, one will have a remainder 1, one a remainder of 2, one a remainder of 3 and one a remainder of 0 when divided by 4.)

Answer: (A)

16. Suppose that the initial radius of the circle is 1. Then its initial area is $\pi(1)^2 = \pi$ and its initial circumference is $2\pi(1) = 2\pi$.
When the radius is tripled, the new radius is 3.
The new area is $\pi(3)^2 = 9\pi$ and the new circumference is $2\pi(3) = 6\pi$ so the area is 9 times as large and the circumference is 3 times as large.

ANSWER: (A)

17. Since each number of votes in this problem is a multiple of 1000, we consider the number of thousands of votes that each potential Idol received, to make the numbers easier with which to work.
There was a total of 5219 thousand votes cast.
Suppose that the winner received x thousand votes. Then his opponents received $x-22$, $x-30$ and $x-73$ thousand votes.
Equating the total numbers of thousand of votes,

$$\begin{aligned} x + (x-22) + (x-30) + (x-73) &= 5219 \\ 4x - 125 &= 5219 \\ 4x &= 5344 \\ x &= 1336 \end{aligned}$$

Therefore, the winner received 1 336 000 votes.

ANSWER: (D)

18. When the number n is doubled, $2n$ is obtained.
When y is added, $2n + y$ is obtained.
When this number is divided by 2, we obtain $\frac{1}{2}(2n+y) = n + \dfrac{y}{2}$.
When n is subtracted, $\dfrac{y}{2}$ is obtained.

ANSWER: (E)

19. To make a fraction as large as possible, we should make the numerator as large as possible and the denominator as small as possible.
Of the four numbers in the diagram, z is the largest and w is the smallest, so the largest possible fraction is $\dfrac{z}{w}$.

ANSWER: (E)

20. *Solution 1*
Lorri's 240 km trip to Waterloo at 120 km/h took $240 \div 120 = 2$ hours.
Lorri's 240 km trip home at 80 km/h took $240 \div 80 = 3$ hours.
In total, Lorri drove 480 km in 5 hours, for an average speed of $\dfrac{480}{5} = 96$ km/h.

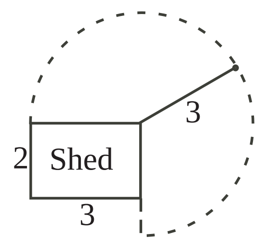
Solution 2
Lorri's 240 km trip to Waterloo at 120 km/h took $240 \div 120 = 2$ hours.
Lorri's 240 km trip home at 80 km/h took $240 \div 80 = 3$ hours.
Over the 5 hours that Lorri drove, her speeds were 120, 120, 80, 80, and 80, so her average speed was $\dfrac{120+120+80+80+80}{5} = \dfrac{480}{5} = 96$ km/h.

ANSWER: (B)

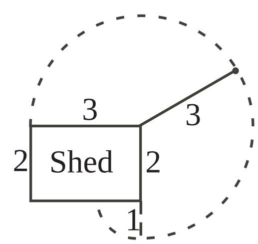
21. The area of rectangle $WXYZ$ is $10 \times 6 = 60$.
Since the shaded area is half of the total area of $WXYZ$, its area is $\frac{1}{2} \times 60 = 30$.
Since AD and WX are perpendicular, then the shaded area has four right angles, so is a rectangle.
Since square $ABCD$ has a side length of 6, then $DC = 6$.
Since the shaded area is 30, then $PD \times DC = 30$ or $PD \times 6 = 30$ or $PD = 5$.
Since $AD = 6$ and $PD = 5$, then $AP = 1$.

Answer: (A)

22. When Chuck has the leash extended to its full length, he can move in a 270° arc, or $\frac{3}{4}$ of a full circle about the point where the leash is attached. (He is blocked from going further by the shed.)



The area that he can play inside this circle is $\frac{3}{4}$ of the area of a full circle of radius 3, or $\frac{3}{4} \times \pi(3^2) = \frac{27}{4}\pi$.
When the leash is extended fully to the left, Chuck just reaches the top left corner of the shed, so can go no further.
When the leash is extended fully to the bottom, Chuck's leash extends 1 m below the length of the shed.
This means that Chuck can play in more area to the left.



This area is a 90° sector of a circle of radius 1, or $\frac{1}{4}$ of this circle. So this additional area is $\frac{1}{4} \times \pi(1^2) = \frac{1}{4}\pi$.
So the total area that Chuck has in which to play is $\frac{27}{4}\pi + \frac{1}{4}\pi = \frac{28}{4}\pi = 7\pi$.

Answer: (A)

23. *Solution 1*
Using \$5 bills, any amount of money that is a multiple of 5 (that is, ending in a 5 or a 0) can be made.
In order to get to \$207 from a multiple of 5 using only \$2 coins, the multiple of \$5 must end in a 5. (If it ended in a 0, adding \$2 coins would still give an amount of money that was an even integer, and so couldn't be \$207.)
Also, from any amount of money ending in a 5 that is less than \$207, enough \$2 coins can always be added to get to \$207.

The positive multiples of 5 ending in a 5 that are less than 207 are 5, 15, 25, . . ., 195, 205. An easy way to count the numbers in this list is to remove the units digits (that is, the 5s) leaving 0, 1, 2, . . ., 19, 20; there are 21 numbers in this list.
These are the only 21 multiples of 5 from which we can use \$2 coins to get to \$207. So there are 21 different ways to make \$207.

Solution 2
We are told that 1 \$2 coin and 41 \$5 bills make \$207. We cannot use fewer \$2 coins, since 0 \$2 coins would not work, so we can only use more \$2 coins.
To do this, we need to "make change" – that is, trade \$5 bills for \$2 coins. We cannot trade 1 \$5 bill for \$2 coins, since 5 is not even. But we can trade 2 \$5 bills for 5 \$2 coins, since each is worth \$10.
Making this trade once gets 6 \$2 coins and 39 \$5 bills.
Making this trade again get 11 \$2 coins and 37 \$5 bills.
We can continue to do this trade until we have only 1 \$5 bill remaining (and so \$202 in \$2 coins, or 101 coins).
So the possible numbers of \$5 bills are 41, 39, 37, . . ., 3, 1. These are all of the odd numbers from 1 to 41. We can quickly count these to get 21 possible numbers of \$5 bills and so 21 possible ways to make \$207.

Answer: (E)

24. To get from $(2, 1)$ to $(12, 21)$, we go 10 units to the right and 20 units up, so we go 2 units up every time we go 1 unit to the right. This means that every time we move 1 unit to the right, we arrive at a lattice point. So the lattice points on this segment are

$$(2, 1), (3, 3), (4, 5), (5, 7), (6, 9), (7, 11), (8, 13), (9, 15), (10, 17), (11, 19), (12, 21)$$

(There cannot be more lattice points in between as we have covered all of the possible x-coordinates.)
To get from $(2, 1)$ to $(17, 6)$, we go 15 units to the right and 5 units up, so we go 3 units to the right every time we go 1 unit up. This means that every time we move 1 unit up, we arrive at a lattice point. So the lattice points on this segment are

$$(2, 1), (5, 2), (8, 3), (11, 4), (14, 5), (17, 6)$$

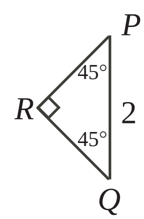
To get from $(12, 21)$ to $(17, 6)$, we go 5 units to the right and 15 units down, so we go 3 units down every time we go 1 unit to the right. This means that every time we move 1 unit to the right, we arrive at a lattice point. So the lattice points on this segment are

$$(12, 21), (13, 18), (14, 15), (15, 12), (16, 9), (17, 12)$$

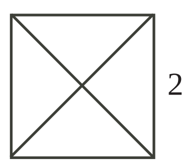
In total, there are $11 + 6 + 6 = 23$ points on our three lists. But 3 points (the 3 vertices of the triangle) have each been counted twice, so there are in fact $23 - 3 = 20$ different points on our lists.
Thus, there are 20 lattice points on the perimeter of the triangle.

Answer: (C)

25. To find the area of quadrilateral $DRQC$, we subtract the area of $\triangle PRQ$ from the area of $\triangle PDC$.
First, we calculate the area of $\triangle PDC$.
We know that $DC = AB = 5$ cm and that $\angle DCP = 90^\circ$.
When the paper is first folded, PC is parallel to AB and lies across the entire width of the paper, so $PC = AB = 5$ cm.
Therefore, the area of $\triangle PDC$ is $\frac{1}{2} \times 5 \times 5 = \frac{25}{2} = 12.5$ cm^2.
Next, we calculate the area of $\triangle PRQ$.
We know that $\triangle PDC$ has $PC = 5$ cm, $\angle PCD = 90^\circ$, and is isosceles with $PC = CD$.
Thus, $\angle DPC = 45^\circ$.
Similarly, $\triangle ABQ$ has $AB = BQ = 5$ cm and $\angle BQA = 45^\circ$.
Therefore, since $BC = 8$ cm and $PB = BC - PC$, then $PB = 3$ cm. Similarly, $AC = 3$ cm.
Since $PQ = BC - BP - QC$, then $PQ = 2$ cm.
Also, $\angle RPQ = \angle DPC = 45^\circ$ and $\angle RQP = \angle BQA = 45^\circ$.



Using four of these triangles, we can create a square of side length 2 cm (thus area 4 cm^2).



The area of one of these triangles (for example, $\triangle PRQ$) is $\frac{1}{4}$ of the area of the square, or 1 cm^2.
So the area of quadrilateral $DRQC$ is therefore $12.5 - 1 = 11.5$ cm^2.

Answer: (D)

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Grade 8 solutions follow the Grade 7 solutions

2006 Gauss Contests

(Grades 7 and 8)

Wednesday, May 10, 2006

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Grade 8 solutions follow the Grade 7 solutions

Grade 7

1. Calculating, $(8 \times 4) + 3 = 32 + 3 = 35$.

   Answer: (D)

2. Since the angles along a straight line add to 180°, then $x^\circ + 40^\circ = 180^\circ$ or $x + 40 = 180$ or $x = 140$.

   Answer: (B)

3. To determine the number of \$50 bills, we divide the total amount of money by 50, to get $10\,000 \div 50 = 200$ bills.
   Therefore, Mikhail has 200 \$50 bills.

   Answer: (B)

4. The figure has 8 sides, each of equal length.
   Since the length of each side is 2, then the perimeter of the figure is $8 \times 2 = 16$.

   Answer: (A)

5. Using a common denominator, $\frac{2}{5} + \frac{1}{3} = \frac{6}{15} + \frac{5}{15} = \frac{11}{15}$.

   Answer: (C)

6. Calculating by determining each product first,
   $$6 \times 100\,000 + 8 \times 1000 + 6 \times 100 + 7 \times 1 = 600\,000 + 8000 + 600 + 7 = 608\,607$$

   Answer: (C)

7. Since $3 + 5x = 28$, then $5x = 28 - 3 = 25$ so $x = \dfrac{25}{5} = 5$.

   Answer: (C)

8. Calculating, $9^2 - \sqrt{9} = 9 \times 9 - \sqrt{9} = 81 - 3 = 78$.

   Answer: (E)

9. In total, there are $2 + 5 + 4 = 11$ balls in the bag.
   Since there are 5 yellow balls, then the probability of choosing a yellow ball is $\frac{5}{11}$.

   Answer: (B)

10. Since the left edge of the block is at the "3" on the ruler and the right edge of the block is between the "5" and "6", then the length of the block is between 2 and 3.
    Looking at the possible choices, the only choice between 2 and 3 is (C) or 2.4 cm.
    (Looking again at the figure, the block appears to end roughly halfway between the "5" and the "6", so 2.4 cm is reasonable.)

    Answer: (C)

11. *Solution 1*
    Since the sales tax is 15%, then the total price for the CD including tax is $1.15 \times \$14.99 = \$17.2385$ which rounds to \$17.24.

    Solution 2
    Since the sales tax is 15%, then the amount of tax on the CD which costs \$14.99 is $0.15 \times \$14.99 = \$2.2485$, which rounds to \$2.25.
    Therefore, the total price of the CD including tax is $\$14.99 + \$2.25 = \$17.24$.

    Answer: (A)

12. *Solution 1*
Since the pool has dimensions 6 m by 12 m by 4 m, then its total volume is $6 \times 12 \times 4 = 288$ m^3.
Since the pool is only half full of water, then the volume of water in the pool is $\frac{1}{2} \times 288$ m^3 or 144 m^3.

Solution 2
Since the pool is half full of water, then the depth of water in the pool is $\frac{1}{2} \times 4 = 2$ m.
Therefore, the portion of the pool which is filled with water has dimensions 6 m by 12 m by 2 m, and so has volume $6 \times 12 \times 2 = 144$ m^3.

ANSWER: (E)

13. To determine the number that must be added 8 to give the result -5, we subtract 8 from -5 to get $(-5) - 8 = -13$. Checking, $8 + (-13) = -5$.

ANSWER: (D)

14. *Solution 1*
Since AOB is a diameter of the circle, then $\angle AOB = 180^\circ$.
We are told that the angle in the "Winter" sector is a right angle (or 90°). Also, we are told that the angle in the "Spring" sector is 60°.
Therefore, the angle in the "Fall" sector is $180^\circ - 90^\circ - 60^\circ = 30^\circ$.
What fraction of the complete circle is 30°?
Since the whole circle has 360°, then the fraction is $\dfrac{30^\circ}{360^\circ} = \dfrac{1}{12}$.
Therefore, $\frac{1}{12}$ of the students chose fall as their favourite season, or $\frac{1}{12} \times 600 = 50$ students in total.

Solution 2
Since AOB is a diameter of the circle, then $\frac{1}{2}$ of the students chose summer as their favourite season, or $\frac{1}{2} \times 600 = 300$ students in total.
Since the angle in the "Winter" sector is a right angle (or 90°), then $\frac{1}{4}$ of the students (since 4 right angles make up a complete circle) chose Winter as their favourite season, or $\frac{1}{4} \times 600 = 150$ students in total.
Since the angle in the "Spring" sector is 60°, then $\dfrac{60^\circ}{360^\circ} = \dfrac{1}{6}$ of the students chose Spring as their favourite season, or $\frac{1}{6} \times 600 = 100$ students in total.
Since there were 600 students in total, then the number who chose Fall as their favourite season was $600 - 300 - 150 - 100 = 50$.

ANSWER: (B)

15. Since Harry charges 50% more for each additional hour as he did for the previous hour, then he charges 1.5 or $\frac{3}{2}$ times as much as he did for the previous hour.
Harry charges \$4 for the first hour.
Harry then charges $\frac{3}{2} \times \$4 = \$6$ for the second hour.
Harry then charges $\frac{3}{2} \times \$6 = \$9$ for the third hour.
Harry then charges $\frac{3}{2} \times \$9 = \$\frac{27}{2} = \$13.50$ for the fourth hour.
Therefore, for 4 hours of babysitting, Harry would earn $\$4 + \$6 + \$9 + \$13.50 = \$32.50$.

ANSWER: (C)

16. *Solution 1*

We obtain fractions equivalent to $\frac{5}{8}$ by multiplying the numerator and denominator by the same number.
The sum of the numerator and denominator of $\frac{5}{8}$ is 13, so when we multiply the numerator and denominator by the same number, the sum of the numerator and denominator is also multiplied by this same number.
Since $91 = 13 \times 7$, then we should multiply the numerator and denominator both by 7 to get a fraction $\frac{5\times7}{8\times7} = \frac{35}{56}$ equivalent to $\frac{5}{8}$ whose numerator and denominator add up to 91.
The difference between the denominator and numerator in this fraction is $56 - 35 = 21$.

Solution 2

We make a list of the fractions equivalent to $\frac{5}{8}$ by multiplying the numerator and denominator by the same number, namely 2, 3, 4, and so on:

$$\frac{5}{8}, \frac{10}{16}, \frac{15}{24}, \frac{20}{32}, \frac{25}{40}, \frac{30}{48}, \frac{35}{56}, \ldots$$

Since the numerator and denominator of $\frac{35}{56}$ add to 91 (since $35 + 56 = 91$), then this is the fraction for which we are looking.
The difference between the denominator and numerator is $56 - 35 = 21$.

Solution 3

We obtain fractions equivalent to $\frac{5}{8}$ by multiplying the numerator and denominator by the same number.
If this number is n, then a fraction equivalent to $\frac{5}{8}$ is $\frac{5n}{8n}$.
For the numerator and denominator to add up to 91, we must have $5n + 8n = 91$ or $13n = 91$ or $n = 7$.
Therefore, the fraction for which we are looking is $\frac{5\times7}{8\times7} = \frac{35}{56}$.
The difference between the denominator and numerator is $56 - 35 = 21$.

Answer: (A)

17. *Solution 1*

Since the shoe is 28 cm long and fits 15 times along one edge of the carpet, then one dimension of the carpet is $15 \times 28 = 420$ cm.
Since the shoe fits 10 times along another edge of the carpet, then one dimension of the carpet is $10 \times 28 = 280$ cm.
Therefore, then area of the carpet is $420 \times 280 = 117\,600$ cm^2.

Solution 2

Since the shoe fits along one edge of the carpet 15 times and along another edge 10 times, then the area of the carpet is $15 \times 10 = 150$ square shoes.
Since the length of the shoe is 28 cm, then

$$1 \text{ square shoe} = 1 \text{ shoe} \times 1 \text{ shoe} = 28 \text{ cm} \times 28 \text{ cm} = 784 \text{ cm}^2$$

Therefore, the area of the carpet in cm^2 is $150 \times 784 = 117\,600$ cm^2.

Answer: (E)

18. *Solution 1*

Since Keiko takes 120 seconds to run 3 times around the track, then it takes her $\frac{1}{3} \times 120 = 40$ seconds to run 1 time around the track.

Since Leah takes 160 seconds to run 5 times around the track, then it takes her $\frac{1}{5} \times 160 = 32$ seconds to run 1 time around the track.
Since Leah takes less time to run around the track than Keiko, then she is the faster runner.
Since Leah takes 32 seconds to run the 150 m around the track, then her speed is $\dfrac{150\text{ m}}{32\text{ s}} = 4.6875\text{ m/s} \approx 4.69\text{ m/s}$.
Therefore, Leah is the faster runner and her speed is approximately 4.69 m/s.

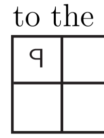
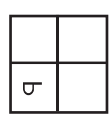
Solution 2
In 120 seconds, Keiko runs 3 times around the track, or $3 \times 150 = 450$ m in total. Therefore, her speed is $\dfrac{450\text{ m}}{120\text{ s}} = 3.75\text{ m/s}$.
In 160 seconds, Leah runs 5 times around the track, or $5 \times 150 = 750$ m in total. Therefore, her speed is $\dfrac{750\text{ m}}{160\text{ s}} = 4.6875\text{ m/s} \approx 4.69\text{ m/s}$.
Since Leah's speed is larger, she is the faster runner and her speed is approximately 4.69 m/s.

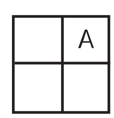
ANSWER: (D)

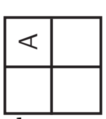
19. *Solution 1*
In one minute, there are 60 seconds.
In one hour, there are 60 minutes, so there are $60 \times 60 = 3600$ seconds.
In one day, there are 24 hours, so there are $24 \times 3600 = 86\,400$ seconds.
Therefore, 10^6 seconds is equal to $\dfrac{10^6}{86\,400} \approx 11.574$ days, which of the given choices is closest to 10 days.

Solution 2
Since there are 60 seconds in one minute, then 10^6 seconds is $\dfrac{10^6}{60} \approx 16\,666.67$ minutes.
Since there are 60 minutes in one hour, then 16 666.67 minutes is $\dfrac{16\,666.67}{60} \approx 277.78$ hours.
Since there are 24 hours in one day, then 277.78 hours is $\dfrac{277.78}{24} \approx 11.574$ days, which of the given choices is closest to 10 days.

ANSWER: (B)

20. One possible way to transform the initial position of the "P" to the final position of the "P" is to reflect the grid in the vertical line in the middle to obtain  and then rotate the grid 90° counterclockwise about the centre to obtain  .

Applying these transformations to the grid containing the "A", we obtain 

and then  . (There are many other possible combinations of transformations which will produce the same resulting image with the "P"; each of these combinations will produce the same result with the "A".)

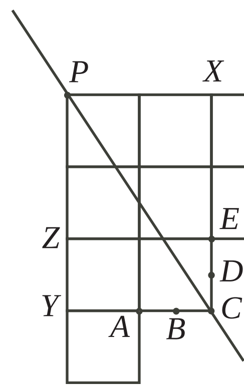
ANSWER: (B)

21. *Solution 1*
Between x a.m. and x p.m. there are 12 hours. (For example, between 10 a.m. and 10 p.m. there are 12 hours.)
Therefore, Gail works for 12 hours on Saturday.

Solution 2
From x a.m. until 12 noon, the number of hours which Gail works is $12 - x$.
From 12 noon until x p.m., she works x hours.
Thus, the total number of hours that Gail works is $(12 - x) + x = 12$.

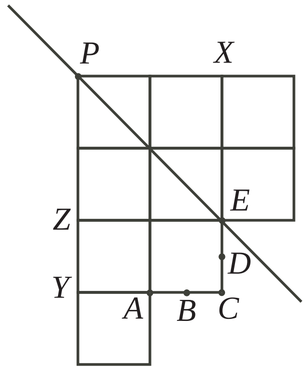
ANSWER: (E)

22. As an initial guess, let us see what happens when the line passes through C.



Since each square is a unit square, then the area of rectangle $PXCY$ is $2 \times 3 = 6$, and so the line through P and C cuts this area in half, leaving 3 square units in the bottom piece and 3 square units in the top piece.
(A fact has been used here that will be used several times in this solution: If a line passes through two diagonally opposite vertices of a rectangle, then it cuts the rectangle into two pieces of equal area (since it cuts the rectangle into two congruent triangles).)
In the bottom piece, we have not accounted for the very bottom unit square, so the total area of the bottom piece is 4 square units.
In the top piece, we have not accounted for the two rightmost unit squares, so the total area of the top piece is 5 square units.
So putting the line through C does not produce two pieces of equal area, so C is not the correct answer.
Also, since the area of the bottom piece is larger than the area of the top piece when the line passes through C, we must move the line up to make the areas equal (so neither A nor B can be the answer).

Should the line pass through E?



If so, then the line splits the square $PXEZ$ (of area 4) into two pieces of area 2.
Then accounting for the remaining squares, the area of the bottom piece is $2+3=5$ and the area of the top piece is $2+2=4$.
So putting the line through E does not produce two pieces of equal area, so E is not the correct answer.
By elimination, the correct answer should be D.

(We should verify that putting the line through D does indeed split the area in half.
The total area of the shape is 9, since it is made up of 9 unit squares.
If the line goes through D, the top piece consists of $\triangle PXD$ and 2 unit squares.
The area of $\triangle PXD$ is $\frac{1}{2}\times 2\times\frac{5}{2}=\frac{5}{2}$, since $PX=2$ and $XD=\frac{5}{2}$.
Thus, the area of the top piece is $\frac{5}{2}+2=\frac{9}{2}$, which is exactly half of the total area, as required.)

Answer: (D)

23. We label the blank spaces to make them easier to refer to.

$$\begin{array}{r}\boxed{A}\,\boxed{B}\\ +\ \boxed{C}\,\boxed{D}\\ \hline \boxed{E}\,\boxed{F}\,\boxed{G}\end{array}$$

Since we are adding two 2-digit numbers, then their sum cannot be 200 or greater, so E must be a 1 if the sum is to have 3 digits.

Where can the digit 0 go?
Since no number can begin with a 0, then neither A nor C can be 0.
Since each digit is different, then neither B and D can be 0, otherwise both D and G or B and G would be the same.
Therefore, only F or G could be 0.

Since we are adding two 2-digit numbers and getting a number which is at least 100, then $A+C$ must be at least 9. (It could be 9 if there was a "carry" from the sum of the units digits.)
This tells us that A and C must be 3 and 6, 4 and 5, 4 and 6, or 5 and 6.

If G was 0, then B and D would have to 4 and 6 in some order. But then the largest that A and C could be would be 3 and 5, which are not among the possibilities above.
Therefore, G is not 0, so $F=0$.

$$\begin{array}{rccc} & & \boxed{A} & \boxed{B} \\ + & & \boxed{C} & \boxed{D} \\ \hline & \boxed{1} & \boxed{0} & \boxed{G} \end{array}$$

So the sum of A and C is either 9 or 10, so A and C are 3 and 6, 4 and 5, or 4 and 6.
In any of these cases, the remaining possibilities for B and D are too small to give a carry from the units column to the tens column.
So in fact, A and C must add to 10, so A and C are 4 and 6 in some order.
Let's try $A = 4$ and $C = 6$.

$$\begin{array}{rccc} & & \boxed{4} & \boxed{B} \\ + & & \boxed{6} & \boxed{D} \\ \hline & \boxed{1} & \boxed{0} & \boxed{G} \end{array}$$

The remaining digits are 2, 3 and 5. To make the addition work, B and D must be 2 and 3 and G must be 5. (We can check that either order for B and D works, and that switching the 4 and 6 will also work.)

So the units digit of the sum must be 5, as in the example

$$\begin{array}{rccc} & & \boxed{4} & \boxed{2} \\ + & & \boxed{6} & \boxed{3} \\ \hline & \boxed{1} & \boxed{0} & \boxed{5} \end{array}$$

(Note that we could have come up with this answer by trial and error instead of this logical procedure.)

Answer: (D)

24. The sum of any two sides of a triangle must be bigger than the third side.
(When two sides are known to be equal, we only need to check if the sum of the two equal sides is longer than the third side, since the sum of one of the equal sides and the third side will always be longer than the other equal side.)
If the equal sides were both equal to 2, the third side must be shorter than $2 + 2 = 4$. The 1 possibility from the list not equal to 2 (since we cannot have three equal sides) is 3. So here there is 1 possibility.
If the equal sides were both equal to 3, the third side must be shorter than $3 + 3 = 6$. The 2 possibilities from the list not equal to 3 (since we cannot have three equal sides) are 2 and 5. So here there are 2 possibilities.
If the equal sides were both equal to 5, the third side must be shorter than $5 + 5 = 10$. The 3 possibilities from the list not equal to 5 (since we cannot have three equal sides) are 2, 3 and 7. So here there are 3 possibilities.
If the equal sides were both equal to 7, the third side must be shorter than $7 + 7 = 14$. The 4 possibilities from the list not equal to 7 (since we cannot have three equal sides) are 2, 3, 5

and 11. So here there are 4 possibilities.
If the equal sides were both equal to 11, the third side must be shorter than $11 + 11 = 22$. The 4 possibilities from the list not equal to 11 (since we cannot have three equal sides) are 2, 3, 5 and 7. So here there are 4 possibilities.
Thus, in total there are $1 + 2 + 3 + 4 + 4 = 14$ possibilities.

Answer: (E)

25. The five scores are N, 42, 43, 46, and 49.
If $N < 43$, the median score is 43.
If $N > 46$, the median score is 46.
If $N \geq 43$ and $N \leq 46$, then N is the median.
We try each case.

If $N < 43$, then the median is 43, so the mean should be 43.
Since the mean is 43, then the sum of the 5 scores must be $5 \times 43 = 215$.
Therefore, $N + 42 + 43 + 46 + 49 = 215$ or $N + 180 = 215$ or $N = 35$, which is indeed less than 43.
We can check that the median and mean of 35, 42, 43, 46 and 49 are both 43.

If $N > 46$, then the median is 46, so the mean should be 46.
Since the mean is 46, then the sum of the 5 scores must be $5 \times 46 = 230$.
Therefore, $N + 42 + 43 + 46 + 49 = 230$ or $N + 180 = 230$ or $N = 50$, which is indeed greater than 46.
We can check that the median and mean of 42, 43, 46, 49, and 50 are both 46.

If $N \geq 43$ and $N \leq 46$, then the median is N, so the mean should be N.
Since the mean is N, then the sum of the 5 scores must be $5N$.
Therefore, $N + 42 + 43 + 46 + 49 = 5N$ or $N + 180 = 5N$ or $4N = 180$, or $N = 45$, which is indeed between 43 and 46.
We can check that the median and mean of 42, 43, 45, 46 and 49 are both 45.

Therefore, there are 3 possible values for N.

Answer: (A)

Grade 8

1. Calculating using the correct order of operations, $30 - 5^2 = 30 - 25 = 5$.

   Answer: (E)

2. *Solution 1*
   Since $98 \div 2 = 49$, $98 \div 7 = 14$, $98 \div 14 = 7$, $98 \div 49 = 2$, and $98 \div 4 = 24.5$, then the one of the five choices which does not divide exactly into 98 is 4.

   Solution 2
   The prime factorization of 98 is $98 = 2 \times 7 \times 7$.
   Of the given possibilities, only $4 = 2 \times 2$ cannot be a divisor of 98, since there are not two 2's in the prime factorization of 98.

   Answer: (B)

3. Since the tax is 15% on the \$200.00 camera, then the tax is $0.15 \times \$200.00 = \$30.00$.

   Answer: (A)

4. Since $1 + 1.1 + 1.11 = 3.21$, then to get a sum of 4.44, we still need to add $4.44 - 3.21 = 1.23$. Thus, the number which should be put in the box is 1.23.

   Answer: (B)

5. In total, there are $2 + 5 + 4 = 11$ balls in the bag.
   Since there are 5 yellow balls, then the probability of choosing a yellow ball is $\frac{5}{11}$.

   Answer: (B)

6. We check each number between 20 and 30.
   None of 20, 22, 24, 26, 28, and 30 is a prime number, because each has a factor of 2.
   Neither 21 nor 27 is a prime number, since each has a factor of 3.
   Also, 25 is not a prime number, since it has a factor of 5.
   We do not need to look for any larger prime factors, since this leaves only 23 and 29, each of which is a prime number. Therefore, there are 2 prime numbers between 20 and 30.

   Answer: (C)

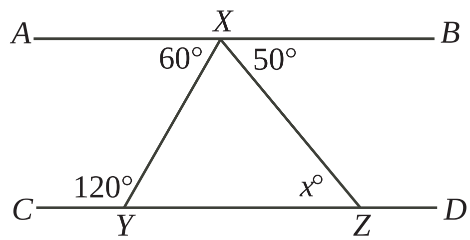
7. Since the volume of a rectangular block is equal to the area of its base times its height, then the height of this particular rectangular block is $\dfrac{120 \text{ cm}^3}{24 \text{ cm}^2} = 5$ cm.

   Answer: (A)

8. Since the rate of rotation of the fan doubles between the slow and medium settings and the fan rotates 100 times in 1 minute on the slow setting, then it rotates $2 \times 100 = 200$ times in 1 minute on the medium setting.
   Since the rate of rotation of the fan doubles between the medium and high setting, then it rotates $2 \times 200 = 400$ times in 1 minute on the high setting.
   Therefore, in 15 minutes, it will rotate $15 \times 400 = 6000$ times.

   Answer: (C)

9. *Solution 1*



Since $\angle AXB = 180°$, then $\angle YXZ = 180° - 60° - 50° = 70°$.
Also, $\angle XYZ = 180° - \angle CYX = 180° - 120° = 60°$.
Since the angles in $\triangle XYZ$ add to $180°$, then $x° = 180° - 70° - 60° = 50°$, so $x = 50$.

Solution 2
Since $\angle CYX + \angle AXY = 180°$, then AB is parallel to CD.
Therefore, $\angle YZX = \angle ZXB$ or $x° = 50°$ or $x = 50$.

Answer: (A)

10. *Solution 1*
When we divide 8362 by 12, we obtain

$$\begin{array}{r|l} & \ \ 696 \\ \hline 12 & 8362 \\ & \underline{72} \\ & 116 \\ & \underline{108} \\ & \ \ \ 82 \\ & \ \ \ \underline{72} \\ & \ \ \ 10 \end{array}$$

so $\dfrac{8362}{12} = 696\dfrac{10}{12}$. In other words, 8362 lollipops make up 696 complete packages leaving 10 lollipops left over.

Solution 2
When we divide 8362 by 12, we obtain approximately 696.83, so the maximum possible number of packages which can be filled is 696.
In total, 696 packages contain 8352 lollipops, leaving 10 remaining from the initial 8362 lollipops.

Answer: (E)

11. Since the sound of thunder travels at 331 m/s and the thunder is heard 12 seconds after the lightning flash, then Joe is 12 s $\times$ 331 m/s $=$ 3972 m $=$ 3.972 km from the lightning flash, or, to the nearest tenth of a kilometre, 4.0 km.

Answer: (C)

12. The shaded triangle has a base of length 10 cm.
Since the triangle is enclosed in a rectangle of height 3 cm, then the height of the triangle is 3 cm.
(We know that the enclosing shape is a rectangle, because any figure with 4 sides, including 2 pairs of equal opposite sides, and 2 right angles must be a rectangle.)
Therefore, the area of the triangle is $\frac{1}{2} \times 3 \times 10 = 15$ cm^2.

Answer: (C)

13. We need to find two consecutive numbers the first of which is a multiple of 7 and the second of which is a multiple of 5.
We try the multiples of 7 and the numbers after each.
Do 7 and 8 work? No, since 8 is not a multiple of 5.
Do 14 and 15 work? Yes, since 15 is a multiple of 5.
This means that this year, Kiril is 15 years old.
So it will be 11 years until Kiril is 26 years old.
(Of course, Kiril could also be 50 years old or 85 years old this year, but then he would have already been 26 years old in the past.)

ANSWER: (A)

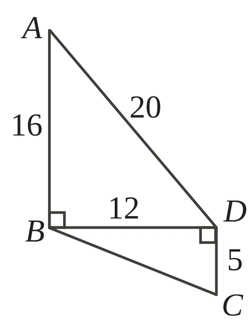
14. We are told that the second term in the sequence is 260.
Using the rule for the sequence, to get the third term, we divide the second term by 2 to obtain 130 and then add 10 to get 140, which is the third term.
To get the fourth term, we divide the third term by 2 to obtain 70 and then add 10 to get 80.

ANSWER: (E)

15. When the original shape  is reflected in Line 1 (that is, reflected vertically), the shape  is obtained.
When this new shape is reflected in Line 2 (that is, reflected horizontally), the shape that results is  .

ANSWER: (D)

16. By the Pythagorean Theorem in $\triangle ABD$, we have $BD^2 + 16^2 = 20^2$ or $BD^2 + 256 = 400$ or $BD^2 = 144$. Therefore, $BD = 12$.



By the Pythagorean Theorem in $\triangle BDC$, we have $BC^2 = 12^2 + 5^2 = 144 + 25 = 169$, so $BC^2 = 169$ or $BC = 13$.

ANSWER: (A)

17. Since $10^x - 10 = 9990$, then $10^x = 9990 + 10 = 10\,000$.
If $10^x = 10\,000$, then $x = 4$, since $10\,000$ ends in 4 zeroes.

ANSWER: (D)

18. Since the square has perimeter 24, then the side length of the square is $\frac{1}{4} \times 24 = 6$.
Since the square has side length 6, then the area of the square is $6^2 = 36$.
Since the rectangle and the square have the same area, then the area of rectangle is 36.
Since the rectangle has area 36 and width 4, then the length of the rectangle is $\dfrac{36}{4} = 9$.
Since the rectangle has width 4 and length 9, then it has perimeter $4 + 9 + 4 + 9 = 26$.

ANSWER: (A)

19. *Solution 1*

We can write out the possible arrangements, using the first initials of the four people, and remembering that there two different ways in which Dominic and Emily can sit side by side:

DEBC, DECB, EDBC, EDCB, BDEC, CDEB, BEDC, CEDB, BCDE, CBDE, BCED, CBED

There are 12 possible arrangements.

Solution 2

Since Dominic and Emily must sit beside other, then we can combine then into one person – either Domily or Eminic, depending on the order in which they sit – and their two seats into one seat.

We now must determine the number of ways of arranging Bethany, Chun and Domily into three seats, and the number of ways of arranging Bethany, Chun and Eminic into three seats.

With three people, there are three possible choices for who sits in the first seat, and for each of these choices there are two possible choices for the next seat, leaving the choice for the final seat fixed. So with three people, there are $3 \times 2 = 6$ possible ways that they can sit in three chairs.

So for each of the 2 ways in which Dominic and Emily can sit, there are 6 ways that the seating can be done, for a total of 12 seating arrangements.

Answer: (C)

20. We label the blank spaces to make them easier to refer to.

$$\begin{array}{cccc} & & \boxed{A} & \boxed{B} \\ + & & \boxed{C} & \boxed{D} \\ \hline & \boxed{E} & \boxed{F} & \boxed{G} \end{array}$$

Since we are adding two 2-digit numbers, then their sum cannot be 200 or greater, so E must be a 1 if the sum is to have 3 digits.

Where can the digit 0 go?

Since no number can begin with a 0, then neither A nor C can be 0.

Since each digit is different, then neither B and D can be 0, otherwise both D and G or B and G would be the same.

Therefore, only F or G could be 0.

Since we are adding two 2-digit numbers and getting a number which is at least 100, then $A+C$ must be at least 9. (It could be 9 if there was a "carry" from the sum of the units digits.) This tells us that A and C must be 3 and 6, 4 and 5, 4 and 6, or 5 and 6.

If G was 0, then B and D would have to 4 and 6 in some order. But then the largest that A and C could be would be 3 and 5, which are not among the possibilities above.

Therefore, G is not 0, so $F = 0$.

$$\begin{array}{cccc} & & \boxed{A} & \boxed{B} \\ + & & \boxed{C} & \boxed{D} \\ \hline & \boxed{1} & \boxed{0} & \boxed{G} \end{array}$$

So the sum of A and C is either 9 or 10, so A and C are 3 and 6, 4 and 5, or 4 and 6.
In any of these cases, the remaining possibilities for B and D are too small to give a carry from the units column to the tens column.
So in fact, A and C must add to 10, so A and C are 4 and 6 in some order.
Let's try $A = 4$ and $C = 6$.

$$\begin{array}{cccc} & & \boxed{4} & \boxed{B} \\ + & & \boxed{6} & \boxed{D} \\ \hline & \boxed{1} & \boxed{0} & \boxed{G} \end{array}$$

The remaining digits are 2, 3 and 5. To make the addition work, B and D must be 2 and 3 and G must be 5. (We can check that either order for B and D works, and that switching the 4 and 6 will also work.)

So the units digit of the sum must be 5, as in the example

$$\begin{array}{cccc} & & \boxed{4} & \boxed{2} \\ + & & \boxed{6} & \boxed{3} \\ \hline & \boxed{1} & \boxed{0} & \boxed{5} \end{array}$$

(Note that we could have come up with this answer by trial and error instead of this logical procedure.)

Answer: (D)

21. *Solution 1*
If Nathalie had exactly 9 quarters, 3 dimes and 1 nickel, she would have
$9 \times 25 + 3 \times 10 + 5 = 260$ cents or \$2.60 in total.
Since the coins she has are in this same ratio $9 : 3 : 1$, then we can split her coins up into sets of 9 quarters, 3 dimes and 1 nickel, with each set having a value of \$2.60.
Since the total value of her coins is \$18.20 and $\dfrac{\$18.20}{\$2.60} = 7$, then she has 7 of these sets of coins.
Since each of these sets contains 13 coins, then she has $7 \times 13 = 91$ coins in total.

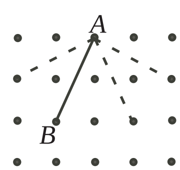
Solution 2
Suppose Nathalie had n nickels. Since the ratio of the number of quarters to the number of dimes to the number of nickels that she has is $9 : 3 : 1$, then she must have $3n$ dimes and $9n$ quarters.
This tells us that the total value of her coins in cents is $(9n \times 25) + (3n \times 10) + (n \times 5) = 260n$.
Since we know that the total value of her coins is 1820 cents, then $260n = 1820$ or $n = 7$.
Therefore, she has 7 nickels, 21 dimes and 63 quarters, or 91 coins in total.

Answer: (D)

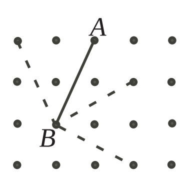
22. Label the first 8 people at the party as A, B, C, D, E, F, G, and H.
Then A shakes hands with the 7 others, B accounts for 6 more handshakes (since we have already included B shaking hands with A), C accounts for 5 more handshakes (since we have already included C shaking hands with A and B), D accounts for 4 more handshakes, E accounts for 3 more handshakes, F accounts for 2 more handshakes, and G accounts for 1 more handshake. (H's handshakes with each of the others have already been included.)
So there are $7+6+5+4+3+2+1 = 28$ handshakes which take place before the ninth person arrives.
Since there are a total of 32 handshakes which take place, then the ninth person shakes hands with $32 - 28 = 4$ people.

Answer: (B)

23. In how many ways can we place C so that $AB = AC$?
To get from A to B we go 1 unit in one direction and 2 units in the perpendicular direction from A.
To choose C so that $AB = AC$, we must also go 1 unit in one direction and 2 units in the perpendicular direction.
This gives the 3 possible points:



In how many ways can we place C so that $BA = BC$?
To choose C so that $BA = BC$, we again also go 1 unit in one direction and 2 units in the other direction from B.
This gives the 3 possible points:



So in total we have 6 possible points so far, which is the largest of the possible answers.
So 6 must be the answer.
(There is one more possibility: can C be placed so that $CA = CB$? Since 6 must be the answer, then the answer to this question is no. We can also check this by trying to find such points in the diagram.)

Answer: (A)

24. In the 1st row, every box is shaded.
In the 2nd row, the boxes whose column numbers are multiples of 2 are shaded.
In the 3rd row, the boxes whose column numbers are multiples of 3 are shaded.
In the nth row, the boxes whose column numbers are multiples of n are shaded.
So in a particular column, the boxes which are shaded are those which belong to row numbers which are divisors of the column number.
(We can see this for instance in columns 4 and 6 where the boxes in rows 1, 2 and 4, and 1, 2,

3, and 6, respectively, are shaded.)
So to determine which of the given columns has the largest number of shaded boxes, we must determine which of the given numbers has the greatest number of divisors.
To find the divisors of 144, for instance, it is easier to find the prime factors of 144 first.
To do this, we see that $144 = 16 \times 9 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 = 2^4 \times 3^2$.
We can then write out the divisors of 144, which are 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144, or a total of 15 divisors.
Similarly, $120 = 2^3 \times 3 \times 5$, so we can find the divisors of 120, which are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120, or a total of 16 divisors.
Since $150 = 2 \times 3 \times 5^2$, the divisors of 150 are 1, 2, 3, 5, 6, 10, 15, 25, 30, 50, 75, 150, or a total of 12 divisors.
Since $96 = 2^5 \times 3$, the divisors of 96 are 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96, or a total of 12 divisors.
Since $100 = 2^2 \times 5^2$, the divisors of 100 are 1, 2, 4, 5, 10, 20, 25, 50, 100, or a total of 9 divisors.
So the number with the most divisors (and thus the column with the most shaded boxes) is 120.

Answer: (B)

25. The numbers which have yet to be placed in the grid are 10, 11, 12, 13, 14, 15, 16, 17, 18, and 19.
We look at the numbers in the corners first, because they have the fewest neighbours.
Look first at the 25 in the bottom left corner. It must be the sum of two of its neighbours, so it must be $24 + 1$ or $9 + 16$. Since the 1 has already been placed in the grid, then the empty space next to the 25 must be filled with the 16.

Look next at the 21 in the top right corner. It must be equal to $20 + 1$ (which it cannot be since the 1 has already been placed) or $4 + 17$, so the 17 must be placed in the space next to the 21.

			20	21
	6	5	4	17
23	7	1	3	?
16	9	8	2	
25	24			22

Since the 17 must be the sum of two of its neighbours, then the 17 must be $4 + 13$ or $3 + 14$, so the "?" must be replaced by either the 13 or the 14.

Consider the 22 in the corner. Since the 20 is already placed, 22 cannot be $2 + 20$. So the 22 is the sum of its two missing neighbours.
Since the possible missing neighbours are 10, 11, 12, 13, 14, 15, and 18, then the two neighbours must be 10 and 12 in some order.
However, the 10 cannot go above the 22, since there it could not be the sum of two of its neighbours (since the 7 and 8 have already been placed).
Therefore, the 12 goes above the 22, so we have

			20	21
	6	5	4	17
23	7	1	3	?
16	9	8	2	12
25	24		10	22

The 13 now cannot replace the "?" since 13 is not the sum of any two of 17, 4, 3, 2, and 12. Therefore, the "?" is replaced by the 14.

At this stage, we are already finished, but we could check that the grid does complete as follows:

19	11	15	20	21
13	6	5	4	17
23	7	1	3	14
16	9	8	2	12
25	24	18	10	22

Answer: (C)

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Grade 8 solutions follow the Grade 7 solutions

2005 Gauss Contests

(Grades 7 and 8)

Wednesday, May 11, 2005

Solutions

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Grade 7

Grade 8 solutions follow the Grade 7 solutions

1. Calculating the numerator first, $\frac{3 \times 4}{6} = \frac{12}{6} = 2$.

   Answer: (B)

2. Calculating, $0.8 - 0.07 = 0.80 - 0.07 = 0.73$.

   Answer: (E)

3. Since the arrow is pointing between 9.6 and 9.8, it is pointing to a rating closest to 9.7.

   Answer: (C)

4. Twelve million is written as 12 000 000 and twelve thousand is written as 12 000, so the sum of these two numbers is 12 012 000.

   Answer: (A)

5. To figure out which number is largest, we look first at the number in the tenths position. Since four of the given numbers have a 1 in the tenths position and 0.2 has a 2, then 0.2 is the largest.

   Answer: (B)

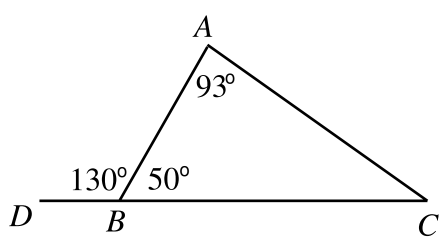
6. Since Meghan chooses a prize from 27 in the bag and the probability of her choosing a book is $\frac{2}{3}$, then $\frac{2}{3}$ of the prizes in the bag must be books.
   Therefore, the number of books in the bag is $\frac{2}{3} \times 27 = 18$.

   Answer: (E)

7. Since 83% in decimal form is 0.83, then the number of people who voted for Karen is equal to $0.83 \times 1\,480\,000 = 1\,228\,400$.

   Answer: (B)

8. Since $\angle ABC + \angle ABD = 180^\circ$ (in other words, $\angle ABC$ and $\angle ABD$ are supplementary) and $\angle ABD = 130^\circ$, then $\angle ABC = 50^\circ$.

   Since the sum of the angles in triangle ABC is 180° and we know two angles 93° and 50° which add to 143°, then $\angle ACB = 180^\circ - 143^\circ = 37^\circ$.

   Answer: (B)

9. There are six odd-numbered rows (rows 1, 3, 5, 7, 9, 11).
   These rows have $6 \times 15 = 90$ seats in total.
   There are five even-numbered rows (rows 2, 4, 6, 8, 10).
   These rows have $5 \times 16 = 80$ seats in total.
   Therefore, there are $90 + 80 = 170$ seats in total in the theatre.

   Answer: (D)

10. When it is 5:36 p.m. in St. John's, it is 90 minutes or $1\frac{1}{2}$ hours earlier in Smiths Falls, so it is 4:06 p.m. in Smiths Falls.
When it is 4:06 p.m. in Smiths Falls, it is 3 hours earlier in Whitehorse, so it is 1:06 p.m. in Whitehorse.

ANSWER: (A)

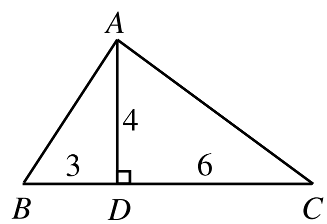
11. On each day, the temperature range is the difference between the daily high and daily low temperatures.
On Monday, the range is $6 - (-4) = 10$ degrees Celsius.
On Tuesday, the range is $3 - (-6) = 9$ degrees Celsius.
On Wednesday, the range is $4 - (-2) = 6$ degrees Celsius.
On Thursday, the range is $4 - (-5) = 9$ degrees Celsius.
On Friday, the range is $8 - 0 = 8$ degrees Celsius.
The day with the greatest range is Monday.

ANSWER: (A)

12. Since the bamboo plant grows at a rate of 105 cm per day and there are 7 days from May 1st and May 8th, then it grows $7 \times 105 = 735$ cm in this time period.
Since 735 cm = 7.35 m, then the height of the plant on May 8th is $2 + 7.35 = 9.35$ m.

ANSWER: (E)

13. Since $BD = 3$ and DC is twice the length of BD, then $DC = 6$.



Therefore, triangle ABC has a base of length 9 and a height of length 4.
Therefore, the area of triangle ABC is $\frac{1}{2}bh = \frac{1}{2}(9)(4) = \frac{1}{2}(36) = 18$.

ANSWER: (D)

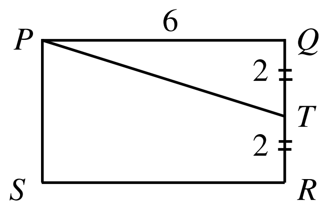
14. *Solution 1*
Since the sum of the numbers on opposite faces on a die is 7, then 1 and 6 are on opposite faces, 2 and 5 are on opposite faces, and 3 and 4 are on opposite faces.
On the first die, the numbers on the unseen faces opposite the 6, 2 and 3 are 1, 5 and 4, respectively.
On the second die, the numbers on the unseen faces opposite the 1, 4 and 5 are 6, 2 and 3, respectively.
The sum of the missing numbers is $1 + 5 + 4 + 6 + 2 + 3 = 21$.

Solution 2
The sum of the numbers on a die is $1 + 2 + 3 + 4 + 5 + 6 = 21$ and so the sum of the numbers on two die is $2 \times 21 = 42$.
Since there is a sum of 21 showing on the six visible faces, the sum of the numbers on the six unseen faces is $42 - 21 = 21$.

ANSWER: (C)

15. *Solution 1*

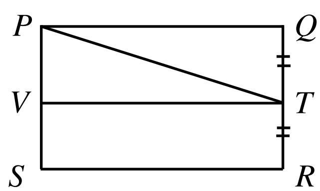
Since the area of rectangle $PQRS$ is 24, let us assume that $PQ = 6$ and $QR = 4$.
Since $QT = QR$, then $QR = 2QT$ so $QT = 2$.



Therefore, triangle PQT has base PQ of length 6 and height QT of length 2, so has area $\frac{1}{2}(6)(2) = \frac{1}{2}(12) = 6$.
So the area of quadrilateral $PTRS$ is equal to the area of rectangle $PQRS$ (which is 24) minus the area of triangle PQT (which is 6), or 18.

Solution 2

Draw a line through T parallel to PQ across the rectangle parallel so that it cuts PS at point V.



Since T is halfway between Q and R, then V is halfway between P and S.
Therefore, $SVTR$ is a rectangle which has area equal to half the area of rectangle $PQRS$, or 12.
Similarly, $VPQT$ is a rectangle of area 12, and $VPQT$ is cut in half by PT, so triangle PVT has area 6.
Therefore, the area of $PTRS$ is equal to the sum of the area of rectangle $SVTR$ and the area of triangle PVT, or $12 + 6 = 18$.

Answer: (A)

16. Nicholas sleeps for an hour and a half, or 90 minutes.
Since three sleep cross the road per minute, then $3 \times 90 = 270$ sheep cross while he is asleep.
Since 42 sheep crossed before he fell asleep, then $42 + 270 = 312$ sleep have crossed the road in total when he wakes up.
Since this is half of the total number of sheep in the flock, then the total number in the flock is $2 \times 312 = 624$.

Answer: (D)

17. *Solution 1*

When we calculate the value of the symbol, we add the product of the numbers on each of the two diagonals.
The product of the entries on the diagonal with the 1 and the 6 is 6.
Since the symbol is evaluated as 16, then the product of the entries on the other diagonal is 10.
Since one of the entries on the other diagonal is 2, then the missing entry must be 5.

Solution 2
Let the missing number be x.
Using the definition for the evaluation of the symbol, we know that $2 \times x + 1 \times 6 = 16$ or $2x + 6 = 16$ or $2x = 10$ or $x = 5$.

ANSWER: (E)

18. When a die is rolled, there are six equally likely possibilities (1 through 6).
In order for the game to be fair, half of the six possibilities, or three possibilities, must be winning possibilities.
In the first game, only rolling a 2 gives a win, so this game is not fair.
In the second game, rolling a 2, 4 or 6 gives a win, so this game is fair.
In the third game, rolling a 1, 2 or 3 gives a win, so this game is fair.
In the fourth game, rolling a 3 or 6 gives a win, so this game is not fair.
Therefore, only two of the four games are fair.

ANSWER: (C)

19. At each distance, two throws are made: the 1st and 2nd throws are made at 1 m, the 3rd and 4th are made at 2 m, and so on, with the 27th and 28th throws being made at 14 m.
Therefore, the 29th throw is the first throw made at 15 m.
At each distance, the first throw is made by Pat to Chris, so Chris misses catching the 29th throw at a distance of 15 m.

ANSWER: (A)

20. Since Sally's car travels 80 km/h, it travels 80 000 m in one hour.
Since there are 60 minutes in an hour, the car travels $\frac{1}{60} \times 80\,000$ m in one minute.
Since there are 60 seconds in a minute, the car travels $\frac{1}{60} \times \frac{1}{60} \times 80\,000$ in one second.
Therefore, in 4 seconds, the car travels $4 \times \frac{1}{60} \times \frac{1}{60} \times 80\,000 \approx 88.89$ m.
Of the possible choices, this is closest to 90 m.

ANSWER: (E)

21. Since the price of the carpet is reduced by 10% every 15 minutes, then the price is multiplied by 0.9 every 15 minutes.
At 9:15, the price was \$9.00.
At 9:30, the price fell to $0.9 \times \$9.00 = \$8.10$.
At 9:45, the price fell to $0.9 \times \$8.10 = \$7.29$.
So the price fell below \$8.00 at 9:45 a.m., so Emily bought the carpet at 9:45 a.m.

ANSWER: (A)

22. *Solution 1*
We start by assuming that there are 20 oranges. (We pick 20 since the ratio of apples to oranges is $1 : 4$ and the ratio of oranges to lemons is $5 : 2$, so we pick a number of oranges which is divisible by 4 and by 5. Note that we did not have to assume that there were 20 oranges, but making this assumption makes the calculations much easier.)
Since there are 20 oranges and the ratio of the number of apples to the number of oranges is $1 : 4$, then there are $\frac{1}{4} \times 20 = 5$ apples.
Since there are 20 oranges and the ratio of the number of oranges to the number of lemons is $5 : 2$, then there are $\frac{2}{5} \times 20 = 8$ lemons.
Therefore, the ratio of the number of apples to the number of lemons is $5 : 8$.

Solution 2
Let the number of apples be x.
Since the ratio of the number of apples to the number of oranges is $1:4$, then the number of oranges is $4x$.
Since the ratio of the number of oranges to the number of lemons is $5:2$, then the number of lemons is $\frac{2}{5} \times 4x = \frac{8}{5}x$.
Since the number of apples is x and the number of lemons is $\frac{8}{5}x$, then the ratio of the number of apples to the number of lemons is $1 : \frac{8}{5} = 5 : 8$.

Answer: (C)

23. *Solution 1*
If 4 □ balance 2 ○, then 1 □ would balance the equivalent of $\frac{1}{2}$ ○.
Similarly, 1 △ would balance the equivalent of $1\frac{1}{2}$ ○.
If we take each of the answers and convert them to an equivalent number of ○, we would have:
(**A**): $1\frac{1}{2} + 1 + \frac{1}{2} = 3$ ○
(**B**): $3\left(\frac{1}{2}\right) + 1\frac{1}{2} = 3$ ○
(**C**): $2\left(\frac{1}{2}\right) + 2 = 3$ ○
(**D**): $2\left(1\frac{1}{2}\right) + \frac{1}{2} = 3\frac{1}{2}$ ○
(**E**): $1 + 4\left(\frac{1}{2}\right) = 3$ ○
Therefore, 2 △ and 1 □ do not balance the required.

Solution 2
Since 4 □ balance 2 ○, then 1 ○ would balance 2 □.
Therefore, 3 ○ would balance 6 □, so since 3 ○ balance 2 △, then 6 □ would balance 2 △, or 1 △ would balance 3 □.
We can now express every combination in terms of □ only.
1 △, 1 ○ and 1 □ equals $3 + 2 + 1 = 6$ □.
3 □ and 1 △ equals $3 + 3 = 6$ □.
2 □ and 2 ○ equals $2 + 2 \times 2 = 6$ □.
2 △ and 1 □ equals $2 \times 3 + 1 = 7$ □.
1 ○ and 4 □ equals $2 + 4 = 6$ □.
Therefore, since 1 △, 1 ○ and 1 □ equals 6 □, then it is 2 △ and 1 □ which will not balance with this combination.

Solution 3
We try assigning weights to the different shapes.
Since 3 ○ balance 2 △, assume that each ○ weighs 2 kg and each △ weighs 3 kg.
Therefore, since 4 □ balance 2 ○, which weigh 4 kg combined, then each □ weighs 1 kg.
We then look at each of the remaining combinations.
1 △, 1 ○ and 1 □ weigh $3 + 2 + 1 = 6$ kg.
3 □ and 1 △ weigh $3 + 3 = 6$ kg.
2 □ and 2 ○ weigh $2 + 2 \times 2 = 6$ kg.
2 △ and 1 □ weigh $2 \times 3 + 1 = 7$ kg.
1 ○ and 4 □ weigh $2 + 4 = 6$ kg.
Therefore, it is the combination of 2 △ and 1 □ which will not balance the other combinations.

Answer: (D)

24. Since Alphonse and Beryl always pass each other at the same three places on the track and since they each run at a constant speed, then the three places where they pass must be equally spaced on the track. In other words, the three places divide the track into three equal parts.
We are not told which runner is faster, so we can assume that Beryl is the faster runner.
Start at one place where Alphonse and Beryl meet. (Now that we know the relative positions of where they meet, we do not actually have to know where they started at the very beginning.)
To get to their next meeting place, Beryl runs farther than Alphonse (since she runs faster than he does), so Beryl must run $\frac{2}{3}$ of the track while Alphonse runs $\frac{1}{3}$ of the track in the opposite direction, since the meeting placed are spaced equally at $\frac{1}{3}$ intervals of the track.
Since Beryl runs twice as far in the same length of time, then the ratio of their speeds is $2:1$.
ANSWER: (D)

25. *Solution 1*
We want to combine 48 coins to get 100 cents.
Since the combined value of the coins is a multiple of 5, as is the value of a combination of nickels, dimes and quarters, then the value of the pennies must also be a multiple of 5.

Therefore, the possible numbers of pennies are 5, 10, 15, 20, 25, 30, 35, 40.
We can also see that because there are 48 coins in total, it is not possible to have anything other than 35, 40 or 45 pennies. (For example, if we had 30 pennies, we would have 18 other coins which are worth at least 5 cents each, so we would have at least $30 + 5 \times 18 = 120$ cents in total, which is not possible. We can make a similar argument for 5, 10, 15, 20 and 25 pennies.)

It is also not possible to have 3 or 4 quarters. If we did have 3 or 4 quarters, then the remaining 45 or 44 coins would give us a total value of at least 44 cents, so the total value would be greater than 100 cents. Therefore, we only need to consider 0, 1 or 2 quarters.

<u>Possibility 1:</u> 2 quarters
If we have 2 quarters, this means we have 46 coins with a value of 50 cents.
The only possibility for these coins is 45 pennies and 1 nickel.

<u>Possibility 2:</u> 1 quarter
If we have 1 quarter, this means we have 47 coins with a value of 75 cents.
The only possibility for these coins is 40 pennies and 7 nickels.

<u>Possibility 3:</u> 0 quarters
If we have 0 quarters, this means we have 48 coins with a value of 100 cents.
If we had 35 pennies, we would have to have 13 nickels.
If we had 40 pennies, we would have to have 4 dimes and 4 nickels.
It is not possible to have 45 pennies.

Therefore, there are 4 possible combinations.

Solution 2
We want to use 48 coins to total 100 cents.
Let us focus on the number of pennies.
Since any combination of nickels, dimes and quarters always is worth a number of cents which is divisible by 5, then the number of pennies in each combination must be divisible by 5, since the total value of each combination is 100 cents, which is divisible by 5.

Could there be 5 pennies? If so, then the remaining 43 coins are worth 95 cents. But each of the remaining coins is worth at least 5 cents, so these 43 coins are worth at least $5 \times 43 = 215$ cents, which is impossible. So there cannot be 5 pennies.
Could there be 10 pennies? If so, then the remaining 38 coins are worth 90 cents. But each of the remaining coins is worth at least 5 cents, so these 38 coins are worth at least $5 \times 38 = 190$ cents, which is impossible. So there cannot be 10 pennies.
We can continue in this way to show that there cannot be 15, 20, 25, or 30 pennies.
Therefore, there could only be 35, 40 or 45 pennies.

If there are 35 pennies, then the remaining 13 coins are worth 65 cents. Since each of the remaining coins is worth at least 5 cents, this is possible only if each of the 13 coins is a nickel. So one combination that works is 35 pennies and 13 nickels.

If there are 40 pennies, then the remaining 8 coins are worth 60 cents.
We now look at the number of quarters in this combination.
If there are 0 quarters, then we must have 8 nickels and dimes totalling 60 cents. If all of the 8 coins were nickels, they would be worth 40 cents, so we need to change 4 nickels to dimes to increase our total by 20 cents to 60 cents. Therefore, 40 pennies, 0 quarters, 4 nickels and 4 dimes works.
If there is 1 quarter, then we must have 7 nickels and dimes totalling 35 cents. Since each remaining coin is worth at least 5 cents, then all of the 7 remaining coins must be nickels. Therefore, 40 pennies, 1 quarter, 7 nickels and 0 dimes works.
If there are 2 quarters, then we must have 6 nickels and dimes totalling 10 cents. This is impossible. If there were more than 2 quarters, the quarters would be worth more than 60 cents, so this is not possible.

If there are 45 pennies, then the remaining 3 coins are worth 55 cents in total.
In order for this to be possible, there must be 2 quarters (otherwise the maximum value of the 3 coins would be with 1 quarter and 2 dimes, or 45 cents).
This means that the remaining coin is worth 5 cents, and so is a nickel.
Therefore, 45 pennies, 2 quarters, 1 nickel and 0 dimes is a combination that works.

Therefore, there are 4 combinations that work.

Answer: (B)

Grade 8

1. Using a common deminator of 8, $\frac{1}{4}+\frac{3}{8}=\frac{2}{8}+\frac{3}{8}=\frac{5}{8}$.

   Answer: (B)

2. Calculating, $(-3)(-4)(-1)=(12)(-1)=-12$.

   Answer: (A)

3. Since $V=l\times w\times h$, the required volume is $4\times 2\times 8=64$ cm^3.

   Answer: (C)

4. The mean of these five numbers is $\dfrac{6+8+9+11+16}{5}=\dfrac{50}{5}=10$.

   Answer: (C)

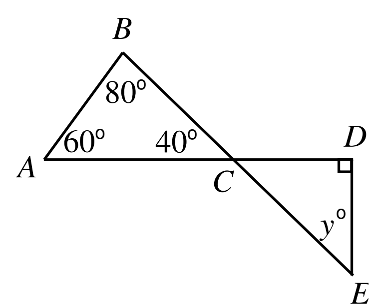
5. 10% of 10 is $0.1\times 10=1$ or $\frac{1}{10}\times 10=1$.
   20% of 20 is $0.2\times 20=4$ or $\frac{1}{5}\times 20=4$.
   Therefore, 10% of 10 times 20% of 20 equals $1\times 4=4$.

   Answer: (E)

6. $8210=8.21\times 1000$ so we must have $10^{\square}=1000$ so the required number is 3.

   Answer: (C)

7. Since $\angle ABC+\angle BAC+\angle BCA=180^\circ$ and $\angle ABC=80^\circ$ and $\angle BAC=60^\circ$, then $\angle BCA=40^\circ$.



   Since $\angle DCE=\angle BCA=40^\circ$, and looking at triangle CDE, we see that $\angle DCE+\angle CED=90^\circ$ then $40^\circ+y^\circ=90^\circ$ or $y=50$.

   Answer: (D)

8. *Solution 1*
   There are 10 numbers (30 to 39) which have a tens digit of 3.
   There are 6 numbers (3, 13, 23, 33, 43, 53) which have a units digit of 3.
   In these two lists, there is one number counted twice, namely 33.
   Therefore, the total number of different numbers in these two lists is $10+6-1=15$.

   Solution 2
   We list out the numbers in increasing order: 3, 13, 23, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 43, 53. There are 15 of these numbers.

   Answer: (D)

9. Since the average monthly rainfall was 41.5 mm in 2003, then the average monthly rainfall in 2004 was $41 + 2 = 43.5$ mm.
Therefore, the total rainfall in 2004 was $12 \times 43.5 = 522$ mm.

Answer: (B)

10. Since Daniel rides at a constant speed, then, in 30 minutes, he rides $\frac{3}{4}$ of the distance that he does in 40 minutes.
Therefore, in 30 minutes, he rides $\frac{3}{4} \times 24 = 18$ km.

Answer: (D)

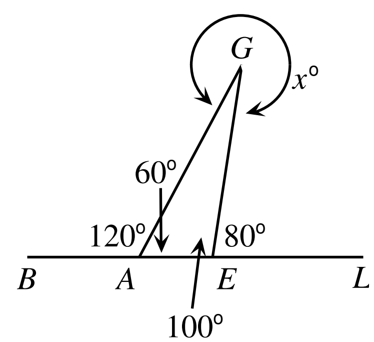
11. Triangle ABC has base AB of length 25 cm and height AC of length 20 cm.
Therefore, the area of triangle ABC is $\frac{1}{2}bh = \frac{1}{2}(25 \text{ cm})(20 \text{ cm}) = \frac{1}{2}(500 \text{ cm}^2) = 250 \text{ cm}^2$.

Answer: (E)

12. To make the sum of the five consecutive even numbers including 10 and 12 as large as possible, we should make 10 and 12 the smallest of these five numbers.
Therefore, to make the sum as large as possible, the numbers should be 10, 12, 14, 16, and 18, which have a sum of 70.

Answer: (E)

13. Since the sum of the angles at any point on a line is 180°, then $\angle GAE = 180^\circ - 120^\circ = 60^\circ$ and $\angle GEA = 180^\circ - 80^\circ = 100^\circ$.



Since the sum of the angles in a triangle is 180°,
$\angle AGE = 180^\circ - \angle GAE - \angle GEA = 180^\circ - 60^\circ - 100^\circ = 20^\circ$.
Since $\angle AGE = 20^\circ$, then the reflex angle at G is $360^\circ - 20^\circ = 340^\circ$, so $x = 340$.

Answer: (A)

14. *Solution 1*
Since the numerators of the five fractions are the same, the largest fraction will be the one with the *smallest* denominator.
To get the smallest denominator, we must subtract (not add) the largest quantity from 2.
Therefore, the largest fraction is $\dfrac{4}{2 - \frac{1}{2}}$.

Solution 2
We evaluate each of the choices:

$$\begin{aligned}
\frac{4}{2-\frac{1}{4}} &= \frac{4}{\frac{7}{4}} = \frac{16}{7} \approx 2.29 \\
\frac{4}{2+\frac{1}{4}} &= \frac{4}{\frac{9}{4}} = \frac{16}{9} \approx 1.78 \\
\frac{4}{2-\frac{1}{3}} &= \frac{4}{\frac{5}{3}} = \frac{12}{5} = 2.4 \\
\frac{4}{2+\frac{1}{3}} &= \frac{4}{\frac{7}{3}} = \frac{12}{7} \approx 1.71 \\
\frac{4}{2-\frac{1}{2}} &= \frac{4}{\frac{3}{2}} = \frac{8}{3} \approx 2.67
\end{aligned}$$

The largest of the five fractions is $\frac{4}{2-\frac{1}{2}}$.

Answer: (E)

15. We first try $x = 1$ in each of the possibilities and we see that it checks for all five possibilities.
Next, we try $x = 2$.
When $x = 2$, $y = x + 0.5 = 2 + 0.5 = 2.5$.
When $x = 2$, $y = 1.5x = 1.5(2) = 3$.
When $x = 2$, $y = 0.5x + 1 = 0.5(2) + 1 = 2$.
When $x = 2$, $y = 2x - 0.5 = 2(2) - 0.5 = 3.5$.
When $x = 2$, $y = x^2 + 0.5 = 2^2 + 0.5 = 4.5$.
So only the second choice agrees with the table when $x = 2$.
When $x = 3$, the second choice gives $y = 1.5x = 1.5(3) = 4.5$ and when $x = 4$,
$y = 1.5x = 1.5(4) = 6$.
Therefore, the second choice is the only one which agrees with the table.

Answer: (B)

16. If the student were to buy 40 individual tickets, this would cost $40 \times \$1.50 = \$60.00$.
If the student were to buy the tickets in packages of 5, she would need to buy $40 \div 5 = 8$ packages, and so this would cost $8 \times \$5.75 = \$46.00$.
Therefore, she would save $\$60.00 - \$46.00 = \$14.00$.

Answer: (C)

17. *Solution 1*
In this solution, we try specific values. This does not guarantee correctness, but it will tell us which answers are wrong.
We try setting $a = 2$ (which is even) and $b = 1$ (which is odd).
Then $ab = 2 \times 1 = 2$, $a + 2b = 2 + 2(1) = 4$, $2a - 2b = 2(2) - 2(1) = 2$, $a + b + 1 = 2 + 1 + 1 = 4$, and $a - b = 2 - 1 = 1$.
Therefore, $a - b$ is the only choice which gives an odd answer.

Solution 2
Since a is even, then ab is even, since an even integer times any integer is even.
Since a is even and $2b$ is even (since 2 times any integer is even), then their sum $a + 2b$ is even.
Since 2 times any integer is even, then both $2a$ and $2b$ are both even, so their difference $2a - 2b$

is even (since even minus even is even).
Since a is even and b is odd, then $a+b$ is odd, so $a+b+1$ is even.
Since a is even and b is odd, then $a-b$ is odd.
Therefore, $a-b$ is the only choice which gives an odd answer.

ANSWER: (E)

18. *Solution 1*
For 100 to divide evenly into N and since $100 = 2^2 \times 5^2$, then N must have at least 2 factors of 2 and at least 2 factors of 5.
From its prime factorization, N already has enough factors of 2, so the box must provide 2 factors of 5.
The only one of the five choices which has 2 factors of 5 is 75, since $75 = 3 \times 5 \times 5$.
Checking, $2^5 \times 3^2 \times 7 \times 75 = 32 \times 9 \times 7 \times 75 = 151\,200$.
Therefore, the only possible correct answer is 75.

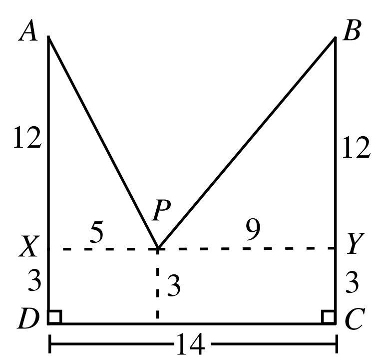
Solution 2
Multiplying out the part of N we know, we get $N = 2^5 \times 3^2 \times 7 \times \square = 32 \times 9 \times 7 \times \square = 2016 \times \square$.
We can then try the five possibilities.
$2016 \times 5 = 10\,080$ which is not divisible by 100.
$2016 \times 20 = 40\,320$ which is not divisible by 100.
$2016 \times 75 = 151\,200$ which is divisible by 100.
$2016 \times 36 = 72\,576$ which is not divisible by 100.
$2016 \times 120 = 241\,920$ which is not divisible by 100.
Therefore, the only possibility which works is 75.

ANSWER: (C)

19. In the diagram, B appears to be about 0.4 and C appears to be about 0.6, so $B \times C$ should be about $0.4 \times 0.6 = 0.24$.
Also, A appears to be about 0.2, so $B \times C$ is best represented by A.

ANSWER: (A)

20. Label the points A, B, C and D, as shown.
Through P, draw a line parallel to DC as shown.
The points X and Y are where this line meets AD and BC.
From this, we see that $AX = YC = 15 - 3 = 12$.
Also, $PY = 14 - 5 = 9$.

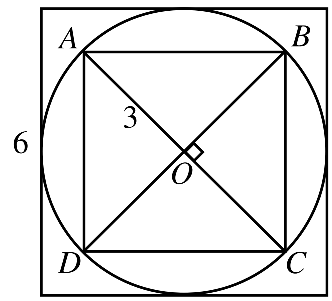


To calculate the length of the rope, we need to calculate AP and BP, each of which is the hypotenuse of a right-angled triangle.

Now, $AP^2 = 12^2 + 5^2 = 169$ so $AP = 13$, and $BP^2 = 12^2 + 9^2 = 225$, so $BP = 15$.
Therefore, the required length of rope is $13 + 15$ or 28 m.

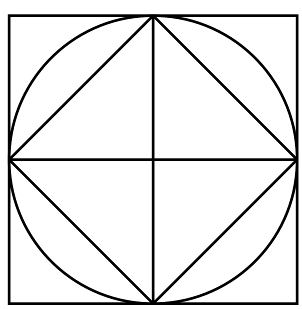
ANSWER: (A)

21. *Solution 1*
Since the area of the large square is 36, then the side length of the large square is 6.
Therefore, the diameter of the circle must be 6 as well, and so its radius is 3.
Label the four vertices of the small square as A, B, C, and D.
Join A to C and B to D.
Since $ABCD$ is a square, then AC and BD are perpendicular, crossing at point O, which by symmetry is the centre of the circle.
Therefore, $AO = BO = CO = DO = 3$, the radius of the circle.



But square $ABCD$ is divided into four identical isosceles right-angled triangles.
The area of each of these triangles is $\frac{1}{2}bh = \frac{1}{2}(3)(3) = \frac{9}{2}$ so the area of the square is $4 \times \frac{9}{2} = 18$.

Solution 2
Rotate the smaller square so that its four corners are at the four points where the circle touches the large square.
Next, join the top and bottom points where the large square and circle touch, and join the left and right points.



By symmetry, these two lines divide the large square into four sections (each of which is square) of equal area.
But the original smaller square occupies exactly one-half of each of these four sections, since each edge of the smaller square is a diagonal of one of these sections.
Therefore, the area of the smaller square is exactly one-half of the area of the larger square, or 18.

ANSWER: (E)

22. *Solution 1*
Since there were 50 students surveyed in total and 8 played neither hockey nor baseball, then 42 students in total played one game or the other.
Since 33 students played hockey and 24 students played baseball, and this totals $33 + 24 = 57$ students, then there must be 15 students who are "double-counted", that is who play both sports.

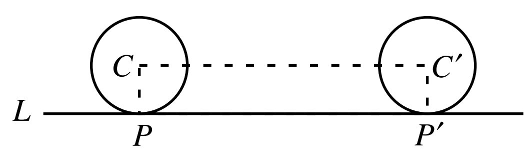
Solution 2
Let x be the number of students who play both hockey and baseball.
Then the number of students who play just hockey is $33 - x$ and the number of students who play just baseball is $24 - x$.
But the total number of students (which is 50) is the sum of the numbers of students who play neither sport, who play just hockey, who play just baseball, and who play both sports.
In other words,

$$\begin{aligned} 8 + (33 - x) + (24 - x) + x &= 50 \\ 65 - 2x + x &= 50 \\ 65 - x &= 50 \\ 65 - 50 &= x \\ x &= 15 \end{aligned}$$

Therefore, the number of students who play both sports is 15.

Answer: (D)

23. We begin by considering a point P which is where the circle first touches a line L.



If a circle makes one complete revolution, the point P moves to P' and the distance PP' is the circumference of the circle, or 2π m.
If we now complete the rectangle, we can see that the distance the centre travels is CC' which is exactly equal to PP' or 2π m.

Answer: (C)

24. Let the three positive integers be x, y and z.
From the given information $(x + y) \times z = 14$ and $(x \times y) + z = 14$.
Let us look at the third number, z, first.
From the first equation $(x + y) \times z = 14$, we see that z must be a factor of 14.
Therefore, the only possible values of z are 1, 2, 7 and 14.

If $z = 1$, then $x + y = 14$ and $xy = 13$. From this, $x = 1$ and $y = 13$ or $x = 13$ and $y = 1$. (We can find this by seeing that $xy = 13$ so the only possible values of x and y are 1 and 13, and then checking the first equation to see if they work.)

If $z = 2$, then $x + y = 7$ and $xy = 12$. From this, $x = 3$ and $y = 4$ or $x = 4$ and $y = 3$.

(We can find these by looking at pairs of positive integers which add to 7 and checking if they multiply to 12.)

If $z = 7$, then $x + y = 2$ and $xy = 7$. There are no possibilities for x and y here. (Since x and y are positive integers and $x + y = 2$ then we must have $x = 1$ and $y = 1$, but this doesn't work in the second equation.)

If $z = 14$, then $x + y = 1$ and $xy = 0$. There are no possibilities for x and y here. (This is because one of them must be 0, which is not a positive integer.)

Therefore, the four possible values for x are 1, 13, 3 and 4.

Answer: (B)

25. Suppose that there are n coins in the purse to begin with.
Since the average value of the coins is 17 cents, then the total value of the coins is $17n$.
When one coin is removed, there are $n - 1$ coins.
Since the new average value of the coins is 18 cents, then the new total value of the coins is $18(n - 1)$.
Since 1 penny was removed, the total value was decreased by 1 cent, or $17n - 1 = 18(n - 1)$ or $17n - 1 = 18n - 18$ or $n = 17$.
Therefore, the original collection of coins has 17 coins worth 289 cents.

Since the value of each type of coin except the penny is divisible by 5, then there must be at least 4 pennies.
Removing these pennies, we have 13 coins worth 285 cents. These coins may include pennies, nickels, dimes and quarters.

How many quarters can there be in this collection of 13 coins?
12 quarters are worth $12 \times 25 = 300$ cents, so the number of quarters is fewer than 12.
Could there be as few as 10 quarters?
If there were 10 quarters, then the value of the quarters is 250 cents, so the remaining 3 coins (which are pennies, nickels and dimes) are worth 35 cents. But these three coins can be worth no more than 30 cents, so this is impossible.
In a similar way, we can see that there cannot be fewer than 10 quarters.

Therefore, there are 11 quarters in the collection, which are worth 275 cents.
Thus, the remaining 2 coins are worth 10 cents, so must both be nickels.
Therefore, the original collection of coins consisted of 11 quarters, 2 nickels and 4 pennies.

Answer: (A)

Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

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Part A

1. Simplifying,

$$\frac{10+20+30+40}{10}=\frac{100}{10}=10$$

Answer: (C)

2. Using a common denominator,

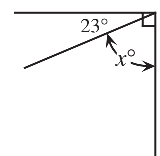
$$\tfrac{1}{2}-\tfrac{1}{8}=\tfrac{4}{8}-\tfrac{1}{8}=\tfrac{3}{8}$$

Answer: (A)

3. Seven thousand twenty-two is $7000+22=7022$.

Answer: (D)

4. From the diagram, $23^\circ + x^\circ = 90^\circ$ so $x^\circ = 67^\circ$ or $x=67$.



Answer: (C)

5. Since Sally was 7 years old five years ago, then she is 12 years old today. Thus, in two more years, she will be 14.

Answer: (B)

6. Since Stuart earns 5 reward points for every \$25 he spends, then when he spends \$200, he earns $\frac{200}{25}\times 5 = 8\times 5 = 40$ points.

Answer: (C)

7. Using a calculator, $\frac{8}{9}=0.888...$, $\frac{7}{8}=0.875$, $\frac{66}{77}=\frac{6}{7}=0.857...$, $\frac{55}{66}=\frac{5}{6}=0.833...$, $\frac{4}{5}=0.8$, so $\frac{8}{9}=0.888...$ is the largest.

Answer: (A)

8. There are 6 balls in the box. 5 of the balls in the box are not grey. Therefore, the probability of selecting a ball that is not grey is $\frac{5}{6}$.

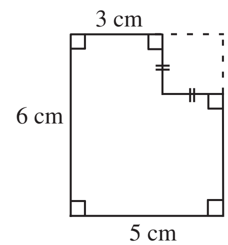
Answer: (E)

9. The sum of the numbers in the second column is $19+15+11=45$, so the sum of the numbers in any row column or diagonal is 45. The sum of the two numbers already in the first row is 33, so the third number in the first row (in the upper right corner) must be 12. Finally, the diagonal from bottom left to top right has x, 15 and 12, so $x+15+12=45$ or $x=18$.

Answer: (E)

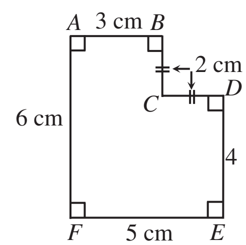
10. *Solution 1*

We notice that if we complete the given figure to form a rectangle, then the perimeter of this rectangle and the original figure are identical. Therefore, the perimeter is $2\times 5+2\times 6=22$ cm.



Solution 2

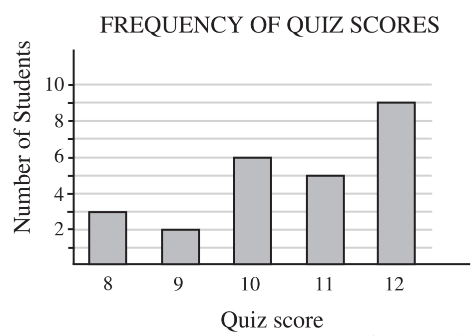
Since the width of the figure is 5 cm, then $AB+CD=5$ cm, so $CD=BC=2$ cm.
Since the height of the figure is 6 cm, then $BC+DE=6$ cm, so $DE=4$ cm.
Therefore, the perimeter is $3+2+2+4+5+6=22$ cm.



Answer: (E)

Part B

11. When we list the quiz scores in ascending order, including repetition, we get 8, 8, 8, 9, 9, 10, 10, 10, 10, 10, 10, 11, 11, 11, 11, 11, 12, 12, 12, 12, 12, 12, 12, 12, 12. Since there are 25 scores, the middle score is the 13th along, so the median is 11.



Answer: (D)

12. In travelling between the two lakes, the total change in elevation is $174.28-75.00=99.28$ m. Since this change occurs over 8 hours, the average change in elevation per hour is $\frac{99.28\text{ m}}{8\text{ h}}=12.41$ m / h.

Answer: (A)

13. We make a chart of the pairs of positive integers which sum to 11 and their corresponding products:

First integer	Second integer	Product
1	10	10
2	9	18
3	8	24
4	7	28
5	6	30

so the greatest possible product is 30.

Answer: (E)

14. Evaluating the exponents, $3^2 = 9$ and $3^3 = 27$, so the even whole numbers between the two given numbers are the even whole numbers from 10 to 26, inclusive. These are 10, 12, 14, 16, 18, 20, 22, 24, and 26, so there are 9 of them.

Answer: (A)

15. If $P = 1000$ and $Q = 0.01$, then

$$P + Q = 1000 + 0.01 = 1000.01$$
$$P \times Q = 1000 \times 0.01 = 10$$
$$\frac{P}{Q} = \frac{1000}{0.01} = 100000$$
$$\frac{Q}{P} = \frac{0.01}{1000} = 0.00001$$
$$P - Q = 1000 - 0.01 = 999.99$$

so the largest is $\frac{P}{Q}$.

Answer: (C)

16. The volume of the box of $40 \times 60 \times 80 = 192000 \text{ cm}^3$. The volume of each of the blocks is $20 \times 30 \times 40 = 24000 \text{ cm}^3$. Therefore, the maximum number of blocks that can fit inside the box is $\frac{192000 \text{ cm}^3}{24000 \text{ cm}^3} = 8$. 8 blocks can indeed be fit inside this box. Can you see how?

Answer: (D)

17 In the recipe, the ratio of volume of flour to volume of shortening is 5 : 1. Since she uses $\frac{2}{3}$ cup of shortening, then to keep the same ratio as called for in the recipe, she must use $5 \times \frac{2}{3} = \frac{10}{3} = 3\frac{1}{3}$ cups of flour.

Answer: (B)

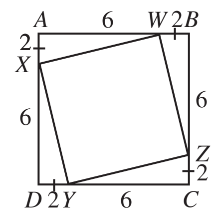
18. The rectangular prism in the diagram is made up of 12 cubes. We are able to see 10 of these 12 cubes. One of the two missing cubes is white and the other is black. Since the four blocks of each colour are attached together to form a piece, then the middle block in the back row in the bottom layer must be white, so the missing black block is the leftmost block of the back row in the bottom layer. Thus, the leftmost block in the back row in the top layer is attached to all three of the other black blocks, so the shape of the black piece is (A). (This is the only one of the 5 possibilities where one block is attached to three other blocks.)

Answer: (A)

19. Since the number is divisible by $8 = 2^3$, by $12 = 2^2 \times 3$, and by $18 = 2 \times 3^2$, then the number must have at least three factors of 2 and two factors of 3, so the number must be divisible by $2^3 \times 3^2 = 72$. Since the number is a two-digit number which is divisible by 72, it must be 72 (it cannot have more than two digits), so it is between 60 and 79.

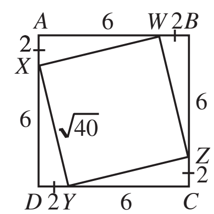
Answer: (D)

20. *Solution 1*



Since the area of square *ABCD* is 64, then the side length of square *ABCD* is 8. Since $AX = BW = CZ = DY = 2$, then $AW = BZ = CY = DX = 6$. Thus, each of triangles *XAW*, *WBZ*, *ZCY* and *YDX* is right-angled with one leg of length 2 and the other of length 6. Therefore, each of these four triangles has area $\frac{1}{2}(2)(6) = 6$. Therefore, the area of square *WXYZ* is equal to the area of square *ABCD* minus the sum of the areas of the four triangles, or $64 - 4(6) = 40$.

Solution 2



Since the area of square *ABCD* is 64, then the side length of square *ABCD* is 8. Since $AX = BW = CZ = DY = 2$, then $AW = BZ = CY = DX = 6$.

By the Pythagorean Theorem,

$XW = WZ = ZY = YX = \sqrt{2^2 + 6^2} = \sqrt{4+36} = \sqrt{40}$.

Therefore, the area of square *WXYZ* is $\left(\sqrt{40}\right)^2 = 40$.

Answer: (D)

Part C

21. In the diagram, we will refer to the horizontal dimension as the width of the room and the vertical dimension as the length of the room. Since the living room is square and has an area of 16 m^2, then it has a length of 4 m and a width of 4 m. Since the laundry room is square and has an area of 4 m^2, then it has a length of 2 m and a width of 2 m. Since the dining room has a length of 4 m (the same as the length of the living room) and an area of 24 m^2, then it has a width of 6 m. Thus, the entire ground floor has a width of 10 m, and so the kitchen has a width of 8 m (since the width of the laundry room is 2 m) and a length of 2 m(since the length of the laundry room is 2 m), and so the kitchen has an area of 16 m^2.

Answer: (B)

22. Let the volume of a large glass be *L* and of a small glass be *S*. Since the jug can exactly fill either 9 small glasses and 4 large glasses, or 6 small glasses and 6 large glasses, then $9S + 4L = 6S + 6L$ or $3S = 2L$. In other words, the volume of 3 small glasses equals the volume of 2 large glasses. (We can also see this without using algebra – if we compare the two cases, we can see that if we remove 3 small glasses then we increase the volume by 2 large glasses.) Therefore, the volume of 9 small glasses equals the volume of 6 large glasses. Thus, the volume of 9 small glasses and 4 large glasses equals the volume of 6 large glasses and 4 large glasses, or 10 large glasses in total, and so the jug can fill 10 large glasses in total.

Answer: (C)

23. In her 40 minutes (or $\frac{2}{3}$ of an hour) on city roads driving at an average speed of 45 km/h, Sharon drives $\left(\frac{2}{3}\text{ h}\right)\times(45\text{ km / h}) = 30\text{ km}$. So the distance that she drives on the highway must be $59\text{ km} - 30\text{ km} = 29\text{ km}$. Since she drives this distance in 20 minutes (or $\frac{1}{3}$ of an hour), then her average speed on the highway is $\dfrac{29\text{ km}}{\frac{1}{3}\text{ h}} = (29\times 3)\text{ km / h} = 87\text{ km / h}$.

Answer: (C)

24. We consider each possible number of silver medals starting with 8.
Could she have won 8 silver medals? This would account for 24 points in 8 events, but since she won 27 points in 8 events, this is not possible.
Could she have won 7 silver medals? This would account for 21 points in 7 events, and so in the remaining 1 event, she would have won 6 points, which is impossible, since she could not score more than 5 points (a gold medal) on this event.
Could she have won 6 silver medals? This would account for 18 points in 6 events, and so in the remaining 2 events, she would have won 9 points, which is impossible, since we cannot combine either two 5s, two 1s or a 1 and a 5 to get 9.
Could she have won 5 silver medals? This would account for 15 points in 5 events, and so in the remaining 3 events, she would have won 12 points, which is impossible. (Try combining up to three 5s and 1s to get 12. We need at least two 5s and two 1s to make 12.)
Could she have won 4 silver medals? This would account for 12 points in 4 events, and so in the remaining 4 events, she would have won 15 points. This is possible – she could win gold on 3 of the 4 remaining events (for 15 points in total) and no medal on the last event (there are 6 competitors and only 3 medals for each event, so there are competitors who do not win medals).
Thus, the maximum number of silver medals she could have won is 4.

Answer: (D)

Answer: (D)

25. *Solution 1*
Start with a grid with two columns and ten rows. There are 10 ways to place the domino horizontally (one in each row) and 18 ways to place the column vertically (nine in each column), so 28 ways overall.
How many more positions are added when a new column is added? When a new column is added, there are 9 new vertical positions (since the column has ten squares) and 10 new horizontal positions (one per row overlapping the new column and the previous column). So there are 19 new positions added.
How many times do we have to add 19 to 28 to get to 2004? In other words, how many times does 19 divide into $2004 - 28 = 1976$? Well, $1976 \div 19 = 104$, so we have to add 104 new columns to the original 2 columns, for 106 columns in total.

Solution 2
Let the number of columns be n.
In each column, there are 9 positions for the domino (overlapping squares 1 and 2, 2 and 3, 3 and 4, and so on, down to 9 and 10).
In each row, there are $n-1$ positions for the domino (overlapping squares 1 and 2, 2 and 3, 3 and 4, and so on, along to $n-1$ and n).
Therefore, the total number of positions for the domino equals the number of rows times the number of positions per row plus the number of columns times the number of positions per column, or $n(9)+10(n-1)=19n-10$. We want this to equal 2004, so $19n-10=2004$ or $19n=2014$ or $n=106$. Thus, there are 106 columns.

Answer: (B)

Part A

1. 25% of 2004 is $\frac{1}{4}$ of 2004, or 501.

Answer: (B)

2. Using a common denominator,
$$\tfrac{1}{2}+\tfrac{3}{4}-\tfrac{5}{8}=\tfrac{4}{8}+\tfrac{6}{8}-\tfrac{5}{8}=\tfrac{5}{8}$$

Answer: (C)

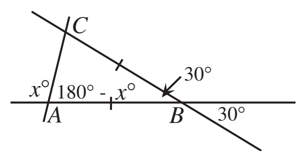
3. Rewriting the given integer,
$$800\,670=800\,000+600+70=8\times10^5+6\times10^2+7\times10^1$$
so $x=5$, $y=2$ and $z=1$, which gives $x+y+z=8$.

Answer: (B)

4. Rewriting the right side with a common denominator,
$$\frac{7863}{13}=\frac{604\times13+\square}{13}=\frac{7852+\square}{13}$$
Therefore, $\square=11$.

Answer: (A)

5. In the diagram, $\angle ABC=\angle XBY=30^\circ$ since they are opposite angles.
Also, $\angle BAC=180^\circ-x^\circ$ by supplementary angles, and so $\angle BCA=180^\circ-x^\circ$ because triangle ABC is isosceles. Looking at the sum of the angles in triangle ABC, we have
$$180^\circ-x^\circ+180^\circ-x^\circ+30^\circ=180^\circ$$
$$210^\circ=2x^\circ$$
$$x=105$$



Answer: (D)

6. Since the perimeter of each of the small equilateral triangles is 6 cm, then the side length of each of these triangles is 2 cm. Since there are three of the small triangles along each side of triangle of ABC, then the side length of triangle ABC is 6 cm, and so its perimeter is 18 cm.

Answer: (A)

7. If $x=-4$ and $y=4$, then
$$\frac{x}{y}=\frac{-4}{4}=-1$$
$$y-1=4-1=3$$
$$x-1=-4-1=-5$$
$$-xy=-(-4)(4)=16$$
$$x+y=-4+4=0$$
Thus, $-xy$ is the largest.

Answer: (D)

8. When two coins are tossed, there are four equally likely outcomes: HEADS and HEADS, HEADS and TAILS, TAILS and HEADS, and TAILS and TAILS. One of these four outcomes has both coins landing as HEADS. Thus, the probability is $\frac{1}{4}$.

Answer: (E)

9. The water surface has an elevation of +180 m, and the lowest point of the lake floor has an elevation of –220 m. Therefore, the actual depth of the lake at this point is $180-(-220)=400$ m.

Answer: (D)

10. We make a chart of the pairs of positive integers which sum to 11 and their corresponding products:

First integer	Second integer	Product
1	10	10
2	9	18
3	8	24
4	7	28
5	6	30

so the greatest possible product is 30.

Answer: (E)

Part B

11. To walk 1.5 km, Ruth takes $\dfrac{1.5\text{ km}}{5\text{ km / h}}=0.3\text{ h}=18\text{ min}$.

Answer: (C)

12. Computing each of the first and fourth of the numbers, we have the four numbers $\sqrt{36}=6$, 35.2, 35.19 and $5^2=25$. Arranging these in increasing order gives 6, 25, 35.19, 35.2, or $\sqrt{36}=6$, $5^2=25$, 35.19, 35.2.

Answer: (D)

13. We number the trees from 1 to 13, with tree number 1 being closest to Trina's house and tree number 13 being closest to her school. On the way to school, she puts a chalk mark on trees 1, 3, 5, 7, 9, 11, 13. On her way home, she puts a chalk mark on trees 13, 10, 7, 4, 1. This leaves trees 2, 6, 8, 12 without chalk marks.

Answer: (B)

14. The rectangular prism in the diagram is made up of 12 cubes. We are able to see 10 of these 12 cubes. One of the two missing cubes is white and the other is black. Since the four blocks of each colour are attached together to form a piece, then the middle block in the back row in the bottom layer must be white, so the missing black block is the leftmost block of the back row in the bottom layer. Thus, the leftmost block in the back row in the top layer is attached to all three of the other black blocks, so the shape of the black piece is (A). (This is the only one of the 5 possibilities where one block is attached to three other blocks.)

Answer: (A)

15. This solid can be pictured as a rectangular prism with dimensions 4 by 5 by 6 with a rectangular prism with dimensions 1 by 2 by 4 removed. Therefore, the volume is $4 \times 5 \times 6 - 1 \times 2 \times 4 = 120 - 8 = 112$.

Answer: (B)

16. Since the number is divisible by $8 = 2^3$, by $12 = 2^2 \times 3$, and by $18 = 2 \times 3^2$, then the number must have at least three factors of 2 and two factors of 3, so the number must be divisible by $2^3 \times 3^2 = 72$. Since the number is a two-digit number which is divisible by 72, it must be 72, so it is between 60 and 79.

Answer: (D)

17. Since $2^3 = 8$ and $2^a = 8$, then $a = 3$. Since $a = 3$ and $a = 3c$, then $c = 1$.

Answer: (C)

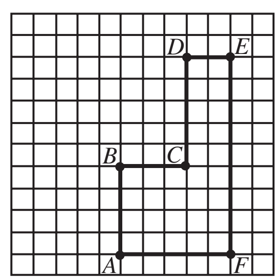
18. Since the range is unchanged after a score is removed, then the score that we remove cannot be the smallest or the largest (since each of these occurs only once). Thus, neither the 6 nor the 10 is removed.
Since the mode is unchanged after a score is removed, then the score that we remove cannot be the most frequently occurring. Thus, the score removed is not an 8.
Therefore, either a 7 or a 9 is removed.
Since we wish to increase the average, we remove the smaller of the two numbers, ie. the 7. (We could calculate that before any number is removed, the average is 7.875, if the 7 is removed, the average is 8, and if the 9 is removed, the average is 7.714.)

Answer: (B)

19. Since the numerical values of CAT and CAR are 8 and 12, then the value of R must be 4 more than the value of T.
Therefore, the value of BAR is 4 more than the value of BAT, so BAR has a numerical value of 10.

Answer: (A)

20. To get from A to E, we go right 5 and up 9, so $AE = \sqrt{5^2 + 9^2} = \sqrt{106} \approx 10.30$, by the Pythagorean Theorem.
To get from C to F, we go right 2 and down 4, so $CF = \sqrt{2^2 + 4^2} = \sqrt{20} \approx 4.47$, and so $CD + CF \approx 5 + 4.47 = 9.47$.
To get from A to C, we go right 3 and up 4, so $AC = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$, and so $AC + CF \approx 5 + 4.47 = 9.47$.
To get from F to D, we go left 2 and up 9, so $FD = \sqrt{2^2 + 9^2} = \sqrt{85} \approx 9.22$.
To get from C to E, we go right 2 and up 5, so $CE = \sqrt{2^2 + 5^2} = \sqrt{29} \approx 5.39$, and so $AC + CE \approx 5 + 5.39 = 10.39$.
Therefore, the longest of these five lengths is $AC + CE$.



Answer: (E)

Part C

21. The scale of the map is equal to the ratio of a distance on the map to the actual distance. Since the distance between Saint John and St. John's is 21 cm on the map and 1050 km in reality, then the scale of the map is equal to

$$21 \text{ cm} : 1050 \text{ km} = 0.21 \text{ m} : 1\,050\,000 \text{ m} = 21 : 105\,000\,000 = 1 : 5\,000\,000$$

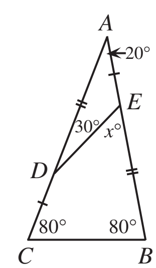
Answer: (E)

22. *Solution 1*
When the pouring stops, $\frac{1}{4}$ of the water in the bottle has been transferred to the glass. This represents $\frac{3}{4}$ of the volume of the glass. Therefore, the volume of the bottle is three times the volume of the glass, so the volume of the glass is 0.5 L.

Solution 2
When the pouring stops, $\frac{1}{4}$ of the water in the bottle or $\frac{1}{4} \times 1.5 = 0.375$ L of water is in the glass. Since this represents $\frac{3}{4}$ of the volume of the glass, then the volume of the glass is $\frac{4}{3} \times 0.375 = 0.5$ L.

Answer: (A)

23. From the diagram, $BE = AD$ and $AE = CD$, so
$AC = AD + CD = BE + AE = AB$ so triangle ABC is isosceles.
Therefore, $\angle ACB = \angle ABC = 80^\circ$ and so
$\angle BAC = 180^\circ - \angle ABC - \angle ACB = 180^\circ - 80^\circ - 80^\circ = 20^\circ$.
Considering triangle AED next,
$\angle AED = 180^\circ - \angle ADE - \angle EAD = 180^\circ - 30^\circ - 20^\circ = 130^\circ$.
But $x^\circ = 180^\circ - \angle AED = 180^\circ - 130^\circ = 50^\circ$ so $x = 50$.



Answer: (C)

24. Since x has digits ABC, then $x = 100A + 10B + C$.
Since y has digits CBA, then $y = 100C + 10B + A$.
Since $x - y = 495$, then

$$\begin{aligned}(100A + 10B + C) - (100C + 10B + A) &= 495\\ 99A - 99C &= 495\\ 99(A - C) &= 495\\ A - C &= 5\end{aligned}$$

and there is no restriction on B.

Thus, there are 10 possibilities for B (0 through 9) and for each of these possibilities we could have A and C equal to 6 and 1, 7 and 2, 8 and 3, or 9 and 4. (For example, $873 - 378 = 495$.)
Therefore, there are 40 possibilities for x.

Answer: (B)

25. Consider the block as n layers each having 11 rows and 10 columns.
First, we consider positions of the 2 by 1 by 1 block which are entirely contained in one layer. In each layer, there are 9 possible positions for the 2 by 1 by 1 block in each row (crossing columns 1 and 2, 2 and 3, 3 and 4, and so on, up to 9 and 10), and there are 10 possible positions in each column (crossing rows 1 and 2, 2 and 3, 3 and 4, and so on, up to 10 and 11). Therefore, within each layer, there are $11(9)+10(10)=199$ positions for the 2 by 1 by 1 block. In the large block, there are thus $199n$ positions of this type for the 2 by 1 by 1 block, since there are n layers.

Next, we consider positions of the 2 by 1 by 1 block which cross between two layers. Since each layer has 110 blocks (in 11 rows and 10 columns) then there are 110 positions for the 2 by 1 by 1 block between each pair of touching layers. Since there are $n-1$ pairs of touching layers, then there are $110(n-1)$ positions of this type.

Thus, overall we have 2362 total positions, so

$$\begin{aligned} 199n+110(n-1) &= 2362 \\ 309n-110 &= 2362 \\ 309n &= 2472 \\ n &= 8 \end{aligned}$$

Answer: (B)

Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2003 Solutions

Gauss Contest

(Grades 7 and 8)

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Part A

1. Multiplying gives $3.26 \times 1.5 = 4.89$. ANSWER: (B)

2. Calculating in the parentheses first,

   $$(9-2)-(4-1) = 7-3 = 4.$$

   ANSWER: (C)

3. Adding gives,

   $$30 + 80\,000 + 700 + 60 = 30 + 80\,760 = 80\,790$$

   ANSWER: (D)

4. $\dfrac{1+2+3}{4+5+6} = \dfrac{6}{15} = \dfrac{2}{5}$ ANSWER: (C)

5. In the survey, a total of 90 people were surveyed.
   According to the graph, 25 people chose a cat, 10 people chose a fish, 15 people chose a bird, and 5 people chose "other", accounting for $25 + 10 + 15 + 5 = 55$ people.
   This leaves $90 - 55 = 35$ people who have chosen a dog. ANSWER: (E)

6. If Travis uses 4 mL of gel every day and a tube of gel contains 128 mL of gel, then it will take him $\dfrac{128}{4} = 32$ days to empty the tube. ANSWER: (A)

7. *Solution 1*
   On the left hand side of the equation, if the 3's are cancelled, we would have

   $$\frac{3 \times 6 \times 9}{3} = 6 \times 9 = \frac{\square}{2}$$

   Therefore, the expression in the box should be $2 \times 6 \times 9$, since we can write $6 \times 9 = \dfrac{2 \times 6 \times 9}{2}$.

   Solution 2
   Evaluating the left side of the expression, we obtain $\dfrac{3 \times 6 \times 9}{3} = 54$.
   Therefore, we must place an expression equal to $54 \times 2 = 108$ in the box to make the equation true.
   Evaluating the five choices, we obtain
   (A) 48 (B) 72 (C) 108 (D) 64 (E) 432
   Therefore, the answer must be $2 \times 6 \times 9$. ANSWER: (C)

8. If we turn the words

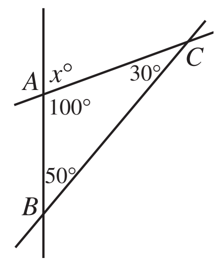
   PUNK CD FOR SALE

   around and look at them as if looking at them through the opposite side of a window, the only letters that appear the same will be U, O, and A. The other letters will all appear differently from the other side of the window. ANSWER: (A)

9. Spencer starts 1000 m from home and walks to a point 800 m from home, a distance of 200 m. He then walks to a point 1000 m from home, for a distance of another 200 m. Finally, he walks home, a distance of 1000 m. So Spencer has walked a total of $200+200+1000=1400$ m.

ANSWER: (E)

10. Since the sum of the angles in a triangle is 180°, then $\angle BAC = 180^\circ - 30^\circ - 50^\circ = 100^\circ$.
Since a straight line makes an angle of 180°, then $x^\circ + 100^\circ = 180^\circ$ or $x = 80$.



ANSWER: (A)

Part B

11. Since there are 12 squares initially, then the number of squares to be removed is
$\frac{1}{2}\times\frac{2}{3}\times 12 = \frac{1}{3}\times 12 = 4$
Therefore, there will be 8 squares remaining. ANSWER: (D)

12. *Solution 1*
Since the perimeter of the field is 3 times the length and the perimeter is 240 m, then the length of the field is 80 m.
Since the perimeter of a rectangle is two times the length plus two times the width, then the length accounts for 160 m of the perimeter, leaving 80 m for two times the width.
Therefore, the width of the field is 40 m.

Solution 2
Let the perimeter of the field be P, the length be l, and the width w.
We are given that $P = 3l$ and $P = 240$, so $l = \frac{1}{3}(240) = 80$.
Since $P = 240$ and $P = 2l + 2w$, we have $240 = 2(80) + 2w$ or $w = 40$. ANSWER: (B)

13. Since Chris runs $\frac{1}{2}$ as fast as his usual running speed, he runs at 5 km/h, and so will take 6 hours to complete the 30 km run.
Since Pat runs at $1\frac{1}{2}$ her usual running speed, she runs at 15 km/h, and so will take 2 hours to complete the 30 km run.
Thus, it takes Chris 4 hours longer to complete the run than it takes Pat. ANSWER: (D)

14. *Solution 1*

Since there are twice as many red disks as green disks and twice as many green disks as blue disks, then there are four times as many red disks as blue disks.
So the total number of disks is seven times the number of blue disks (since the numbers of red and green disks are four and two times the number of blue disks).
Since there are 14 disks in total, there are 2 blue disks, and so there are 4 green disks.

Solution 2

Let the number of green disks be g.
Then the number of red disks is $2g$, and the number of blue disks is $\frac{1}{2}g$.
From the information given,

$$\begin{aligned} 2g+g+\tfrac{1}{2}g &= 14 \\ \tfrac{4}{2}g+\tfrac{2}{2}g+\tfrac{1}{2}g &= 14 \\ \tfrac{7}{2}g &= 14 \\ g &= \tfrac{2}{7}\times 14 \\ g &= 4 \end{aligned}$$

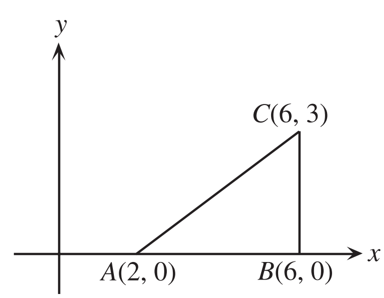
Therefore, the number of green disks is 4. Answer: (B)

15. In the bottle, there are a total of 180 tablets.
Among the 60 stars, there are an equal number of each of the three flavours – strawberry, grape and orange. This tells us that there are 20 grape stars.
If each tablet is equally likely to be chosen from the bottle, the probability of choosing a grape star is

$$\frac{\text{Number of grape stars}}{\text{Total number of tablets}} = \frac{20}{180} = \frac{1}{9}$$

Answer: (A)

16. First, we sketch the triangle.
Since point C is directly above point B (that is, angle ABC is a right angle), then we can look at triangle ABC as having base AB (of length 4) and height BC (of length 3).
Thus, the area of the triangle is $A=\frac{1}{2}bh=\frac{1}{2}(4)(3)=6$ square units.



Answer: (C)

17. If Genna's total bill was \$74.16 and a \$45 fee was charged, this means that her cost based on the distance she drove was $\$74.16-\$45.00=\$29.16$. Thus the number of kilometres driven is $\frac{29.16}{0.12}=243$.

Answer: (C)

18. *Solution 1*

We can calculate the perimeter of the shaded figure by adding the perimeters of the two large squares and subtracting the perimeter of the small square. This is because all of the edges of the two larger squares are included in the perimeter of the shaded figure except for the sides of the smaller square. Since the side length of the larger squares is 5 cm, the perimeter of each of the two larger squares is 20 cm.

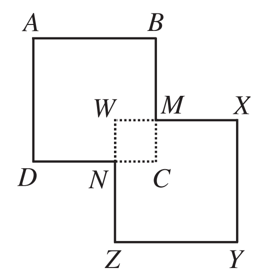
Since the area of the smaller square is 4 cm^2, then its side length is 2 cm, and so its perimeter is 8 cm. Therefore, the perimeter of the shaded figure is $20+20-8=32$ cm.

Solution 2

We label the vertices in the diagram and calculate directly.

We know that the two large squares each have a side length of 5 cm, so $AB=AD=YX=YZ=5$.

The smaller square has an area of 4 cm^2, and so has a side length of 2 cm. Therefore, $WM=MC=CN=NW=2$, and so each of BM, DN, MX, and NZ have a length of 3 cm (the difference between the side length of the two squares).



Thus, the total perimeter of the shaded figure is $4\times5+4\times3=32$ cm.

ANSWER: (B)

19. Abraham's exam had a total of 80 questions. Since he received a mark of 80%, he got $\frac{80}{100}\times80=\frac{8}{10}\times80=64$ questions correct.

We also know that Abraham answered 70% of the 30 algebra questions correctly, or a total of 21 questions.

This tells us that he answered 43 of the geometry questions correctly. ANSWER: (A)

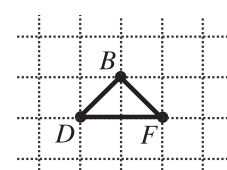
20. We first make a list of all of the possible triangles with an edge lying on *DEF* (that is, with an edge that is one of *DE*, *DF* or *EF*):

DAE, *DBE*, *DCE*, *DAF*, *DBF*, *DCF*, *EAF*, *EBF*, *ECF*

Of these, triangles *DAE*, *DBE*, *DAF*, *DCF*, *EBF*, *ECF* are obviously right-angled, while triangles *DCE* and *EAF* are not right-angled because they each contain an angle of 135°. What about triangle *DBF*?

In fact, triangle *DBF* is right-angled since $\angle DBE=\angle EBF=45^\circ$, and so $\angle DBF=90^\circ$.

Therefore, there are 2 triangles among this group which are not right-angled.



If we now look at the possible triangles with an edge lying on *ABC*, we will again find 2 non right-angled triangles among these triangles. (We can see this by reflecting all of the earlier triangles to move their base from *DEF* to *ABC*.)

We can also see that it is impossible to make a triangle without an edge lying on *ABC* or *DEF*, since the only other possible edges are *AD*, *AE*, *AF*, *BD*, *BE*, *BF*, *CD*, *CE*, and *CF*. No three of these nine edges can be joined to form a triangle since each edge has one end at *A*, *B* or *C* and the other end at *D*, *E* or *F*, and there are no edges joining two points among either set of three points.

Therefore, there are only 4 triangles that can be formed that are not right-angled.

ANSWER: (E)

Part C

21. Since there are ten people scheduled for an operation and each operation begins 15 minutes after the previous one, the tenth operation will begin nine 15 minute intervals after the first operation began at 8:00 a.m.
Nine 15 minute intervals is 135 minutes, or 2 hours 15 minutes.
Thus, the tenth 45 minute operation begins at 10:15 a.m. and so ends at 11:00 a.m.

ANSWER: (D)

22. *Solution 1*

Since Luke has a 95% winning percentage, then he hasn't won 5%, or $\frac{1}{20}$, of his games to date. Since he has played only 20 games, there has only been 1 game that he has not won.

For Luke to have exactly a 96% winning percentage, he must not have won 4%, or $\frac{1}{25}$, of his games. Since he wins every game between these two positions, when he has the 96% winning percentage, he has still not won only 1 game. Therefore, he must have played 25 games in total, or 5 more than initially.

Solution 2

Since Luke has played 20 games and has a 95% winning percentage, then he has won
$\frac{95}{100} \times 20 = \frac{95}{5} = 19$ games.
Let the number of games in a row he wins before reaching the 96% winning percentage be x. Then

$$\text{Winning } \% = \frac{\text{Games Won}}{\text{Games Played}} = \frac{19+x}{20+x} = \frac{96}{100}$$

Cross-multiplying,

$$\begin{aligned} 100(19+x) &= 96(20+x) \\ 1900+100x &= 1920+96x \\ 4x &= 20 \\ x &= 5 \end{aligned}$$

Therefore, we wins 5 more games in a row.

ANSWER: (D)

23. We will determine which letter belongs on the shaded face by "unfolding" the cube.
Using the first of the three positions, we obtain

E	
T	V

Using the third of the three positions, we can add the "A" above the "E" as

∀	
E	
T	V

Using the second of the three positions, the F can be placed above the A and the X to the left of the F to get

X	F	
	∀	
	E	
	T	V

If we refold this cube, with A at the front, F on top, and E on the bottom, the right-hand face will be the V (upside down), and so the shaded face is the V. ANSWER: (E)

24. To get a better understanding of the pattern, let us write each of the numbers in the way that it is obtained:

$$\begin{array}{ccccccccc} & & & 1 & & 2 & & & \\ & & 1 & & 3 & & 2 & & \\ & 1 & & 4 & & 5 & & 2 & \\ 1 & & 5 & & 9 & & 7 & & 2 \\ \iddots & & & & \vdots & & & & \ddots \end{array} \qquad \begin{array}{ccccccccc} & & & 1 & & 2 & & & \\ & & 1 & & 1+2 & & 2 & & \\ & 1 & & 1+3 & & 3+2 & & 2 & \\ 1 & & 1+4 & & 4+5 & & 5+2 & & 2 \\ \iddots & & & & \vdots & & & & \ddots \end{array}$$

We can see from these two patterns side by side that each number in a row is accounted for *twice* in the row below. (The 1 or 2 on the end of a row appears again at the end of the row and as part of the sum in one number in the next row. A number in the middle of a row appears as part of the sum in two numbers in the next row.)

Therefore, the sum of the numbers in a row should be two times the sum of the numbers in the previous row.

We can check this:

Sum of the numbers in the 1st row	3
Sum of the numbers in the 2nd row	6
Sum of the numbers in the 3rd row	12
Sum of the numbers in the 4th row	24

Thus, the sum of the numbers in the thirteenth row should be the sum of the elements in the first row multiplied by 2 twelve times, or $3\times 2^{12} = 3\times 4096 = 12\ 288$. ANSWER: (D)

25. We will present a complete consideration of all of the cases. The answer can be obtained more easily in a trial and error fashion.

First, we rewrite the equation putting letters in each of the boxes

$$\begin{array}{ccc} & A & B \\ & \times & C \\ \hline D & E & F \end{array}$$

We want to replace A, B, C, D, E, and F by the digits 1 through 6.

Could C be 1?

If C was 1, then B and F would have to be same digit, which is impossible since all of the digits are different. Therefore, C cannot be 1.

Could C be 5?

If C was 5, then if B were odd, F would also be 5, which would be impossible. If C was 5 and B even, then F would have to be 0, which is also impossible. Therefore, C cannot be 5.

Could C be 6?

If C was 6, let us list the possibilities for B and the resulting value of F:

B	F	
1	6	Impossible – two 6's
2	2	Impossible – two 2's
3	8	Impossible – no 8
4	4	Impossible – two 4's
5	0	Impossible – no 0

Therefore, C cannot be 6.

Could C be 4?

If C was 4, using a similar chart, we can see that B must be 3 and F must be 2. We will consider this possibility later.

Could C be 3?

If C was 4, using a similar chart, we can see that B must be 2 or 4 and F must be thus 6 or 2.

Could C be 2?

If C was 2, using a similar chart, we can see that B must be 3 and F must be 6.

Let us consider the case of $C = 2$. In this case, we have

$$\begin{array}{rrr} & A & 3 \\ & \times & 2 \\ \hline D & E & 6 \end{array}$$

Since the product has three digits, and the possibilities remaining for A are 1, 4 or 5, then A must be 5. However, this gives a product of $53 \times 2 = 106$, which is impossible.

Similarly, trying the case of $C = 4$, we are left with the possibilities for A being 1, 5, or 6, none of which work.

Therefore, C must be 3, since this is the only possibility left. We should probably check that we can actually get the multiplication to work, though!

Trying the possibilities as above, we can eventually see that $54 \times 3 = 162$ works, and so $C = 3$.

ANSWER: (B)

Part A

1. Performing the calculation,

   $1.000+0.101+0.011+0.001=1.113$ Answer: (B)

2. We group the terms and start with addition, and then do subtraction:

   $$\begin{aligned}&1+2+3-4+5+6+7-8+9+10+11-12\\&=6-4+18-8+30-12\\&=2+10+18\\&=30\end{aligned}$$

   Answer: (A)

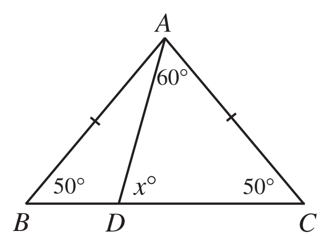
3. The amount that each charity received was the total amount raised divided by the number of charities, or $\$3109 \div 25 = \$124.36$. Answer: (E)

4. The square of the square root of 17 is $\left(\sqrt{17}\right)^2$, which equals 17. The operations of squaring and taking a square root are inverses of each other. Answer: (C)

5. Since triangle ABC is isosceles, $\angle ACB = \angle ABC = 50^\circ$.

   Since the sum of the angles in triangle ACD is 180°, then

   $$\begin{aligned}x^\circ + 60^\circ + 50^\circ &= 180^\circ\\ x &= 70\end{aligned}$$



   Answer: (A)

6. *Solution 1*

   If we "undo" the operations, we would get back to the original number by first subtracting 13 and then dividing by 2, to obtain $\frac{1}{2}(89-13)=38$.

   Solution 2

   Let the number be x. Then $2x+13=89$ or $2x=76$ or $x=38$. Answer: (D)

7. The range of temperature is the difference between the high temperature and the low temperature. We can complete the chart to determine the largest range.

Day	Temperature Range ($^\circ$C)
Monday	$5-(-3)=8$
Tuesday	$0-(-10)=10$
Wednesday	$-2-(-11)=9$
Thursday	$-8-(-13)=5$
Friday	$-7-(-9)=2$

   We see that the temperature range was greatest on Tuesday. Answer: (B)

8. We write each of the five numbers as a decimal in order to be able to arrange them in order from smallest to largest:

$$\sqrt{5} = 2.236...$$
$$2.1 = 2.1$$
$$\tfrac{7}{3} = 2.333...$$
$$2.0\overline{5} = 2.055...$$
$$2\tfrac{1}{5} = 2.2$$

So in order from smallest to largest, we have $2.0\overline{5}$, 2.1, $2\frac{1}{5}$, $\sqrt{5}$, $\frac{7}{3}$. The number in the middle is $2\frac{1}{5}$.

ANSWER: (E)

9. Since one-third of the 30 students in the class are girls, then 10 of the students are girls. This means that 20 of the students are boys. Three-quarters of the 20 boys play basketball, which means that 15 of the boys play basketball. ANSWER: (E)

10. We rewrite the addition in columns, and write each number to three decimals:

$$\begin{array}{r} 15.200 \\ 1.520 \\ 0.15\square \\ +\ \square.128 \\ \hline 20.000 \end{array}$$

Since the sum of the digits in the last column ends in a 0, then the box in the thousandths column must represent a 2.
We can insert the 2 to get

$$\begin{array}{r} 15.200 \\ 1.520 \\ 0.152 \\ +\ \square.128 \\ \hline 20.000 \end{array}$$

If we then perform the addition of the last three columns, we will get a carry of 1 into the units column. Since the sum of the units column plus the carry ends in a 0, then the box in the units column must represent a 3. Therefore, the sum of the digits inserted into the two boxes is 5. (We also check that $15.2 + 1.52 + 0.152 + 3.128 = 20$.) ANSWER: (A)

Part B

11. Reading the data from the graph the numbers of female students in the five classes in order are 10, 14, 7, 9, and 13. The average number of female students is $\frac{10+14+7+9+13}{5}=\frac{53}{5}=10.6$.

ANSWER: (E)

12. The area of the original photo is $20\times 25=500\text{ cm}^2$ and the area of the enlarged photo is $25\times 30=750\text{ cm}^2$. The percentage increase in area is

$$\frac{\text{Final Area}-\text{Initial Area}}{\text{Initial Area}}\times 100\%=\frac{750-500}{500}\times 100\%=\frac{250}{500}\times 100\%=50\%$$

ANSWER: (B)

13. Since the angles are in the ratio 2 : 3 : 4, then we can represent the angles as $2x$, $3x$, and $4x$ (in degrees) for some number x. Since these three angles are the angles of a triangle then

$$\begin{aligned}2x+3x+4x&=180^\circ\\ 9x&=180^\circ\\ x&=20^\circ\end{aligned}$$

Thus the largest angle is $4x$, or 80°.

ANSWER: (C)

14. *Solution 1*

Since George recorded his highest mark higher than it was, then if he writes his seven marks in order from lowest to highest, the order will not be affected.

So the minimum test mark is not affected, nor is the median test mark (which is the middle of the seven different marks).

Is either of the range or mean affected?

The range is difference between the highest mark and the lowest mark, so by recording the highest mark higher, the range has become larger.

The mean is the sum of the seven marks divided by 7, so since the highest mark is higher, the sum of the 7 marks will be higher, and so the mean will be higher.

So only the mean and range are affected.

Solution 2

Suppose that George's marks were 80, 81, 82, 83, 84, 85, and 86, but that he wistfully recorded the 86 as 100.

Originally, with marks 80, 81, 82, 83, 84, 85, 86, the statistics are

Mean	$\frac{80+81+82+83+84+85+86}{7}=83$
Median	83
Minimum test score	80
Range	$86-80=6$

With marks 80, 81, 82, 83, 84, 85, 100, the statistics are

Mean	$\frac{80+81+82+83+84+85+100}{7}=85$
Median	83
Minimum test score	80
Range	$100-80=20$

So the mean and range are the only statistics altered. ANSWER: (C)

15. The volume of the entire pit is

$$(10 \text{ m})\times(50 \text{ cm})\times(2 \text{ m})=(10 \text{ m})\times(0.5 \text{ m})\times(2 \text{ m})=10 \text{ m}^3$$

Since the pit starts with 5 m^3 in it, an additional 5 m^3 of sand is required to fill it.

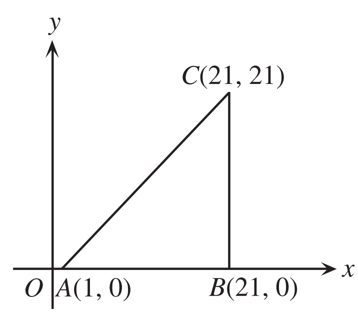
ANSWER: (B)

16. We evaluate this "continued fraction" step by step

$$\frac{1}{1+\frac{1}{1+\frac{1}{2}}}=\frac{1}{1+\frac{1}{\left(\frac{3}{2}\right)}}=\frac{1}{1+\frac{2}{3}}=\frac{1}{\left(\frac{5}{3}\right)}=\frac{3}{5}$$

ANSWER: (A)

17. The perimeter of the triangle is the sum of the lengths of the sides. We make a sketch of the triangle to help us with our calculations. Since side AB is along the x-axis and side BC is parallel to the y-axis, then the triangle is right-angled, and we can use Pythagoras' Theorem to calculate the length of AC.
The length of AB is 20 and the length of BC is 21.
Calculating AC,



$$\begin{aligned}AC^2 &= AB^2+BC^2\\ AC^2 &= 20^2+21^2\\ AC^2 &= 400+441\\ AC^2 &= 841\\ AC &= 29\end{aligned}$$

Therefore, the perimeter is $20+21+29=70$. ANSWER: (A)

18. *Solution 1*
If $-3x^2<-14$, then $3x^2>14$ or $x^2>\frac{14}{3}=4\frac{2}{3}$.
Since we are only looking at x being a whole number, then x^2 is also a whole number. Since x^2 is a whole number and x^2 is greater than $4\frac{2}{3}$, then x^2 must be at least 5.
Of the numbers in the set, the ones which satisfy this condition are $-5,-4,-3$, and 3.

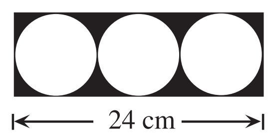
Solution 2

For each number in the set, we substitute the number for x and calculate $-3x^2$:

x	$-3x^2$
-5	-75
-4	-48
-3	-27
-2	-12
-1	-3
0	-3
0	-12

As before, the four numbers $-5, -4, -3$, and 3 satisfy the inequality. ANSWER: (D)

19. Since three circles touch each other and touch the vertical and horizontal sides of the rectangle, then the width of the rectangle is three times the diameter of the circle, and the height of the rectangle is equal to the diameter of the circle.



Since the width of the rectangle is 24 cm, then the diameter of each circle is 8 cm.
Since the diameter of each circle is 8 cm, then the height of the rectangle is 8 cm, and the radius of each circle is 4 cm.
Therefore, the area of the shaded region is

$$\begin{aligned}
&\text{Area of shaded region}\\
&= \text{Area of rectangle} - \text{Area of three circles}\\
&= (24\times 8) - 3\left[\pi(4)^2\right]\\
&= 192 - 48\pi\\
&\approx 192 - 150.80 \qquad (\text{using } \pi \approx 3.14)\\
&= 41.20
\end{aligned}$$

Thus, the area of the shaded region is closest to 41 cm^2. ANSWER: (A)

20. This question is made more difficult by the fact that there are 2 letters that are the same. To overcome this problem, let us change the second S to a T, so that the letters on the tiles are now G, A, U, S, and T, and we want to calculate the probability that Amy chooses the S and the T when she chooses two tiles.
Let us say that Amy chooses the tiles one at a time. How many choices does she have for the first tile she chooses? Since there are 5 tiles, she has 5 choices. How many choices does Amy have for the second tile? There are 4 tiles left, so she has 4 choices. Now for *each* of the 5 ways of choosing the first tile, she has 4 ways of choosing the second tile, for a total of 20 possibilities. (Try writing out the 20 possibilities and convince yourself that it is 5×4 and not $5+4$.)
In these 20 pairs, there are two pairs that contain an *S* and *T*. If the *T* was changed back to an *S*, there would be two pairs out of 20 containing 2 S's so the probability would be $\frac{2}{20}=\frac{1}{10}$ of selecting the 2 S's. ANSWER: (D)

Part C

21. Let's look at a couple of examples of four consecutive whole numbers that add to a multiple of 5, and see what possibilities we can eliminate.
First, we can look at 1, 2, 3, 4 (whose sum is 10).
Using this example, we can eliminate choice (A) (since the sum ends in a 0) and choice (B) (since the largest number ends in a 4).
With a bit more work, we can see that 6, 7, 8, 9 (whose sum is 30) is another example.
Using this second example, we can eliminate choice (C) (since the smallest number is even) and choice (E) (since none of the numbers ends in a 3).
Therefore, the only remaining choice is (D).

Why is (D) always true? This is a bit more tricky to figure out, and requires some algebra.
Let the smallest of the numbers be n. Then the other three numbers are $n+1$, $n+2$, and $n+3$, and their sum is $n+n+1+n+2+n+3=4n+6$.
We know that their sum is a multiple of 5. Since the sum is also $4n+6$, this sum is even and so must end in a 0 if it is also to be a multiple of 5.
Since $4n+6$ ends in a 0, then $4n$ ends in a 4. What can the units digit of n be? The only possibilities for the units digit of n are 1 and 6, and so the four numbers either end with 1, 2, 3, and 4, or 6, 7, 8, and 9, and none of these four numbers is a multiple of 5. ANSWER: (D)

22. *Solution 1*
When Carmina trades her nickels for dimes and her dimes for nickels, she gains \$1.80. Since trading a dime for a nickel results in a loss of 5 cents, and trading a nickel for a dime results in a gain of 5 cents, then by doing her trade she gains 5 cents in $\frac{180}{5}=36$ more cases than she loses 5 cents. Thus, she must have 36 more nickels than dimes.
These extra 36 nickels account for \$1.80. So her initial coins are worth \$1.80 and she has an equal number of nickels and dimes. A nickel and dime together are worth 15 cents, so she must have $\frac{180}{15}=12$ sets of a nickel and dime, or 12 nickels and 12 dimes.
So in total, Carmina has 48 nickels and 12 dimes, or 60 coins.

Solution 2
Suppose that Carmina has n nickels and d dimes.
Then looking at the total number of cents Carmina has, $5n+10d=360$.
If we reverse the nickels and dimes and again look at the total number of cents that Carmina has, we see that $10n+5d=540$.
If we add these two equations together, we get

$$\begin{aligned} 5n+10d+10n+5d &= 360+540 \\ 15n+15d &= 900 \\ n+d &= 60 \end{aligned}$$

So the total number of coins is 60. (Notice that we didn't in fact have to calculate the number of nickels or the number of dimes!)

Solution 3

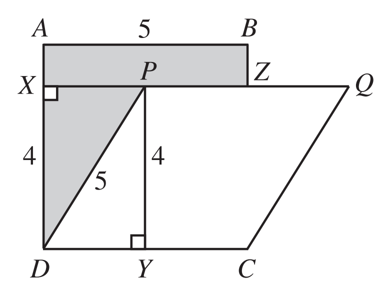
Suppose that Carmina has n nickels and d dimes.
Then looking at the total number of cents Carmina has, $5n + 10d = 360$ or $n + 2d = 72$ or $n = 72 - 2d$.
If we reverse the number of nickels and the number of dimes and look at the number of cents,

$$\begin{aligned} 5d + 10n &= 540 \\ 5d + 10(72 - 2d) &= 540 \\ 5d + 720 - 20d &= 540 \\ 180 &= 15d \\ d &= 12 \end{aligned}$$

Since $d = 12$, then $n = 72 - 2d = 48$.
Therefore, the total number of nickels and dimes that Carmina has is 60. ANSWER: (D)

23. Suppose that Gabriella's twelve plants had 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 tomatoes. How many tomatoes would she have in total? Adding these numbers up, she would have 78 tomatoes.
But we know that she has 186 tomatoes, so there are 108 tomatoes unaccounted for.
Since the number of tomatoes on her plants are twelve consecutive whole numbers, then her plants must *each* have the same number of extra tomatoes more than our initial assumption. How many extra tomatoes should each plant have? 108 tomatoes spread over 12 plants gives 9 extra tomatoes each.
Therefore, the plants have 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, and 21 tomatoes, and the last one has 21 tomatoes. (We can check that there are indeed 186 tomatoes by adding $10 + 11 + \ldots + 21$.)
ANSWER: (D)

24. Since $ABCD$ is a square and has an area of 25 cm^2, then the square has a side length of 5 cm. Since $PQCD$ is a rhombus, then it is a parallelogram, so its area is equal to the product of its base and its height.
Join point P to X on AD so that PX makes a right angle with AD, and to Y on DC so that PY makes a right angle with DC.



Then the area of the shaded region is the area of rectangle $ABZX$ plus the area of triangle PXD.
Since the area of $PQCD$ is 20 cm^2 and its base has length 5 cm, then its height, PY, must have length 4 cm.
Therefore, we can now label $DX = 4$, $DP = 5$ (since $PQCD$ is a rhombus), $AX = 1$, and $AB = 5$.
So $ABZX$ is a 1 by 5 rectangle, and so has area 5 cm^2.
Triangle PXD is right-angled at D, and has $DP = 5$ and $DX = 4$, so by Pythagoras' Theorem, $PX = 3$. Therefore, the area of triangle PXD is $\frac{1}{2}(3)(4) = 6\text{ cm}^2$.
So, in total, the area of the shaded region is 11 cm^2. ANSWER: (C)

25. Since all three numbers on the main diagonal are filled in, we can immediately determine what the product of the entries in any row, column or diagonal is, namely $6 \times 12 \times 24 = 1728$.
We can immediately start to fill in the square by filling in the top-centre, left-centre and bottom-right entries, since we have two entries in each of these rows, columns or diagonals, so the remaining entry is the overall product divided by the two entries already present.

Thus, we obtain

N	$\frac{1728}{24N}$	24
$\frac{1728}{6N}$	12	
6		$\frac{1728}{12N}$

Simplifying, we get

N	$\frac{72}{N}$	24
$\frac{288}{N}$	12	
6		$\frac{144}{N}$

In a similar way, we can fill in the two remaining entries to get

N	$\frac{72}{N}$	24
$\frac{288}{N}$	12	$\frac{1}{2}N$
6	$2N$	$\frac{144}{N}$

Now we are told that each of the nine entries is a positive integer, so each of N, $2N$, $\frac{1}{2}N$, $\frac{72}{N}$, $\frac{144}{N}$, and $\frac{288}{N}$ is a positive integer.
Do we need to check each of these conditions?
Well, if N is an integer, then $2N$ is an integer, so we don't need to check this second condition.
The fact that $\frac{1}{2}N$ is an integer tells us that N has to be an even integer.
The fact that $\frac{72}{N}$ is an integer tells us that N is a factor of 72.
The fact that $\frac{144}{N}$ is an integer tells us that N is a factor of 144, but since N is already a factor of 72 and $144 = 2 \times 72$, then N being a factor of 72 tells us that N is a factor of 144.
Similarly, N being a factor of 72 tells us that N is a factor of 288, so $\frac{288}{N}$ is an integer.
In summary, we are looking for positive integers N which are even and factors of 72.
Writing out the positive factors of 72, we get 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72, of which nine are even.

ANSWER: (C)

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2002 Solutions

Gauss Contest

(Grades 7 and 8)

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Part A

1. When the numbers 8, 3, 5, 0, 1 are arranged from smallest to largest, the middle number is

   (**A**) 5 (**B**) 8 (**C**) 3 (**D**) 0 (**E**) 1

   Solution

   If we rearrange the given numbers from smallest to largest, we would have 0, 1, 3, 5, 8.
   The middle number is 3. ANSWER: (C)

2. The value of $0.9 + 0.99$ is

   (**A**) 0.999 (**B**) 1.89 (**C**) 1.08 (**D**) 1.98 (**E**) 0.89

   Solution

   Adding,

   $$\begin{array}{r} 0.9 \\ +\ 0.99 \\ \hline 1.89 \end{array}$$

   ANSWER: (B)

3. $\dfrac{2+1}{7+6}$ equals

   (**A**) $\frac{3}{13}$ (**B**) $\frac{21}{76}$ (**C**) $\frac{1}{21}$ (**D**) $\frac{2}{13}$ (**E**) $\frac{1}{14}$

   Solution

   Evaluating,

   $\dfrac{2+1}{7+6} = \dfrac{3}{13}$. ANSWER: (A)

4. 20% of 20 is equal to

   (**A**) 400 (**B**) 100 (**C**) 5 (**D**) 2 (**E**) 4

   Solution

   20% of 20 equals 0.2 ¥ 20 = 4. Alternatively, 20% of 20 is $\frac{1}{5}$ of 20, or 4.

   ANSWER: (E)

5. Tyesha earns $5 per hour babysitting, and babysits for 7 hours in a particular week. If she starts the week with $20 in her bank account, deposits all she earns into her account, and does not withdraw any money, the amount she has in her account at the end of the week is

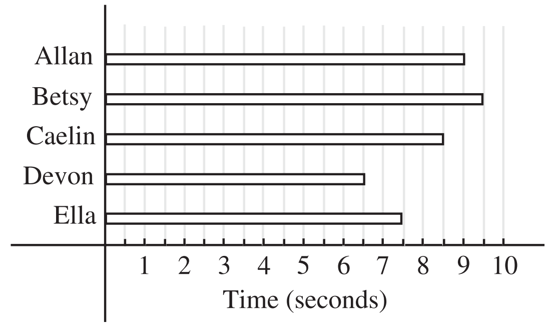
   (**A**) $35 (**B**) $20 (**C**) $45 (**D**) $55 (**E**) $65

Solution

If Tyesha earns $5 per hour and works for 7 hours, then she earns 7 ¥ $5 = $35 in total. If she started with $20 in her bank account and adds the $35, she will have $20 + $35 = $55 in her account. ANSWER: (D)

6. Five rats competed in a 25 metre race. The graph shows the time that each rat took to complete the race. Which rat won the race?

(**A**) Allan (**B**) Betsy (**C**) Caelin
(**D**) Devon (**E**) Ella



Solution

Since each of the rats completed the race, then the rat taking the least amount of time won the race. Since Devon took the least amount of time, she was the winner. ANSWER: (D)

7. The mean (average) of the numbers 12, 14, 16, and 18, is

(**A**) 30 (**B**) 60 (**C**) 17 (**D**) 13 (**E**) 15

Solution

The mean of the given numbers is

$$\frac{12+14+16+18}{4} = \frac{60}{4} = 15.$$

ANSWER: (E)

8. If $P = 1$ and $Q = 2$, which of the following expressions is **not** equal to an integer?

(**A**) $P + Q$ (**B**) P ¥ Q (**C**) $\frac{P}{Q}$ (**D**) $\frac{Q}{P}$ (**E**) P^Q

Solution

Evaluating the choices,

(A) $P + Q = 3$ **(B)** P ¥ $Q = 2$ **(C)** $\frac{P}{Q} = \frac{1}{2}$ **(D)** $\frac{Q}{P} = \frac{2}{1} = 2$ **(E)** $P^Q = 1^2 = 1$

ANSWER: (C)

9. Four friends equally shared $\frac{3}{4}$ of a pizza, which was left over after a party. What fraction of a whole pizza did each friend get?

(**A**) $\frac{3}{8}$ (**B**) $\frac{3}{16}$ (**C**) $\frac{1}{12}$ (**D**) $\frac{1}{16}$ (**E**) $\frac{1}{8}$

Solution

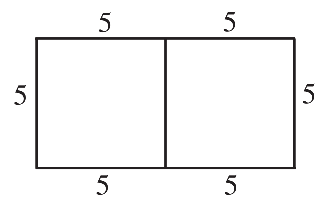
If $\frac{3}{4}$ of a pizza was shared by 4 friends, they would each receive $\frac{1}{4}$ of $\frac{3}{4}$, or $\frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$ of the pizza.

Answer: (B)

10. Two squares, each with an area of $25\,\text{cm}^2$, are placed side by side to form a rectangle. What is the perimeter of this rectangle?
(**A**) 30 cm (**B**) 25 cm (**C**) 50 cm (**D**) 20 cm (**E**) 15 cm

Solution

If the two squares are placed side by side, the rectangle shown would be formed.
The perimeter of this newly formed rectangle is 30 cm.



Answer: (A)

Part B

11. After running 25% of a race, Giselle had run 50 metres. How long was the race, in metres?
(**A**) 100 (**B**) 1250 (**C**) 200 (**D**) 12.5 (**E**) 400

Solution

If 25% of a race is 50 metres, then 100% of the race is $\frac{100}{25} \times 50 = 200$ metres.

Answer: (C)

12. Qaddama is 6 years older than Jack. Jack is 3 years younger than Doug. If Qaddama is 19 years old, how old is Doug?
(**A**) 17 (**B**) 16 (**C**) 10 (**D**) 18 (**E**) 15

Solution

If Qaddama is 6 years older than Jack and she is 19 years old, then Jack is 13 years old. If Jack is 3 years younger than Doug, then Doug must be 16 years of age.

Answer: (B)

13. A palindrome is a positive integer whose digits are the same when read forwards or backwards. For example, 2002 is a palindrome. What is the smallest number which can be added to 2002 to produce a larger palindrome?
(**A**) 11 (**B**) 110 (**C**) 108 (**D**) 18 (**E**) 1001

Solution

The best way to analyze this problem is by asking the question, "What is the next palindrome bigger than 2002?" Since the required palindrome should be of the form $2aa2$, where the middle two digits (both a) do not equal 0, it must be the number 2112. Thus, the number that must be added to 2002 is $2112 - 2002 = 110$. Answer: (B)

14. The first six letters of the alphabet are assigned values A = 1, B = 2, C = 3, D = 4, E = 5, and F = 6. The value of a word equals the sum of the values of its letters. For example, the value of BEEF is 2 + 5 + 5 + 6 = 18. Which of the following words has the greatest value?

(**A**) BEEF (**B**) FADE (**C**) FEED (**D**) FACE (**E**) DEAF

Solution

Each of the five given words contains both an "E" and an "F", so we can eliminate these letters for the purposes of making the comparison. So after eliminating these letters we are looking at the 5 "words",

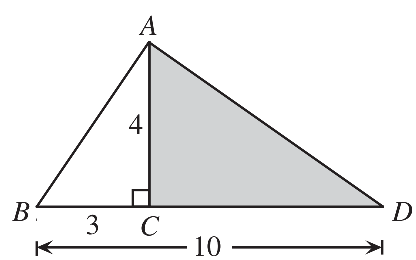
(A) BE **(B)** AD **(C)** ED **(D)** AC **(E)** DA

The highest value of these five words is ED, which has a value of 9, thus implying that FEED has the highest value of the original five words.

Alternatively, we could have calculated the value of each of the five words, and again seen that FEED has the highest value. Answer: (C)

15. In the diagram, $AC = 4$, $BC = 3$, and $BD = 10$. The area of the shaded triangle is

(**A**) 14 (**B**) 20 (**C**) 28
(**D**) 25 (**E**) 12



Solutionn

If $BD = 10$ and $BC = 3$, then $CD = 7$. The area of the shaded triangle is $\frac{1}{2}(7)(4) = 14$.

Answer: (A)

16. In the following equations, the letters a, b and c represent different numbers.

$$1^3 = 1$$
$$a^3 = 1 + 7$$
$$3^3 = 1 + 7 + b$$
$$4^3 = 1 + 7 + c$$

The numerical value of $a + b + c$ is

(**A**) 58 (**B**) 110 (**C**) 75 (**D**) 77 (**E**) 79

Solution

Since $2^3 = 8 = 1 + 7$, then $a = 2$.

Since $3^3 = 27$, then $27 = 8 + b$ or $b = 19$.

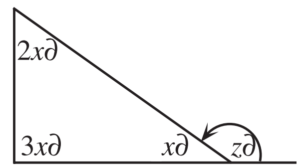
Since $4^3 = 64$, then $64 = 8 + c$ or $c = 56$.

Thus, $a + b + c = 2 + 19 + 56 = 77$. ANSWER: (D)

17. In the diagram, the value of z is

(**A**) 150 (**B**) 180 (**C**) 60

(**D**) 90 (**E**) 120



Solution

Since the angles of a triangle add to 180∂,

$$2x\partial + 3x\partial + x\partial = 180\partial$$
$$6x\partial = 180\partial$$
$$x = 30$$

Now since the angles $x\partial$ and $z\partial$ together form a straight line, the

$$x\partial + z\partial = 180\partial$$
$$30\partial + z\partial = 180\partial$$
$$z = 150$$

ANSWER: (A)

18. A perfect number is an integer that is equal to the sum of all of its positive divisors, except itself. For example, 28 is a perfect number because $28 = 1 + 2 + 4 + 7 + 14$. Which of the following is a perfect number?

(**A**) 10 (**B**) 13 (**C**) 6 (**D**) 8 (**E**) 9

Solution

We must check each of the answers:

	Number	Positive divisors	Sum of all positive divisors except itself
(A)	10	1, 2, 5, 10	$1 + 2 + 5 = 8$
(B)	13	1, 13	1
(C)	6	1, 2, 3, 6	$1 + 2 + 3 = 6$
(D)	8	1, 2, 4, 8	$1 + 2 + 4 = 7$
(E)	9	1, 3, 9	$1 + 3 = 4$

The only number from this set that is a perfect number is 6. (Note that the next two perfect number bigger than 28 are 496 and 8128.) ANSWER: (C)

19. Subesha wrote down Davina's phone number in her math binder. Later that day, while correcting her homework, Subesha accidentally erased the last two digits of the phone number, leaving 893-44_ _. Subesha tries to call Davina by dialing phone numbers starting with 893-44. What is the least number of phone calls that she has to make to be guaranteed to reach Davina's house?

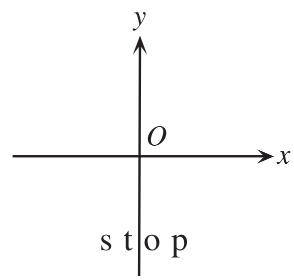
(**A**) 100 (**B**) 90 (**C**) 10 (**D**) 1000 (**E**) 20

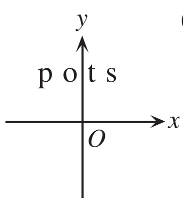
Solution

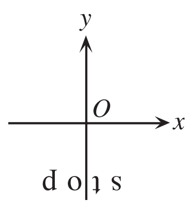
Davina could have a telephone number between and including 893-4400 and 893-4499. Since there are 100 numbers between and including these two numbers, this is precisely the number of calls that Subesha would have to make to be assured that she would reach Davina's house. An alternate way of seeing this is to realize that Davina's number is of the form 8 9 3 - 4 4 $\underline{a}\,\underline{b}$, where there are 10 possibilities for a and for each of these possibilities, there are 10 possibilities for b. Thus there are $10 \times 10 = 100$ different possibilities in total.

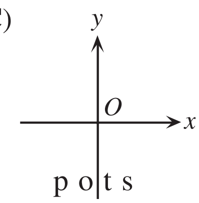
ANSWER: (A)

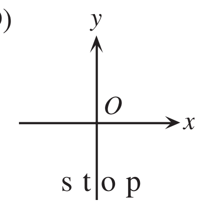
20. The word "stop" starts in the position shown in the diagram to the right. It is then rotated 180° clockwise about the origin, O, and this result is then reflected in the x-axis. Which of the following represents the final image?

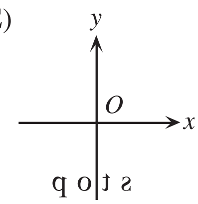


(**A**)


(**B**)


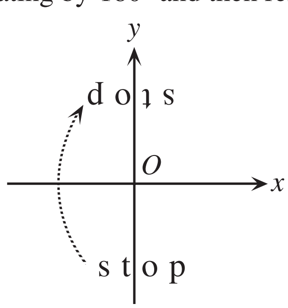
(**C**)


(**D**)


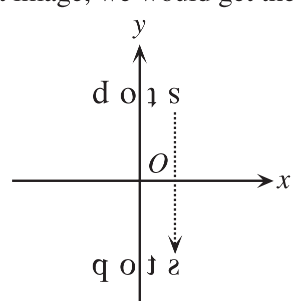
(**E**)


Solution

If we start by rotating by 180° and then reflecting that image, we would get the following:



Rotation of 180°



Reflection in x-axis

ANSWER: (E)

Part C

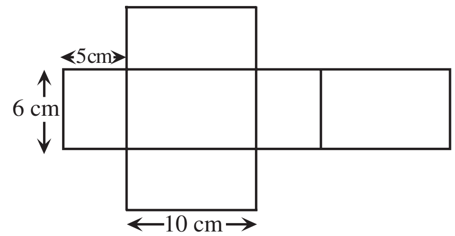
21. Five people are in a room for a meeting. When the meeting ends, each person shakes hands with each of the other people in the room exactly once. The total number of handshakes that occurs is

(**A**) 5 (**B**) 10 (**C**) 12 (**D**) 15 (**E**) 25

Solution

Each of the five people in the room will shake four others' hands. This gives us 20 handshakes, except each handshake is counted twice (Person X shakes Person Y's hand *and* Person Y shakes Person X's hand), so we have to divide the total by 2, to obtain 10 handshakes in total. ANSWER: (B)

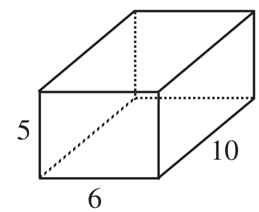
22. The figure shown can be folded along the lines to form a rectangular prism. The surface area of the rectangular prism, in cm^2, is



(**A**) 312 (**B**) 300 (**C**) 280
(**D**) 84 (**E**) 600

Solution

The required surface area is $2(5\times 6+5\times 10+6\times 10)=280\text{ cm}^2$.
Alternatively, if we fold the net into a rectangular box, we would obtain the following diagram. From this we can see that the faces of the box are two rectangles of area 30 cm^2, two rectangles of area 50 cm^2, and two rectangles of area 60 cm^2. This gives a total surface area of 280 cm^2.



ANSWER: (C)

23. Mark has a bag that contains 3 black marbles, 6 gold marbles, 2 purple marbles, and 6 red marbles. Mark adds a number of white marbles to the bag and tells Susan if she now draws a marble at random from the bag, the probability of it being black or gold is $\frac{3}{7}$. The number of white marbles that Mark adds to the bag is

(**A**) 5 (**B**) 2 (**C**) 6 (**D**) 4 (**E**) 3

Solution

Since the probability of selecting a black or gold marble is $\frac{3}{7}$, this implies that the total number of marbles in the bag is a multiple of 7. That is to say, there are possibly 7, 14, 21, 28, etc. marbles in the bag. The only acceptable number of marbles in the bag is 21, since there are 9 marbles in total which are black or gold, and $\frac{9}{21}=\frac{3}{7}$. If there are 21 marbles in the bag, this means that 4 marbles must have been added, since there are 17 already accounted for.
Alternatively, we could say that the number of white marbles in the bag was w (an unknown number), and form the equation

$$\frac{6+3}{17+w}=\frac{3}{7}$$

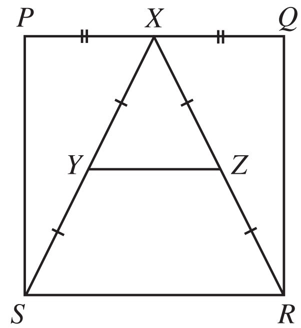
$$\frac{9}{17+w}=\frac{3}{7}$$

$$\frac{9}{17+w}=\frac{9}{21},$$ changing to a numerator of 9.

Thus, $17+w=21$ or $w=4$, and so the number of white marbles is 4. ANSWER: (D)

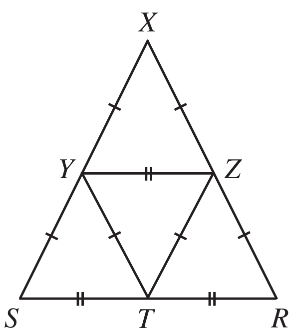
24. *PQRS* is a square with side length 8. *X* is the midpoint of side *PQ*, and *Y* and *Z* are the midpoints of *XS* and *XR*, respectively, as shown. The area of trapezoid *YZRS* is

(**A**) 24 (**B**) 16 (**C**) 20
(**D**) 28 (**E**) 32



Solution

If *PQRS* is a square with side length 8, it must have an area of 64 square units. The area of D*XSR* is thus $\frac{1}{2}(8)(8) = 32$. If we take the point *T* to be the midpoint of *SR* and join *Y* and *Z* to *T*, we would have the following diagram.



Each of the four smaller triangles contained within D*XSR* has an equal area, which is therefore $\frac{1}{4}(32) = 8$. Since the area of trapezoid *YZRS* is made up of three of these triangles, it has an area of 3¥8 = 24.

ANSWER: (A)

25. Each of the integers 226 and 318 have digits whose product is 24. How many three-digit positive integers have digits whose product is 24?

(**A**) 4 (**B**) 18 (**C**) 24 (**D**) 12 (**E**) 21

Solution

First, we determine all of the possible ways to write 24 as the product of single-digit numbers.

(i) 24 = 1¥4¥6
(ii) 24 = 1¥3¥8
(iii) 24 = 2¥3¥4
(iv) 24 = 2¥2¥6

The cases numbered (i), (ii) and (iii) each give 6 possible arrangements. For example, if we consider 24 = 1¥4¥6, the 6 possibilities are then 146, 164, 416, 461, 614, and 641. So for cases (i), (ii) and (iii), we have a total of 18 possibilities.

For the fourth case, there are only 3 possibilities, which are 226, 262 and 622.

In total there are $18 + 3 = 21$ possibilities. ANSWER: (E)

*****

Part A

1. The value of $\frac{1}{2}+\frac{1}{4}$ is

   (**A**) 1 (**B**) $\frac{1}{8}$ (**C**) $\frac{1}{6}$ (**D**) $\frac{2}{6}$ (**E**) $\frac{3}{4}$

   Solution
   Using a common denominator,
   $\frac{1}{2}+\frac{1}{4}=\frac{2}{4}+\frac{1}{4}=\frac{3}{4}$. ANSWER: (E)

2. The expression $6\times 1000+5\times 100+6\times 1$ is equivalent to
   (**A**) 656 (**B**) 6506 (**C**) 6056 (**D**) 60 506 (**E**) 6560

   Solution
   Expanding,
   $6\times 1000+5\times 100+6\times 1=6000+500+6=6506$. ANSWER: (B)

3. The value of $3^2-(4\times 2)$ is
   (**A**) 4 (**B**) 17 (**C**) 1 (**D**) –2 (**E**) 0

   Solution
   By order of operations,
   $3^2-(4\times 2)=9-(4\times 2)=9-8=1$. ANSWER: (C)

4. An integer is divided by 7 and the remainder is 4. An example of such an integer is
   (**A**) 14 (**B**) 15 (**C**) 16 (**D**) 17 (**E**) 18

   Solution
   Since 14 is a multiple of 7, then 18 (which is 4 more than 14) gives a remainder of 4 when divided by 7. ANSWER: (E)

5. Which of the following expressions is equal to an odd integer?

   (**A**) $3(5)+1$ (**B**) $2(3+5)$ (**C**) $3(3+5)$ (**D**) $3+5+1$ (**E**) $\frac{3+5}{2}$

   Solution
   Evaluating the choices,

   (**A**) $3(5)+1=16$ (**B**) $2(3+5)=16$ (**C**) $3(3+5)=24$ (**D**) $3+5+1=9$ (**E**) $\frac{3+5}{2}=4$

   Choice **(D)** gives the only odd integer. ANSWER: (D)

6. Qaddama is 6 years older than Jack. Jack is 3 years younger than Doug. If Qaddama is 19 years old, how old is Doug?

(**A**) 17 (**B**) 16 (**C**) 10 (**D**) 18 (**E**) 15

Solution

If Qaddama is 6 years older than Jack and she is 19 years old, then Jack is 13 years old. If Jack is 3 years younger than Doug, then Doug must be 16 years of age. ANSWER: (B)

7. The volume of a rectangular box is 144 cm^3. If its length is 12 cm and its width is 6 cm, what is its height?

(**A**) 126 cm (**B**) 72 cm (**C**) 4 cm (**D**) 8 cm (**E**) 2 cm

Solution

We know that $\text{Volume} = \text{Length} ¥ \text{Width} ¥ \text{Height}$, the volume is 144 cm^3, and $\text{Length} ¥ \text{Width} = 72\text{ cm}^2$. Thus, $144\text{ cm}^3 = 72\text{ cm}^2 ¥ \text{Height}$, or $\text{Height} = 2\text{ cm}$.

ANSWER: (E)

8. In a jar, the ratio of the number of oatmeal cookies to the number of chocolate chip cookies is 5:2. If there are 20 oatmeal cookies, the number of chocolate chip cookies in the jar is

(**A**) 28 (**B**) 50 (**C**) 8 (**D**) 12 (**E**) 18

Solution

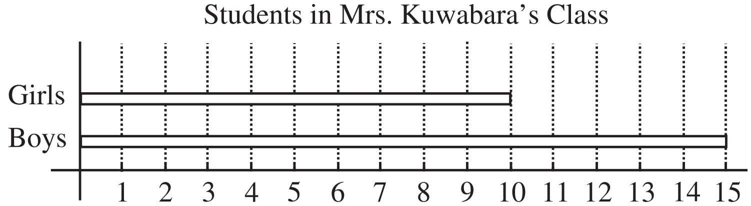
The ratio 5:2 indicates that there are 5 oatmeal cookies for every 2 chocolate chip cookies. Since there are 20 oatmeal cookies, there are four groups of 5 oatmeal cookies. Thus there are $4 ¥ 2 = 8$ chocolate chip cookies.

Algebraically, we could let x represent the number of chocolate chip cookies. Then $5:2 = 20:x$, or $\frac{5}{2} = \frac{20}{x}$. If we want to write $\frac{5}{2}$ as a fraction with a numerator of 20, we multiply both the numerator and denominator by 4, ie. $\frac{5}{2} = \frac{5 ¥ 4}{2 ¥ 4} = \frac{20}{8}$. Therefore, $x = 8$. ANSWER: (C)

9. The bar graph shows the numbers of boys and girls in Mrs. Kuwabara's class. The percentage of students in the class who are girls is

(**A**) 40% (**B**) 15% (**C**) 25% (**D**) 10% (**E**) 60%

Students in Mrs. Kuwabara's Class



Number of Students

Solution
From the graph, there are 10 girls and 15 boys in the class. Then, there are 25 students in total in the class, so the percentage of girls is $\frac{10}{25} \times 100\% = 40\%$. Answer: (A)

10. Which of the following statements is **not** true?
(**A**) A quadrilateral has four sides.
(**B**) The sum of the angles in a triangle is 180°.
(**C**) A rectangle has four 90° angles.
(**D**) A triangle can have two 90° angles.
(**E**) A rectangle is a quadrilateral.

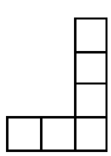
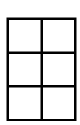
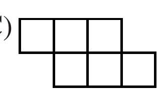
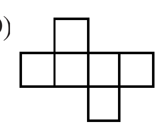
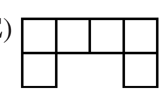
Solution
A quadrilateral has four sides, by definition.
The sum of the angles in a triangle is 180°.
A rectangle has four 90° angles, by definition.
A rectangle is a quadrilateral, since it has four sides.
However, a triangle cannot have two 90°, since its three angles add to 180°, and its third angle cannot be 0°. Answer: (D)

Part B

11. A palindrome is a positive integer whose digits are the same when read forwards or backwards. 2002 is a palindrome. What is the smallest number which can be added to 2002 to produce a larger palindrome?
(**A**) 11 (**B**) 110 (**C**) 108 (**D**) 18 (**E**) 1001

Solution
The best way to analyze this problem is by asking the question, "What is the next palindrome bigger than 2002?" Since the required palindrome should be of the form $2aa2$, where the middle two digits (both a) do not equal 0, it must be the number 2112. Thus, the number that must be added to 2002 is $2112 - 2002 = 110$. Answer: (B)

12. Which of the following can be folded along the lines to form a cube?

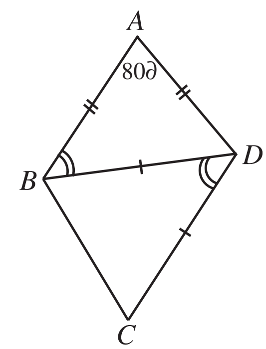
(**A**) 
(**B**) 
(**C**) 
(**D**) 
(**E**) 

Solution
Only choice (**D**) can be folded to form a cube. (Try constructing these nets to check this answer.)
Answer: (D)

13. If $a+b=12$, $b+c=16$, and $c=7$, what is the value of a?
(**A**) 1 (**B**) 5 (**C**) 9 (**D**) 7 (**E**) 3

Solution
Since $c=7$ and $b+c=16$, then $b+7=16$, or $b=9$.
Since $b=9$ and $a+b=12$, then $a+9=12$, or $a=3$. ANSWER: (E)

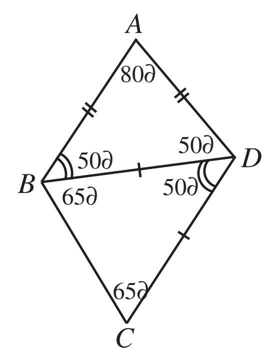
14. In the diagram, $\angle ABD = \angle BDC$ and $\angle DAB = 80^\circ$. Also, $AB = AD$ and $DB = DC$. The measure of $\angle BCD$ is
(**A**) 65° (**B**) 50° (**C**) 80°
(**D**) 60° (**E**) 70°



Solution
Since $\triangle ABD$ is isosceles, then $\angle ABD = \angle ADB$.
Therefore,

$$\begin{aligned} 80^\circ + \angle ABD + \angle ADB &= 180^\circ \\ 2 \times \angle ABD &= 100^\circ \quad (\text{since } \angle ABD = \angle ADB) \\ \angle ABD &= 50^\circ \end{aligned}$$

Thus, $\angle BDC = 50^\circ$ as well, since $\angle ABD = \angle BDC$. Since $\triangle BDC$ is also isosceles, then if we repeat a similar calculation to above, we obtain that $\angle BCD = 65^\circ$.



ANSWER: (A)

15. A perfect number is an integer that is equal to the sum of all of its positive divisors, except itself. For example, 28 is a perfect number because $28 = 1+2+4+7+14$. Which of the following is a perfect number?
(**A**) 10 (**B**) 13 (**C**) 6 (**D**) 8 (**E**) 9

Solution
We must check each of the answers:

	Number	Positive divisors	Sum of all positive divisors except itself
(A)	10	1, 2, 5, 10	$1+2+5=8$
(B)	13	1, 13	1
(C)	6	1, 2, 3, 6	$1+2+3=6$
(D)	8	1, 2, 4, 8	$1+2+4=7$
(E)	9	1, 3, 9	$1+3=4$

The only number from this set that is a perfect number is 6. (Note that the next two perfect number bigger than 28 are 496 and 8128.) Answer: (C)

16. Three pennies are flipped. What is the probability that they all land with heads up?

(**A**) $\frac{1}{8}$ (**B**) $\frac{1}{6}$ (**C**) $\frac{1}{4}$ (**D**) $\frac{1}{3}$ (**E**) $\frac{1}{2}$

Solution

If we toss one penny, the probability that it lands with heads up is $\frac{1}{2}$.

Since we want three heads up, we must multiply these probabilities together. That is, the probability is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

Alternatively, we could list all of the possibilities for the 3 pennies, using H to represent heads and T to represent tails:

HHH	THH
HHT	THT
HTH	TTH
HTT	TTT

This means that there are 8 equally likely possibilities, one of which is the desired possibility.

Therefore, the probability of three heads coming up is $\frac{1}{8}$. Answer: (A)

17. If P is a negative integer, which of the following is always positive?

(**A**) P^2 (**B**) $\frac{1}{P}$ (**C**) $2P$ (**D**) $P-1$ (**E**) P^3

Solution

If we try $P = -1$,

(**A**) $P^2 = 1$ (**B**) $\frac{1}{P} = -1$ (**C**) $2P = -2$ (**D**) $P - 1 = -2$ (**E**) $P^3 = -1$

and so the only possibility for the correct answer is (**A**). (In fact, P^2 is always greater than or equal to 0, regardless of the choice for P.) Answer: (A)

18. When expanded, the number of zeros in 1000^{10} is

(**A**) 13 (**B**) 30 (**C**) 4 (**D**) 10 (**E**) 1000

Solution

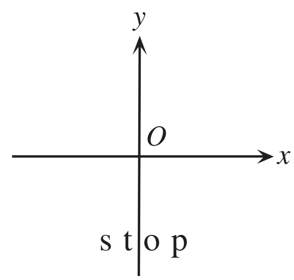
Using exponent laws,

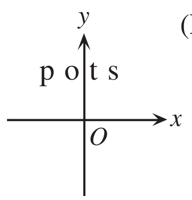
$$1000^{10} = \left(10^3\right)^{10} = 10^{3 \times 10} = 10^{30}$$

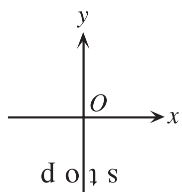
so if we were to write the number out in full, there should be 30 zeros.

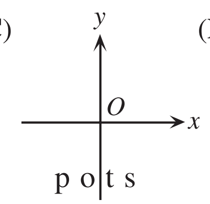
Answer: (B)

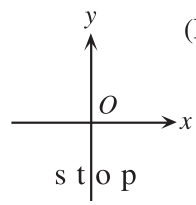
19. The word "stop" starts in the position shown in the diagram to the right. It is then rotated 180∂ clockwise about the origin, O, and this result is then reflected in the x-axis. Which of the following represents the final image?

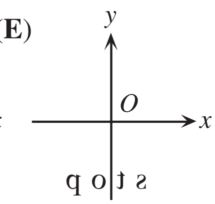


(**A**)


(**B**)


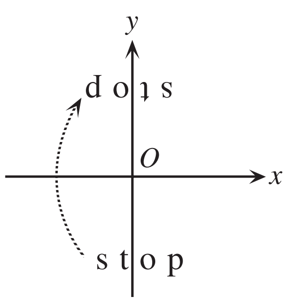
(**C**)


(**D**)


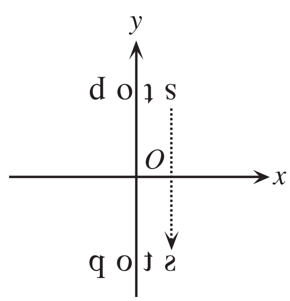
(**E**)


Solution

If we start by rotating by 180° and then reflecting that image, we would get the following:



Rotation of 180°



Reflection in x-axis

Answer: (E)

20. The units digit (that is, the last digit) of 7^{62} is

(**A**) 7 (**B**) 1 (**C**) 3 (**D**) 9 (**E**) 5

Solution

If we write out the first few powers of 7,

$$7^1 = 7,\ 7^2 = 49,\ 7^3 = 343,\ 7^4 = 2401,\ 7^5 = 16\,807, \ldots$$

we can see that the units digit follows the pattern 7, 9, 3, 1, 7, 9, 3, 1, 7, … (That is to say, the units digit of a product depends only on the units digits of the numbers being multiplied together. This tells us that we only need to look at the units digit of the previous power to determine the units digit of a given power.)

So the pattern 7, 9, 3, 1, repeats in blocks of four. Since 60 is a multiple of 4, this means that 7^{60} has a units digit of 1, and so 7^{62} has a units digit of 9. Answer: (D)

Part C

21. A rectangle has sides of integer length (when measured in cm) and an area of 36 cm^2. What is the maximum possible perimeter of the rectangle?
(**A**) 72 cm (**B**) 80 cm (**C**) 26 cm (**D**) 74 cm (**E**) 48 cm

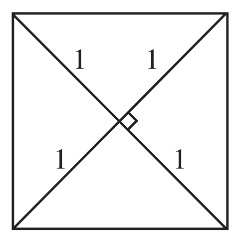
Solution
Since the area is 36 cm^2 and the sides have integer length, then we make a table of the possibilities:

Side lengths	Perimeter
1, 36	$2(1+36)=74$
2, 18	$2(2+18)=40$
3, 12	$2(3+12)=30$
4, 9	$2(4+9)=26$
6, 6	$2(6+6)=24$

So the maximum possible perimeter is 74 cm. Answer: (D)

22. If each diagonal of a square has length 2, then the area of the square is
(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

Solution
We draw the square and its two diagonals.
The diagonals of a square cut each other into two equal parts, and intersect at right angles. So we can decompose the square into 4 identical triangles with base 1 and height 1. So the area of the square is $4\left[\frac{1}{2}(1)(1)\right]=4\left[\frac{1}{2}\right]=2$.



Answer: (B)

23. A map is drawn to a scale of 1:10 000. On the map, the Gauss Forest occupies a rectangular region measuring 10 cm by 100 cm. What is the actual area of the Gauss Forest, in km^2?
(**A**) 100 (**B**) 1 000 000 (**C**) 1000 (**D**) 1 (**E**) 10

Solution
The actual lengths of the sides of the Gauss Forest are 10 000 times the lengths of the sides on the map. So the one side has length

$$10\ 000 ¥ 10\text{ cm} = 100\ 000\text{ cm} = 1000\text{ m} = 1\text{ km}$$

and the other side has length

$$10\ 000 ¥ 100\text{ cm} = 1\ 000\ 000\text{ cm} = 10\ 000\text{ m} = 10\text{ km}.$$

The actual area of the Gauss Forest is therefore $1\text{ km} ¥ 10\text{ km} = 10\text{ km}^2$.

Answer: (E)

24. Veronica has 6 marks on her report card.
The mean of the 6 marks is 74.
The mode of the 6 marks is 76.
The median of the 6 marks is 76.
The lowest mark is 50.
The highest mark is 94.
Only one mark appears twice and no mark appears more than twice.
Assuming all of her marks are integers, the number of possibilities for her second lowest mark is
(**A**) 17 (**B**) 16 (**C**) 25 (**D**) 18 (**E**) 24

Solution

Since the mode of Veronica's 6 marks is 76, and only one mark appears more than once (and no marks appear more than twice), then two of the marks must be 76. This tells us that four of her marks were 50, 76, 76, 94.
Since the median of her marks is 76 and she has six marks in total (that is, an *even* number of marks), then the two marks of 76 must be 3rd and 4th when the marks are arranged in increasing order.
Let the second lowest mark be M, and the second highest be N. So the second lowest mark M is between (but not equal to) 50 and 76, and the second highest mark N is between (but not equal to) 76 and 94.
We still need to use the fact that the mean of Veronica's marks is 74, so

$$\begin{aligned}\frac{50+M+76+76+N+94}{6} &= 74\\ M+N+296 &= 444\\ M+N &= 148\\ M &= 148-N \quad (*)\end{aligned}$$

We know already that M is one of 51 through 75, but the possibilities for N and the equation (*) restrict these possibilities further.
Since N can be any of 77 through 93, there are exactly 17 possibilities for N. The largest value of M corresponds to $N = 77$ (ie. $M = 71$) and the smallest value for M is when $N = 93$ (ie. $M = 55$). Thus the possibilities for M are 55 through 71, ie. there are 17 possibilities in total for M, the second smallest mark.

ANSWER: (A)

25. Emily has created a jumping game using a straight row of floor tiles that she has numbered 1, 2, 3, 4, … . Starting on tile 2, she jumps along the row, landing on every second tile, and stops on the second last tile in the row. Starting from this tile, she turns and jumps back toward the start, this time landing on every third tile. She stops on tile 1. Finally, she turns again and jumps along the row, landing on every fifth tile. This time, she again stops on the second last tile. The number of tiles in the row could be
(**A**) 39 (**B**) 40 (**C**) 47 (**D**) 49 (**E**) 53

Solution

Since Emily first starts on tile 2 and jumps on every second tile, then she lands only on even numbered tiles. Since she stops on the second last tile, the total number of tiles is odd.

Next, Emily jumps back along the row by 3's and ends on tile 1. So every tile that she lands on this time has a number which is 1 more than a multiple of 3 (eg. 1, 4, 7, etc.) So the second last tile has a number that is 1 more than a multiple of 3. This tells us that the overall number of tiles in the row is 2 more than a multiple of 3.

These two conditions tell us that the total number of tiles cannot be 39, 40 or 49.

Lastly, Emily jumps by 5's along the row starting at 1. This says each tile that she lands on has a number that is 1 more than a multiple of 5. By the same reasoning as above, the total number of tiles in the row is 2 more than a multiple of 5.

Of the two remaining possibilities (47 and 53), the only one that satisfies this last condition is 47, and so 47 satisfies all 3 of the required conditions.

(Work back through Emily's steps using the fact that she starts with 47 tiles to check that this does work.) ANSWER: (C)

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Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2001 Solutions
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Part A

1. The largest number in the set $\{0.01, 0.2, 0.03, 0.02, 0.1\}$ is

(**A**) 0.01 (**B**) 0.2 (**C**) 0.03 (**D**) 0.02 (**E**) 0.1

Solution

If we write each of these numbers to two decimal places by adding a '0' in the appropriate hundredths column, the numbers would be 0.01, 0.20, 0.03, 0.02 and 0.10. The largest is 0.2.

ANSWER: (B)

2. In 1998, the population of Canada was 30.3 million. Which number is the same as 30.3 million?

(**A**) 30 300 000 (**B**) 303 000 000 (**C**) 30 300 (**D**) 303 000 (**E**) 30 300 000 000

Solution

In order to find what number best represents 30.3 million, it is necessary to multiply 30.3 by $1\,000\,000$. This gives the number $30\,300\,000$.

ANSWER: (A)

3. The value of $0.001 + 1.01 + 0.11$ is

(**A**) 1.111 (**B**) 1.101 (**C**) 1.013 (**D**) 0.113 (**E**) 1.121

Solution

If we add these numbers 0.001, 1.01 and 0.11, we get the sum 1.121. We can most easily do this by calculator or by adding them in column form,

$$\begin{array}{r} 0.001 \\ 1.01 \\ +0.11 \\ \hline 1.121 \end{array}$$

ANSWER: (E)

4. When the number 16 is doubled and the answer is then halved, the result is

(**A**) 2^1 (**B**) 2^2 (**C**) 2^3 (**D**) 2^4 (**E**) 2^8

Solution

When the number 16 is doubled the result is 32.

When this answer is halved we get back to 16, our starting point. Since $16 = 2^4$, the correct answer is 2^4.

ANSWER: (D)

5. The value of $3 \times 4^2 - (8 \div 2)$ is

(**A**) 44 (**B**) 12 (**C**) 20 (**D**) 8 (**E**) 140

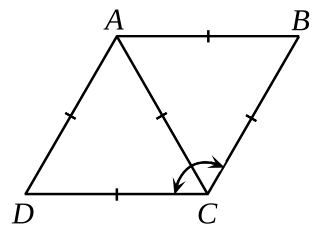
Solution

Evaluating, $3\times 4^2-(8\div 2)$
$= 48-4$
$= 44.$

ANSWER: (A)

6. In the diagram, $ABCD$ is a rhombus. The size of $\angle BCD$ is

(**A**) 60° (**B**) 90° (**C**) 120°
(**D**) 45° (**E**) 160°



Solution

We are given that ΔADC is equilateral which means that $\angle DAC=\angle ACD=\angle ADC=60°$. Similarly, each of the angles in ΔABC equals 60°. This implies that $\angle BCD$ equals 120° since $\angle BCD=\angle BCA+\angle DCA$ and $\angle BCA=\angle DCA=60°$. ANSWER: (C)

7. A number line has 40 consecutive integers marked on it. If the smallest of these integers is –11, what is the largest?

(**A**) 29 (**B**) 30 (**C**) 28 (**D**) 51 (**E**) 50

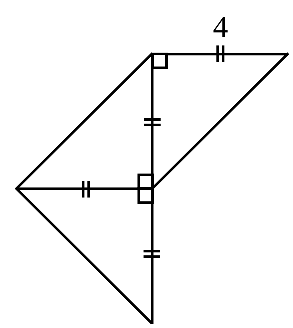
Solution

We note, first of all, that 0 is an integer. This means from –11 to 0, including 0, that there are 12 integers. The remaining 28 mark from 1 to 28 on the number line. The largest integer is 28.

ANSWER: (C)

8. The area of the entire figure shown is

(**A**) 16 (**B**) 32 (**C**) 20
(**D**) 24 (**E**) 64

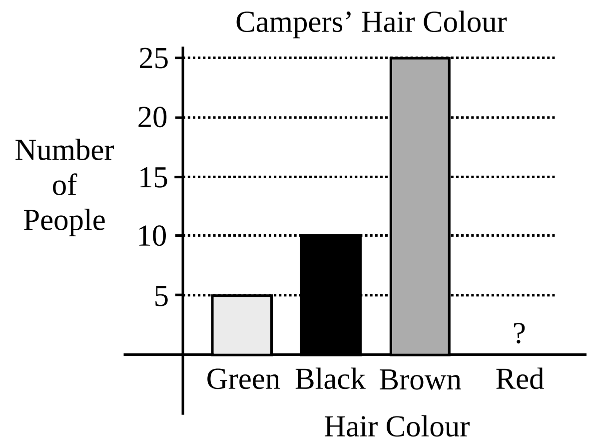


Solution

Each of the three small triangles is an isosceles right angled triangle having a side length of 4. The area of each small triangle is thus $\frac{1}{2}(4)(4)=8$. The total area is 3×8 or 24. ANSWER: (D)

9. The bar graph shows the hair colours of the campers at Camp Gauss. The bar corresponding to redheads has been accidentally removed. If 50% of the campers have brown hair, how many of the campers have red hair?

(**A**) 5 (**B**) 10 (**C**) 25
(**D**) 50 (**E**) 60



Solution

From the graph, we can see that there are 25 campers with brown hair. We are told that this represents 50% of the total number of campers. So in total then there are 2×25 or 50 campers. There is a total of 15 campers who have either green or black hair. This means that $50-(25+15)$ or 10 campers have red hair.

ANSWER: (B)

10. Henri scored a total of 20 points in his basketball team's first three games. He scored $\frac{1}{2}$ of these points in the first game and $\frac{1}{10}$ of these points in the second game. How many points did he score in the third game?

(**A**) 2 (**B**) 10 (**C**) 11 (**D**) 12 (**E**) 8

Solution

Henri scored $\frac{1}{\cancel{2}_1} \times \cancel{20}^{10}$ or 10 points in his first game. In his second game, he scored $\frac{1}{\cancel{10}_1} \times \cancel{20}^{2}$ or 2 points.

In the third game, this means that he will score $20-(2+10)$ or 8 points.

ANSWER: (E)

Part B

11. A fair die is constructed by labelling the faces of a wooden cube with the numbers 1, 1, 1, 2, 3, and 3. If this die is rolled once, the probability of rolling an odd number is

(**A**) $\frac{5}{6}$ (**B**) $\frac{4}{6}$ (**C**) $\frac{3}{6}$ (**D**) $\frac{2}{6}$ (**E**) $\frac{1}{6}$

Solution

There are six different equally likely possibilities in rolling the die. Since five of these are odd numbers, the probability of rolling an odd number is five out of six or $\frac{5}{6}$.

ANSWER: (A)

12. The ratio of the number of big dogs to the number of small dogs at a pet show is 3:17. There are 80 dogs, in total, at this pet show. How many big dogs are there?

(**A**) 12 (**B**) 68 (**C**) 20 (**D**) 24 (**E**) 6

Solution

Since the ratio of the number of big dogs to small dogs is 3:17 this implies that there are 3 large dogs in each group of 20. Since there are 80 dogs, there are four groups of 20. This means that there are 3×4 or 12 large dogs. ANSWER: (A)

13. The product of two whole numbers is 24. The smallest possible sum of these two numbers is

(**A**) 9 (**B**) 10 (**C**) 11 (**D**) 14 (**E**) 25

Solution

If two whole numbers have a product of 24 then the only possibilities are 1×24, 2×12, 3×8 and 4×6. The smallest possible sum is $4 + 6$ or 10. ANSWER: (B)

14. In the square shown, the numbers in each row, column, and diagonal multiply to give the same result. The sum of the two missing numbers is

(**A**) 28 (**B**) 15 (**C**) 30
(**D**) 38 (**E**) 72

12	1	18
9	6	4
		3

Solution

The numbers in each row, column and diagonal multiply to give a product of $(12)(1)(18)$ or 216. We are now looking for two numbers such that $(12)(9)(\quad) = 216$ and $(1)(6)(\quad) = 216$. The required numbers are 2 and 36 which have a sum of 38. ANSWER: (D)

15. A prime number is called a "Superprime" if doubling it, and then subtracting 1, results in another prime number. The number of Superprimes less than 15 is

(**A**) 2 (**B**) 3 (**C**) 4 (**D**) 5 (**E**) 6

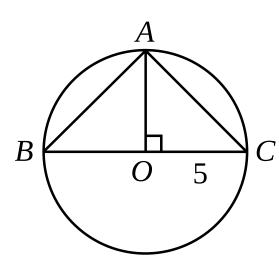
Solution

The only possible candidates for 'Superprimes' are 2, 3, 5, 7, 11 and 13 since they are the only prime numbers less than 15. If we double each of these numbers and then subtract 1 we get 3, 5, 9, 13, 21 and 25. Three of these results are prime numbers. So there are only three Superprimes.

ANSWER: (B)

16. BC is a diameter of the circle with centre O and radius 5, as shown. If A lies on the circle and AO is perpendicular to BC, the area of triangle ABC is

(**A**) 6.25 (**B**) 12.5 (**C**) 25
(**D**) 37.5 (**E**) 50



Solution
If O is the centre of the circle with radius 5 this implies that $OB = AC = OC = 5$. Thus, we are trying to find the sum of the areas of two identical isosceles right angled triangles with a side length of 5. The required area is $2\left[\frac{1}{2}(5)(5)\right]$ or 25.

ANSWER: (C)

17. A rectangular sign that has dimensions 9 m by 16 m has a square advertisement painted on it. The border around the square is required to be at least 1.5 m wide. The area of the largest square advertisement that can be painted on the sign is

(**A**) 78 m^2 (**B**) 144 m^2 (**C**) 36 m^2 (**D**) 9 m^2 (**E**) 56.25 m^2

Solution
If the 9×16 rectangle has a square painted on it such that the square must have a border of width 1.5 m this means that the square has a maximum width of $9 - 1.5 - 1.5 = 6$. So the largest square has an area of $6 \text{ m} \times 6 \text{ m}$ or 36 m^2.

ANSWER: (C)

18. Felix converted $924.00 to francs before his trip to France. At that time, each franc was worth thirty cents. If he returned from his trip with 21 francs, how many francs did he spend?

(**A**) 3 080 (**B**) 3 101 (**C**) 256.2 (**D**) 3 059 (**E**) 298.2

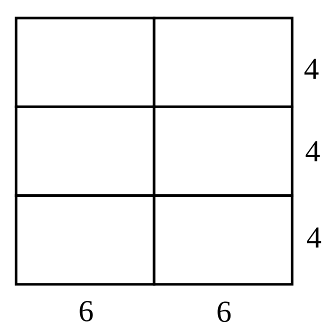
Solution
If each franc has a value of $0.30 then Felix would have been able to purchase $\frac{924}{0.30}$ or $3\,080$ francs. If he returns with 21 francs, he then must have then spent $3\,080 - 21$ or $3\,059$ francs.

ANSWER: (D)

19. Rectangular tiles, which measure 6 by 4, are arranged without overlapping, to create a square. The minimum number of these tiles needed to make a square is

(**A**) 8 (**B**) 24 (**C**) 4 (**D**) 12 (**E**) 6

Solution
Since the rectangles measure 6×4 this means that the lengths of their sides are in a ratio of 3:2. This implies that we need two rectangles that have 6 as their side length to form one side of the square and for each of these rectangles we need three others to form the other side length. In total, we need 2×3 or 6 rectangles.



ANSWER: (E)

20. Anne, Beth and Chris have 10 candies to divide amongst themselves. Anne gets at least 3 candies, while Beth and Chris each get at least 2. If Chris gets at most 3, the number of candies that Beth could get is

(**A**) 2 (**B**) 2 or 3 (**C**) 3 or 4 (**D**) 2, 3 or 5 (**E**) 2, 3, 4, or 5

Solution

If Anne gets at least 3 candies and Chris gets either 2 or 3 this implies that Beth could get as many as 5 candies if Chris gets only 2. If Chris and Anne increase their number of candies this means that Beth could get any number of candies ranging from 2 to 5. ANSWER: (E)

Part C

21. Naoki wrote nine tests, each out of 100. His average on these nine tests is 68%. If his lowest mark is omitted, what is his highest possible resulting average?

(**A**) 76.5% (**B**) 70% (**C**) 60.4% (**D**) 77% (**E**) 76%

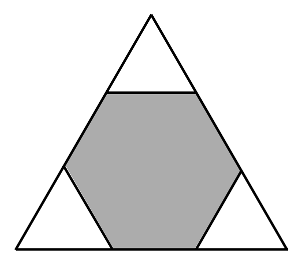
Solution

If Naoki had an average of 68% on nine tests he would have earned a total of 9×68 or 612 marks. Assuming that his lowest test was a '0', this means that Naoki would still have a total of 612 marks only this time on 8 tests. This would mean that Naoki would have an average of $\frac{612}{8}$ or 76.5%.

ANSWER: (A)

22. A regular hexagon is inscribed in an equilateral triangle, as shown. If the hexagon has an area of 12, the area of this triangle is

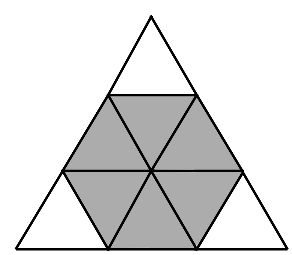
(**A**) 20 (**B**) 16 (**C**) 15
(**D**) 18 (**E**) 24



Solution

First of all, we can see that each of the smaller triangles is equilateral. Also, the side length of each of these is equal to that of the inscribed hexagon. From here, we can divide the hexagon up into six smaller triangles, identical to the white triangles at the three vertices as in the diagram above.

This means that each of the six triangles of the hexagon would have an area of 2, meaning that the large triangle would have an area of 9×2 or 18.



ANSWER: (D)

23. Catrina runs 100 m in 10 seconds. Sedra runs 400 m in 44 seconds. Maintaining these constant speeds, they participate in a 1 km race. How far ahead, to the nearest metre, is the winner as she crosses the finish line?

(**A**) 100 m (**B**) 110 m (**C**) 95 m (**D**) 90 m (**E**) 91 m

Solution

Since Sedra runs 400 m in 44 seconds, then she can run 100 m in 11 seconds. So Catrina runs farther than Sedra.

If Catrina runs 100 m in 10 seconds, she will complete the race in $\frac{1000}{100}\times 10$ or 100 seconds. In 100 seconds, Sedra would have run only $\frac{400}{44}\times 100$ or 909.09 metres. Sedra would then be approximately $1000-909.09$ or 90.91 metres behind as Catrina crossed the finish line. ANSWER: (E)

24. Enzo has fish in two aquariums. In one aquarium, the ratio of the number of guppies to the number of goldfish is 2:3. In the other, this ratio is 3:5. If Enzo has 20 guppies in total, the least number of goldfish that he could have is

(**A**) 29 (**B**) 30 (**C**) 31 (**D**) 32 (**E**) 33

Solution 1

The following tables give the possible numbers of fish in each aquarium.

The three lines join the results which give a total of 20 guppies, namely $2+18$, $8+12$ and $14+6$. The corresponding numbers of goldfish are 33, 32 and 31. The least number of goldfish that he could have is 31.

1st aquarium	
Number of guppies	Number of goldfish
2	3
4	6
6	9
8	12
10	15
12	18
14	21
16	24
18	27

2nd aquarium	
Number of guppies	Number of goldfish
3	5
6	10
9	15
12	20
15	25
18	30

Solution 2

In the first aquarium, the ratio of the number of guppies to goldfish is 2:3 so let the actual number of guppies be $2a$ and the actual number of goldfish be $3a$. In the second aquarium, if we apply the same reasoning, the actual number of guppies is $3b$ and the actual number of goldfish is $5b$. In total, there are 20 guppies so we now have the equation, $2a+3b=20$. We consider the different possibilities for a and b also calculating $3a+5b$, the number of goldfish. From the chart, we see that the smallest possible number of goldfish is 31.

$2a+3b$	a	b	$3a+5b$
20	1	6	33
20	4	4	32
20	7	2	31

ANSWER: (C)

25. A triangle can be formed having side lengths 4, 5 and 8. It is impossible, however, to construct a triangle with side lengths 4, 5 and 9. Ron has eight sticks, each having an integer length. He observes that he cannot form a triangle using any three of these sticks as side lengths. The shortest possible length of the longest of the eight sticks is

(**A**) 20 (**B**) 21 (**C**) 22 (**D**) 23 (**E**) 24

Solution

If Ron wants the three smallest possible lengths with which he cannot form a triangle, he should start with the lengths 1, 1 and 2. (These are the first three Fibonacci numbers). If he forms a sequence by adding the last two numbers in the sequence to form the next term, he would generate the sequence: 1, 1, 2, 3, 5, 8, 13, 21. Notice that if we take any three lengths in this sequence, we can never form a triangle. The shortest possible length of the longest stick is 21. ANSWER: (B)

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Part A

1. In 1998, the population of Canada was 30.3 million. Which number is the same as 30.3 million?

(A) 30 300 000 **(B)** 303 000 000 **(C)** 30 300 **(D)** 303 000 **(E)** 30 300 000 000

Solution
In order to find what number best represents 30.3 million, it is necessary to multiply 30.3 by $1\,000\,000$.
This gives the number $30\,300\,000$. ANSWER: (A)

2. What number should be placed in the box to make $\frac{6+\square}{20}=\frac{1}{2}$?

(A) 10 **(B)** 4 **(C)** -5 **(D)** 34 **(E)** 14

Solution
Expressing $\frac{1}{2}$ as a fraction with denominator 20, we get $\frac{10}{20}$.
So $\frac{6+\square}{20}=\frac{10}{20}$. Comparing the numerators, $6+\square=10$, so the number to place in the box is 4.

ANSWER: (B)

3. The value of $3\times 4^2-(8\div 2)$ is

(A) 44 **(B)** 12 **(C)** 20 **(D)** 8 **(E)** 140

Solution
Evaluating, $3\times 4^2-(8\div 2)$
$= 48-4$
$= 44.$ ANSWER: (A)

4. When a number is divided by 7, the quotient is 12 and the remainder is 5. The number is

(A) 47 **(B)** 79 **(C)** 67 **(D)** 119 **(E)** 89

Solution
Since the quotient is 12 and the remainder is 5, then the number is $(7\times 12)+5=89$.
ANSWER: (E)

5. If $2x-5=15$, the value of x is

(A) 5 **(B)** -5 **(C)** 10 **(D)** 0 **(E)** -10

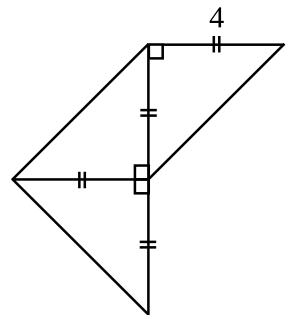
Solution
Since $2x-5=15$, then $2x-5+5=15+5$

$$2x=20$$
$$\frac{2x}{2}=\frac{20}{2}$$
$$x=10.$$

ANSWER: (C)

6. The area of the entire figure shown is

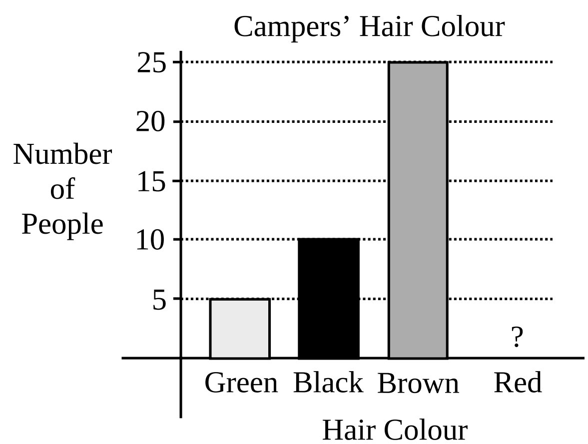
(**A**) 16 (**B**) 32 (**C**) 20
(**D**) 24 (**E**) 64



Solution
Each of the three small triangles is an isosceles right angled triangle having a side length of 4. The area of each small triangle is thus $\frac{1}{2}(4)(4)=8$. The total area is 3×8 or 24. ANSWER: (D)

7. The bar graph shows the hair colours of the campers at Camp Gauss. The bar corresponding to redheads has been accidentally removed. If 50% of the campers have brown hair, how many of the campers have red hair?

(**A**) 5 (**B**) 10 (**C**) 25
(**D**) 50 (**E**) 60



Solution
From the graph, we can see that there are 25 campers with brown hair. We are told that this represents 50% of the total number of campers. So in total there are then 2×25 or 50 campers. There is a total of 15 campers who have either green or black hair. This means that $50-(25+15)$ or 10 campers have red hair. ANSWER: (B)

8. A fair die is constructed by labelling the faces of a wooden cube with the numbers 1, 1, 1, 2, 3, and 3. If this die is rolled once, the probability of rolling an odd number is

(**A**) $\frac{5}{6}$ (**B**) $\frac{4}{6}$ (**C**) $\frac{3}{6}$ (**D**) $\frac{2}{6}$ (**E**) $\frac{1}{6}$

Solution
There are six different equally likely possibilities in rolling the die. Since five of these are odd numbers, the probability of rolling an odd number is five out of six or $\frac{5}{6}$. ANSWER: (A)

9. In the square shown, the numbers in each row, column, and diagonal multiply to give the same result. The sum of the two missing numbers is

12	1	18
9	6	4
		3

(**A**) 28 (**B**) 15 (**C**) 30
(**D**) 38 (**E**) 72

Solution
The numbers in each row, column and diagonal multiply to give a product of $(12)(1)(18)$ or 216. We are now looking for two numbers such that $(12)(9)(\quad) = 216$ and $(1)(6)(\quad) = 216$. The required numbers are 2 and 36 which have a sum of 38. ANSWER: (D)

10. Rowena is able to mow $\frac{2}{5}$ of a lawn in 18 minutes. If she began the job at 10:00 a.m., and mowed at this same constant rate, when did she finish mowing the entire lawn?
(**A**) 10:08 a.m. (**B**) 11:30 a.m. (**C**) 10:40 a.m. (**D**) 10:25 a.m. (**E**) 10:45 a.m.

Solution
Since Rowena can mow $\frac{2}{5}$ of the lawn in 18 minutes, she can mow $\frac{1}{5}$ of the lawn in 9 minutes. This tells us that it takes her $5 \times 9 = 45$ minutes to mow the whole lawn. So if she starts at 10:00 a.m., she finishes at 10:45 a.m. ANSWER: (E)

Part B

11. In a class of 25 students, each student has at most one pet. Three-fifths of the students have cats, 20% have dogs, three have elephants, and the other students have no pets. How many students have no pets?
(**A**) 5 (**B**) 4 (**C**) 3 (**D**) 2 (**E**) 1

Solution
Three-fifths of 25 is $\frac{3}{5} \times 25 = 15$, so 15 students have cats. Twenty percent of 25 is $\frac{20}{100} \times 25 = \frac{1}{5} \times 25 = 5$, so 5 students have dogs. This tells us that $15 + 5 + 3 = 23$ students have pets. So there are 2 students without pets. ANSWER: (D)

12. A prime number is called a “Superprime” if doubling it, and then subtracting 1, results in another prime number. The number of Superprimes less than 15 is
(**A**) 2 (**B**) 3 (**C**) 4 (**D**) 5 (**E**) 6

Solution
The only possible candidates for ‘Superprimes’ are 2, 3, 5, 7, 11 and 13 since they are the only prime numbers less than 15. If we double each of these numbers and then subtract 1 we get 3, 5, 9, 13, 21 and 25. Three of these results are prime numbers. So there are only three Superprimes.
ANSWER: (B)

13. Laura earns \$10/hour and works 8 hours per day for 10 days. She first spends 25% of her pay on food and clothing, and then pays \$350 in rent. How much of her pay does she have left?
(A) \$275 **(B)** \$200 **(C)** \$350 **(D)** \$250 **(E)** \$300

Solution
In 10 days, Laura works $8\times 10 = 80$ hours. So in these 10 days, she earns $80\times \$10 = \$800$. Since $25\% = \frac{1}{4}$, she spends $\frac{1}{4}\times \$800 = \$200$ on food and clothing, leaving her with \$600.
If she spends \$350 on rent, she will then have $\$600 - \$350$ or \$250 left. ANSWER: (D)

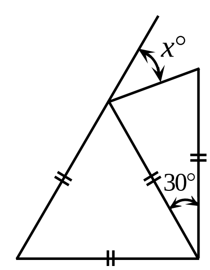
14. A rectangular sign that has dimensions 9 m by 16 m has a square advertisement painted on it. The border around the square is required to be at least 1.5 m wide. The area of the largest square advertisement that can be painted on the sign is
(A) 78 m^2 **(B)** 144 m^2 **(C)** 36 m^2 **(D)** 9 m^2 **(E)** 56.25 m^2

Solution
If the 9×16 rectangle has a square painted on it such that the square must have a border of width 1.5 m this means that the square has a maximum width of $9 - 1.5 - 1.5 = 6$. So the largest square has an area of $6 \text{ m}\times 6 \text{ m}$ or 36 m^2. ANSWER: (C)

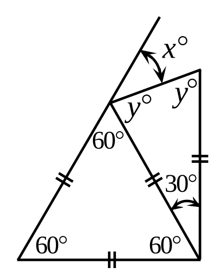
15. The surface area of a cube is 24 cm^2. The volume of this cube is
(A) 4 cm^3 **(B)** 24 cm^3 **(C)** 8 cm^3 **(D)** 27 cm^3 **(E)** 64 cm^3

Solution
A cube has six square faces of equal area, so each of these faces has area $\frac{1}{6}\times 24 \text{ cm}^2 = 4 \text{ cm}^2$. This tells us that the side length of the cube must be 2 cm. The volume of the cube is $2 \text{ cm}\times 2 \text{ cm}\times 2 \text{ cm} = 8 \text{ cm}^3$. ANSWER: (C)

16. In the diagram, the value of x is
(A) 30 **(B)** 40 **(C)** 60
(D) 50 **(E)** 45



Solution
All angles in an equilateral triangle are $60°$.
The triangle next to the equilateral triangle is isosceles, so let each of the remaining angles be $y°$.
The angles in a triangle add to 180°, so
$$y° + y° + 30° = 180°$$
$$2y° + 30° = 180°$$
$$y° = 75°.$$



The angles in a straight line add to $180°$, so

$$x° + y° + 60° = 180°,\ y° = 75°$$
$$x° + 75° + 60° = 180°$$
$$x° = 45°$$

ANSWER: (E)

17. Daniel's age is one-ninth of his father's age. One year from now, Daniel's father's age will be seven times Daniel's age. The difference between their ages is

(A) 24 **(B)** 25 **(C)** 26 **(D)** 27 **(E)** 28

Solution

Let Daniel's age be d. Then Daniel's father's age is $9d$.

In one year, Daniel's age will be $d+1$ and Daniel's father's will be $9d+1$.

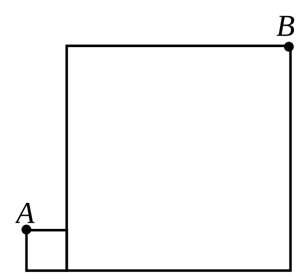
So $9d+1 = 7(d+1)$

$$9d + 1 = 7d + 7$$
$$2d = 6$$
$$d = 3.$$

Therefore, now Daniel's age is 3 and Daniel's father's age is 27, so the difference between their ages is 24.

ANSWER: (A)

18. Two squares are positioned, as shown. The smaller square has side length 1 and the larger square has side length 7. The length of AB is



(A) 14 **(B)** $\sqrt{113}$ **(C)** 10

(D) $\sqrt{85}$ **(E)** $\sqrt{72}$

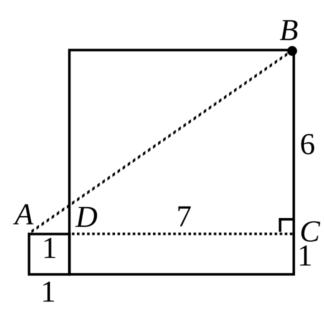
Solution

From A, along the side of the small square, we extend side AD so that it meets the big square at C as shown.

The length of AC is the sum of the side lengths of the squares, i.e. $7+1=8$.

The length of BC is the difference of the side lengths so $BC = 6$.

By Pythagoras, $AB^2 = 8^2 + 6^2 = 100$, $AB = 10$.



ANSWER: (C)

19. Anne, Beth and Chris have 10 candies to divide amongst themselves. Anne gets at least 3 candies, while Beth and Chris each get at least 2. If Chris gets at most 3, the number of candies that Beth could get is

(A) 2 **(B)** 2 or 3 **(C)** 3 or 4 **(D)** 2, 3 or 5 **(E)** 2, 3, 4 or 5

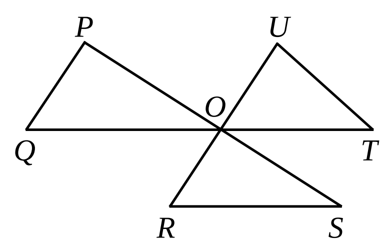
Solution
If Anne gets at least 3 candies and Chris gets either 2 or 3 this implies that Beth could get as many as 5 candies if Chris gets only 2. If Chris and Anne increase their number of candies this means that Beth could get any number of candies ranging from 2 to 5. ANSWER: (E)

20. What number should be placed in the box to make $10^4 \times 100^{\square} = 1000^6$?
(A) 7 **(B)** 5 **(C)** 2 **(D)** $\frac{3}{2}$ **(E)** 10

Solution
Since 1000 has 3 zeros, 1000^6 has 18 zeros, so the left side should have 18 zeros too. The number 10^4 has 4 zeros, so that leaves 14 zeros for the number $100^{\square}$. Since 100 has 2 zeros, the number in the box should be 7. ANSWER: (A)

Part C

21. Lines PS, QT and RU intersect at a common point O, as shown. P is joined to Q, R to S, and T to U, to form triangles. The value of $\angle P + \angle Q + \angle R + \angle S + \angle T + \angle U$ is
(A) 450° **(B)** 270° **(C)** 360°
(D) 540° **(E)** 720°



Solution
We know that $\angle POQ + \angle POU + \angle UOT = 180°$ because when added they form a straight line. Since $\angle POU = \angle ROS$ (vertically opposite angles), therefore $\angle POQ + \angle ROS + \angle UOT = 180°$. Thus the sum of the remaining angles is just $3 \times 180° - 180° = 360°$ since there are 180° in each of the three given triangles. ANSWER: (C)

22. Sixty-four white $1 \times 1 \times 1$ cubes are used to form a $4 \times 4 \times 4$ cube, which is then painted red on each of its six faces. This large cube is then broken up into its 64 unit cubes. Each unit cube is given a score as follows:

Exact number of faces painted red	Score
3	3
2	2
1	1
0	-7

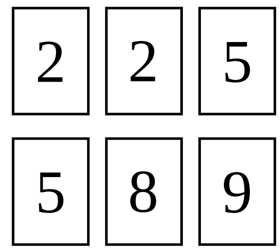
The total score for the $4 \times 4 \times 4$ cube is
(A) 40 **(B)** 41 **(C)** 42 **(D)** 43 **(E)** 44

Solution

First, we deal with the positive points. There is one point assigned for each red face. On each face of the large cube, there will be 16 red faces of $1\times1\times1$ cubes. This gives $6\times16=96$ red faces on $1\times1\times1$ cubes in total. So there are 96 positive points. However, there will be $2\times2\times2=8$ unit cubes which have no paint, and these will account for $8\times(-7)=-56$ points.

Then the point total for the cube is $96+(-56)=40$. (Notice that it was not necessary that we consider cubes with paint on either 3 sides or 2 sides if we use this method.) ANSWER: (A)

23. The integers 2, 2, 5, 5, 8, and 9 are written on six cards, as shown. Any number of the six cards is chosen, and the sum of the integers on these cards is determined. Note that the integers 1 and 30 cannot be obtained as sums in this way. How many of the integers from 1 to 31 cannot be obtained as sums?



(**A**) 4 (**B**) 22 (**C**) 8
(**D**) 10 (**E**) 6

Solution

First, we observe that the sum of the digits on all 6 cards is 31.

Next, we see that if we cannot get a sum of *S*, then we cannot get a sum of $31-S$. This is an important point. If we could get $31-S$, then we could take the cards not used and their digits would add to *S*. Simply stated, our inability to get *S* means that we would be unable to get $31-S$.

So we need to check which sums from 1 to 15 cannot be obtained, and then double this total number. Checking possibilities, we see that we can get 2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15 but cannot get 1, 3, 6. From above, we thus cannot get 31 – 1 = 30, 31 – 3 = 28 or 31 – 6 = 25.

So there are 6 sums that cannot be obtained. ANSWER: (E)

24. A triangle can be formed having side lengths 4, 5 and 8. It is impossible, however, to construct a triangle with side lengths 4, 5 and 9. Ron has eight sticks, each having an integer length. He observes that he cannot form a triangle using any three of these sticks as side lengths. The shortest possible length of the longest of the eight sticks is

(**A**) 20 (**B**) 21 (**C**) 22 (**D**) 23 (**E**) 24

Solution

If Ron wants the three smallest possible lengths with which he cannot form a triangle, he should start with the lengths 1, 1 and 2. (These are the first three Fibonacci numbers). If he forms a sequence by adding the last two numbers in the sequence to form the next term, he would generate the sequence: 1, 1, 2, 3, 5, 8, 13, 21. Notice that if we take any three lengths in this sequence, we can never form a triangle. The shortest possible length of the longest stick is 21. ANSWER: (B)

25. Tony and Maria are training for a race by running all the way up and down a 700 m long ski slope. They each run up the slope at different constant speeds. Coming down the slope, each runs at double his or her uphill speed. Maria reaches the top first, and immediately starts running back down, meeting Tony 70 m from the top. When Maria reaches the bottom, how far behind is Tony?

(**A**) 140 m (**B**) 250 m (**C**) 280 m (**D**) 300 m (**E**) 320 m

Solution

When Tony and Maria meet for the first time, Tony has run $700 - 70 = 630$ m.

At this point, Maria has run 700 m up the hill and then 70 m at double the speed back down the hill. This takes her the same amount of time as if she had run 700 m up the hill and then 35 m more at the same speed.

In effect, she has run 735 m, while Tony has run 630 m. So the ratio of their speeds is $\frac{735}{630} = \frac{7(105)}{6(105)} = \frac{7}{6}$.

This means that for every 6 metres that Tony covers, Maria will cover 7 metres.

If we think of both runners as running at constant speeds, Maria runs $700 + \frac{1}{2}(700) = 1050$ m at a constant speed over the course of the race. In effect, we are saying that she would run the equivalent of 1050 m at the same constant speed at which she ran up the hill.

She runs $\frac{7}{6}$ as fast as Tony. In the time that Maria runs 1050 m, Tony runs $\frac{6}{7} \times 1050 \text{ m} = 900$ m at a constant speed.

So in this new way of looking at the race, Tony is 150 m behind Maria.

But this 150 m is in the new way of looking at things, so Tony is actually $2 \times 150 \text{ m} = 300$ m behind Maria.

ANSWER: (D)

Part A

1. The value of $987+113-1000$ is
(**A**) 90 (**B**) 10 (**C**) 110 (**D**) 2000 (**E**) 100

Solution
$987+113=1100$
$1100-1000=100$

ANSWER: (E)

2. As a decimal, $\frac{9}{10}+\frac{8}{100}$ is
(**A**) 1.098 (**B**) 0.98 (**C**) 0.098 (**D**) 0.0908 (**E**) 9.8

Solution
Since $\frac{9}{10}=0.9$ and $\frac{8}{100}=0.08$, when we add we get $0.9+0.08=0.98$. ANSWER: (B)

3. What integer is closest in value to $7\times\frac{3}{4}$?
(**A**) 21 (**B**) 9 (**C**) 6 (**D**) 5 (**E**) 1

Solution
$7\times\frac{3}{4}=\frac{21}{4}=5\frac{1}{4}$. The integer closest to $5\frac{1}{4}$ is 5. ANSWER: (D)

4. The value of the expression $5^2-4^2+3^2$ is
(**A**) 20 (**B**) 18 (**C**) 21 (**D**) 10 (**E**) 16

Solution
$5^2=25,\ 4^2=16,\ 3^2=9$
Thus, $5^2-4^2+3^2=25-16+9=18$. ANSWER: (B)

5. When a number is divided by 7, it gives a quotient of 4 with a remainder of 6. What is the number?
(**A**) 17 (**B**) 168 (**C**) 34 (**D**) 31 (**E**) 46

Solution
The required number is $4\times7+6=34$. It is easy to verify this by dividing 34 by 7 which gives a quotient of 4 with remainder 6. ANSWER: (C)

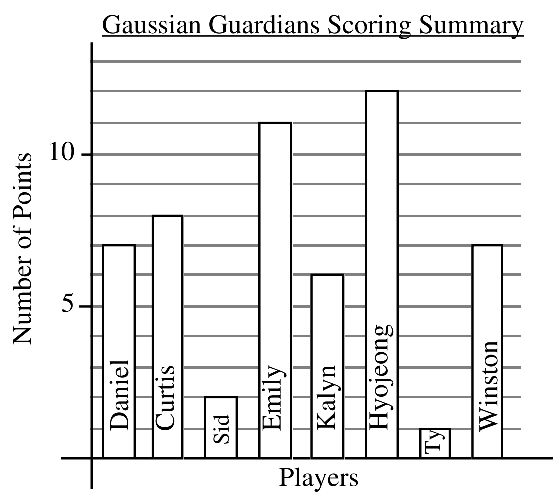
6. In the addition shown, a digit, either the same or different, can be placed in each of the two boxes. What is the sum of the two missing digits?

$$\begin{array}{r} 8\ 6\ 3 \\ \square\ 9\ 1 \\ 7\ \square\ 8 \\ \hline 2\ 1\ 8\ 2 \end{array}$$

(**A**) 9 (**B**) 11 (**C**) 13
(**D**) 3 (**E**) 7

Solution
Adding in the units column gives us, $3+1+8=12$. This means a carry over of 1 into the tens column since $12=1\times 10+2$. In the tens column, we have 1 (carried over) $+6+9+\square=18$. The digit that is placed in this box is 2 with a carry over of 1 unit into the hundreds column. Moving to the hundreds column we have, 1 (carried over) $+8+\square+7=21$. The missing digit here is 5. The two missing digits are 2 and 5 giving a sum of 7. ANSWER: (E)

7. The graph shows the complete scoring summary for the last game played by the eight players on Gaussian Guardians intramural basketball team. The total number of points scored by the Gaussian Guardians was
(**A**) 54 (**B**) 8 (**C**) 12
(**D**) 58 (**E**) 46

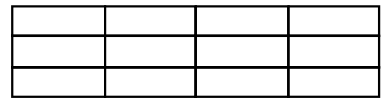


Solution
If we list all the players with their points, we would have the following: Daniel (7), Curtis (8), Sid (2), Emily (11), Kalyn (6), Hyojeong (12), Ty (1) and Winston (7).
The total is, $7+8+2+11+6+12+1+7=54$. ANSWER: (A)

8. If $\frac{1}{2}$ of the number represented by x is 32, what is $2x$?
(**A**) 128 (**B**) 64 (**C**) 32 (**D**) 256 (**E**) 16

Solution
If $\frac{1}{2}$ of the number represented by x is 32, then the number x is $2(32)$ or 64 and $2x$ is $2(64)$ or 128.
ANSWER: (A)

9. In the given diagram, all 12 of the small rectangles are the same size. Your task is to completely shade some of the rectangles until $\frac{2}{3}$ of $\frac{3}{4}$ of the diagram is shaded. The number of rectangles you need to shade is



(**A**) 9 (**B**) 3 (**C**) 4
(**D**) 6 (**E**) 8

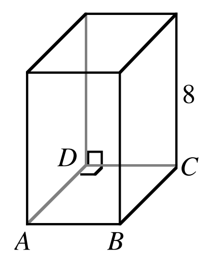
Solution
Since $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12} = \frac{1}{2}$, the number of shaded rectangles is $\frac{1}{2} \times 12 = 6$. Answer: (D)

10. The sum of three consecutive integers is 90. What is the largest of the three integers?
(**A**) 28 (**B**) 29 (**C**) 31 (**D**) 32 (**E**) 21

Solution
Since the integers are consecutive, the middle integer is the average of the three integers. The middle integer is $\frac{90}{3} = 30$. The integers are 29, 30 and 31. The largest is 31. Answer: (C)

Part B

11. A rectangular building block has a square base *ABCD* as shown. Its height is 8 units. If the block has a volume of 288 cubic units, what is the side length of the base?
(**A**) 6 (**B**) 8 (**C**) 36
(**D**) 10 (**E**) 12



Solution
Since the volume of the rectangular block is 288 cubic units and volume is determined by: (Area of base)(Height), then the area of the base is $\frac{288}{8} = 36$. Since we have a square base, it must measure 6×6. The side length of the base is 6 units. Answer: (A)

12. A recipe requires 25 mL of butter to be used along with 125 mL of sugar. If 1000 mL of sugar is used, how much butter would be required?
(**A**) 100 mL (**B**) 500 mL (**C**) 200 mL (**D**) 3 litres (**E**) 400 mL

Solution
If 1000 mL of sugar is used, eight times as much sugar would be used as is required by the recipe. We would use 8×25 or 200 mL of butter. Answer: (C)

13. Karl had his salary reduced by 10%. He was later promoted and his salary was increased by 10%. If his original salary was \$20 000, what is his present salary?
(**A**) \$16 200 (**B**) \$19 800 (**C**) \$20 000 (**D**) \$20 500 (**E**) \$24 000

Solution
If Karl had his salary reduced by 10%, his new salary was $(0.90)(20\,000) = 18\,000$. If his salary was then increased by 10% his new salary is $(1.10)(18\,000) = 19\,800$. His salary after his 'promotion' is \$19 800.
ANSWER: (B)

14. The area of a rectangle is 12 square metres. The lengths of the sides, in metres, are whole numbers. The greatest possible perimeter (in metres) is
(**A**) 14 (**B**) 16 (**C**) 12 (**D**) 24 (**E**) 26

Solution
If the rectangle has an area of 12 square metres and its sides are whole numbers then we have only the following possibilities for the width (w), length (l) and corresponding perimeter:

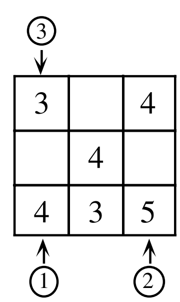
w	l	perimeter
1	12	26
2	6	16
3	4	14

The greatest possible perimeter is 26 m.
ANSWER: (E)

15. In the diagram, all rows, columns and diagonals have the sum 12. What is the sum of the four corner numbers?
(**A**) 14 (**B**) 15 (**C**) 16
(**D**) 17 (**E**) 12

		4
	4	
	3	

Solution
If we fill in the four corners in the indicated order the sum of the numbers at the corners is $4+3+4+5=16$. (This is, of course, not the only way to find the desired number. We could also have started by adding up along the centre column.)

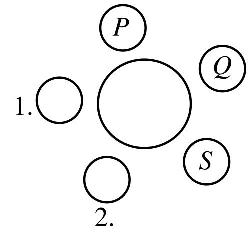


ANSWER: (C)

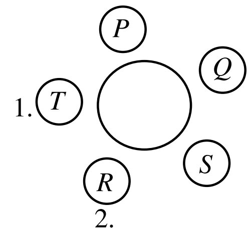
16. Paul, Quincy, Rochelle, Surinder, and Tony are sitting around a table. Quincy sits in the chair between Paul and Surinder. Tony is not beside Surinder. Who is sitting on either side of Tony?
(**A**) Paul and Rochelle (**B**) Quincy and Rochelle (**C**) Paul and Quincy
(**D**) Surinder and Quincy (**E**) Not possible to tell

Solution

If Quincy sits in the chair between Paul and Surinder then these three people would be seated as shown.



Since Tony does not sit beside Surinder then he must sit in the position labelled 1, and Rochelle must sit in the position labelled 2.



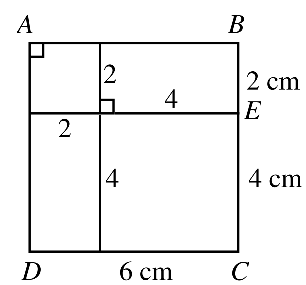
Thus, Tony is seated between Paul and Rochelle as shown in the diagram. ANSWER: (A)

17. $ABCD$ is a square that is made up of two identical rectangles and two squares of area 4 cm^2 and 16 cm^2. What is the area, in cm^2, of the square $ABCD$?

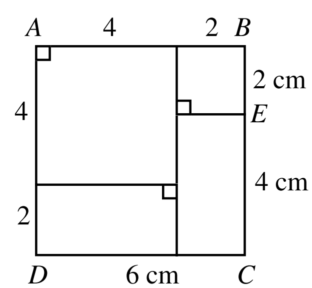
(**A**) 64 (**B**) 49 (**C**) 25 (**D**) 36 (**E**) 20

Solution

One way to draw the required square is shown in the diagram. The smaller square has a side length of 2 cm and the larger a side length of 4 cm. This gives the side length of the larger square to be 6 cm and an area of 36 cm^2.



Note that it is also possible to divide the square up as follows:

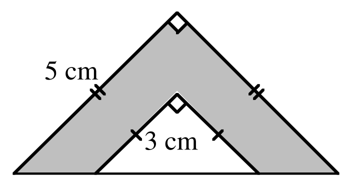


ANSWER: (D)

18. The month of April, 2000, had five Sundays. Three of them fall on even numbered days. The eighth day of this month is a
(**A**) Saturday (**B**) Sunday (**C**) Monday (**D**) Tuesday (**E**) Friday

Solution
Since three of the Sundays fall on even numbered days and two on odd numbered days this implies that the first Sunday of the month must fall on an even numbered day. Note that it is not possible for a Sunday to fall on the 4th day of the month because the 5th Sunday would then have to fall on the 32nd day of the month. The five Sundays will fall on the following days of the calendar: 2, 9, 16, 23, 30. April 8 must be a Saturday. ANSWER: (A)

19. The diagram shows two isosceles right-triangles with sides as marked. What is the area of the shaded region?
(**A**) $4.5\ \text{cm}^2$ (**B**) $8\ \text{cm}^2$ (**C**) $12.5\ \text{cm}^2$
(**D**) $16\ \text{cm}^2$ (**E**) $17\ \text{cm}^2$



Solution
The area of the larger triangle is $\frac{1}{2}(5)(5)$.

The area of the smaller triangle is $\frac{1}{2}(3)(3)$.

The shaded area is, $\frac{1}{2}(5)(5)-\frac{1}{2}(3)(3)$

$$=\frac{1}{2}(25-9)$$

$$= 8.$$

Thus the required area is $8\ \text{cm}^2$. ANSWER: (B)

20. A dishonest butcher priced his meat so that meat advertised at \$3.79 per kg was actually sold for \$4.00 per kg. He sold 1 800 kg of meat before being caught and fined \$500. By how much was he ahead or behind where he would have been had he not cheated?
(**A**) \$478 loss (**B**) \$122 loss (**C**) Breaks even (**D**) \$122 gain (**E**) \$478 gain

Solution
The butcher gained \$0.21 on each kg he sold and thus he dishonestly made $(0.21)(1800) = \$378.00$. After paying the \$500 fine, he would have a loss of $\$500 - \$378 = \$122$. ANSWER: (B)

Part C

21. In a basketball shooting competition, each competitor shoots ten balls which are numbered from 1 to 10. The number of points earned for each successful shot is equal to the number on the ball. If a competitor misses exactly two shots, which one of the following scores is not possible?
(**A**) 52 (**B**) 44 (**C**) 41 (**D**) 38 (**E**) 35

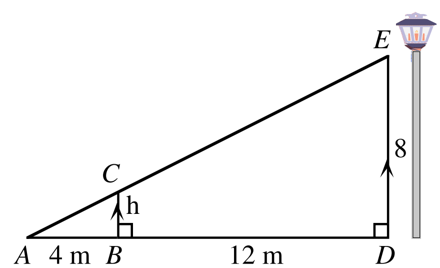
Solution
If all ten balls scored, a score of 55 is possible.
If ball 1 and 2 is missed the maximum possible score is 52. Similarly, if 9 and 10 are missed, the minimum score is 36. Every score between 36 and 52 is also possible. Of the listed scores, 35 is the only score that is not possible.

ANSWER: (E)

22. Sam is walking in a straight line towards a lamp post which is 8 m high. When he is 12 m away from the lamp post, his shadow is 4 m in length. When he is 8 m from the lamp post, what is the length of his shadow?

(**A**) $1\frac{1}{2}$ m (**B**) 2 m (**C**) $2\frac{1}{2}$ m (**D**) $2\frac{2}{3}$ m (**E**) 3 m

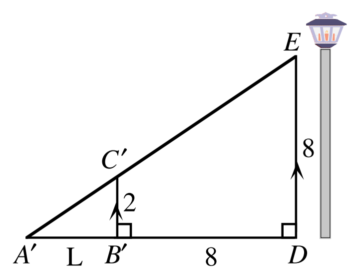
Solution
As Sam approaches the lamp post, we can visualize his position, as shown.
Since ΔABC and ΔADE are similar, the lengths of their corresponding sides are proportional. To determine Sam's height h, we solve $\frac{h}{4} = \frac{8}{16}$, and therefore $h = 2$ m.



As Sam moves to a position that is 8 m from the lamp post we now have the situation, as shown.
Using similar triangles as before, we can now calculate, L, the length of the shadow.

Thus, $\frac{L}{2} = \frac{L+8}{8}$.

Using the property of equivalent fractions, $\frac{L}{2} = \frac{4L}{8} = \frac{L+8}{8}$.



Thus, $4L = L + 8$

$3L = 8$

$L = 2\frac{2}{3}$ m

ANSWER: (D)

23. The total area of a set of different squares, arranged from smallest to largest, is 35 km^2. The smallest square has a side length of 500 m. The next larger square has a side length of 1000 m. In the same way, each successive square has its side length increased by 500 m. What is the total number of squares?

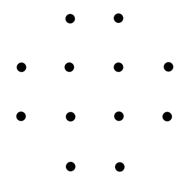
(**A**) 5 (**B**) 6 (**C**) 7 (**D**) 8 (**E**) 9

Solution
We complete the following chart, one row at a time, until 35 appears in the third column.

Number of the square	Length of the square	Area of the square	Cumulative sum of areas
1	0.5 km	0.25 km^2	0.25 km^2
2	1.0 km	1.00 km^2	1.25 km^2
3	1.5 km	2.25 km^2	3.50 km^2
4	2.0 km	4.00 km^2	7.50 km^2
5	2.5 km	6.25 km^2	13.75 km^2
6	3.0 km	9.00 km^2	22.75 km^2
7	3.5 km	12.25 km^2	35.00 km^2

Since there are seven rows, we conclude that there are seven squares. ANSWER: (C)

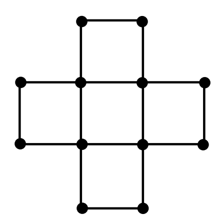
24. Twelve points are marked on a rectangular grid, as shown. How many squares can be formed by joining four of these points?



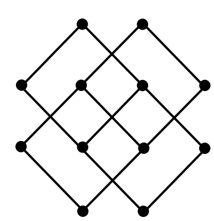
(**A**) 6 (**B**) 7 (**C**) 9
(**D**) 11 (**E**) 13

Solution
In total there are 11 possible squares as shown.

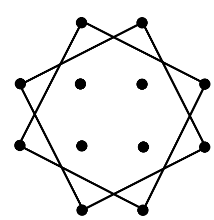
5 small squares:



4 large squares:

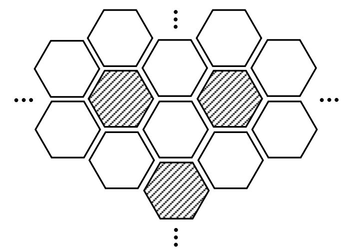


2 that are larger yet:



ANSWER: (D)

25. A square floor is tiled, as partially shown, with a large number of regular hexagonal tiles. The tiles are coloured blue or white. Each blue tile is surrounded by 6 white tiles and each white tile is surrounded by 3 white and 3 blue tiles. Ignoring part tiles, the ratio of the number of blue tiles to the number of white tiles is closest to



(**A**) 1:6 (**B**) 2:3 (**C**) 3:10
(**D**) 1:4 (**E**) 1:2

Solution
Let's start by considering seven tile configurations made up of one blue tile surrounded by six white tiles. If we look just at this tiling only in this way, it appears that there are six times as many white tiles as blue tiles. However, each white tile is adjacent to three different blue tiles. This means that every white tile is part of three different seven tile configurations. Thus, if we count white tiles as simply six times the number counted we will miss the fact that each white tile has been triple counted. Hence the number of white tiles is six times the number of blue tiles divided by three, or twice the number of blue tiles. The ratio of the number of blue tiles to the number of white tiles is 1:2.

ANSWER: (E)

✺ ✺ ✺ ✺ ✺

Part A

1. The value of 2^5+5 is

(**A**) 20 (**B**) 37 (**C**) 11 (**D**) 13 (**E**) 21

Solution

$2\times2\times2\times2\times2+5=37$

ANSWER: (B)

2. A number is placed in the box to make the following statement true: $8+\frac{7}{\square}+\frac{3}{1000}=8.073$. What is this number?

(**A**) 1000 (**B**) 100 (**C**) 1 (**D**) 10 (**E**) 70

Solution

Since $8.073=8+\frac{0}{10}+\frac{7}{100}+\frac{3}{100}$, the missing number is 100.

ANSWER: (B)

3. The value of $\dfrac{5+4-3}{5+4+3}$ is

(**A**) -1 (**B**) $\frac{1}{3}$ (**C**) 2 (**D**) $\frac{1}{2}$ (**E**) $-\frac{1}{2}$

Solution

$$\frac{5+4-3}{5+4+3}=\frac{6}{12}=\frac{1}{2}$$

ANSWER: (D)

4. In the addition shown, a digit, either the same or different, can be placed in each of the two boxes. What is the sum of the two missing digits?

$$\begin{array}{r} 8\ 6\ 3 \\ \square\ 9\ 1 \\ 7\ \square\ 8 \\ \hline 2\ 1\ 8\ 2 \end{array}$$

(**A**) 9 (**B**) 11 (**C**) 13
(**D**) 3 (**E**) 7

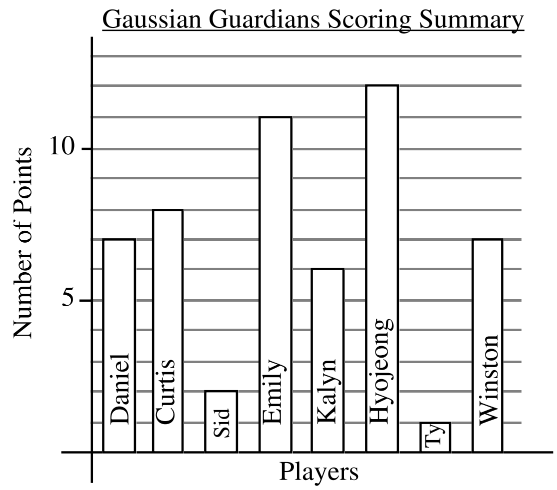
Solution

Adding in the units column gives us, $3+1+8=12$. This means a carry over of 1 into the tens column since $12=1\times10+2$. In the tens column, we have 1 (carried over) $+6+9+\square=18$. The digit that is placed in this box is 2 with a carry over of 1 unit into the hundreds column. Moving to the hundreds column we have, 1 (carried over) $+8+\square+7=21$. The missing digit here is 5. The two missing digits are 2 and 5 giving a sum of 7.

ANSWER: (E)

5. The graph shows the complete scoring summary for the last game played by the eight players on Gaussian Guardians intramural basketball team. The total number of points scored by the Gaussian Guardians was

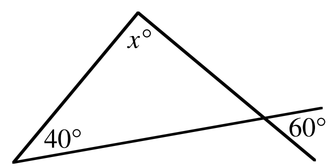
(**A**) 54 (**B**) 8 (**C**) 12
(**D**) 58 (**E**) 46



Solution
If we list all the players with their points, we would have the following: Daniel (7), Curtis (8), Sid (2), Emily (11), Kalyn (6), Hyojeong (12), Ty (1) and Winston (7).
The total is, $7+8+2+11+6+12+1+7=54$. ANSWER: (A)

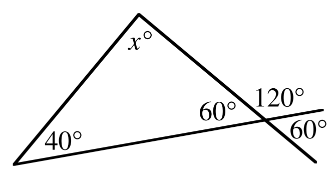
6. In the given diagram, what is the value of x?

(**A**) 20 (**B**) 80 (**C**) 100
(**D**) 120 (**E**) 60



Solution
From the given diagram, we can label the supplementary angle 120° and the vertically opposite angle 60°.
Since the angles in a triangle have a sum of 180°,

$$x = 180 - (40 + 60)$$
$$x = 80.$$



ANSWER: (B)

7. During the week, the Toronto Stock Exchange made the following gains and losses:

Monday	-150	Thursday	$+182$
Tuesday	$+106$	Friday	-210
Wednesday	-47		

What was the net change for the week?

(**A**) a loss of 119 (**B**) a gain of 119 (**C**) a gain of 91
(**D**) a loss of 91 (**E**) a gain of 695

Solution
$-150+106-47+182-210=-119$
Thus, the net change was a loss of 119 for the week. ANSWER: (A)

8. If $x*y=x+y^2$, then $2*3$ equals
(**A**) 8 (**B**) 25 (**C**) 11 (**D**) 13 (**E**) 7

Solution
$2*3^2=2+3^2=11$ ANSWER: (C)

9. Of the following five statements, how many are correct?
(i) 20% of $40=8$ (ii) $2^3=8$ (iii) $7-3\times 2=8$ (iv) $3^2-1^2=8$ (v) $2(6-4)^2=8$
(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

Solution
(i) True, $\frac{1}{5}\times 40=8$
(ii) True, $2^3=2\times 2\times 2=8$
(iii) False, $7-3\times 2=7-6=1$
(iv) True, $9-1=8$
(v) True, $2(2)^2=8$
Only (iii) is false. There are four correct statements. ANSWER: (D)

10. Karl had his salary reduced by 10%. He was later promoted and his salary was increased by 10%. If his original salary was \$20 000, what is his present salary?
(**A**) \$16 200 (**B**) \$19 800 (**C**) \$20 000 (**D**) \$20 500 (**E**) \$24 000

Solution
If Karl had his salary reduced by 10%, his new salary was $(0.90)(20\,000)=18\,000$. If his salary was then increased by 10% his new salary is $(1.10)(18\,000)=19\,800$. His salary after his 'promotion' is \$19 800. ANSWER: (B)

Part B

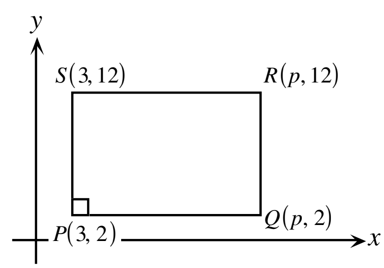
11. Pat planned to place patio stones in a rectangular garden that has dimensions 15 m by 2 m. If each patio stone measures 0.5 m by 0.5 m, how many stones are needed to cover the garden?
(**A**) 240 (**B**) 180 (**C**) 120 (**D**) 60 (**E**) 30

Solution
The garden has an area of 30 m^2.
Each patio stone has an area of $(0.5)(0.5) = 0.25 \text{ m}^2$.
Pat will need $\frac{30}{0.25}$ or 120 patio stones. ANSWER: (C)

12. The prime numbers between 10 and 20 are added together to form the number Q. What is the largest prime divisor of Q?
(**A**) 2 (**B**) 3 (**C**) 5 (**D**) 7 (**E**) 11

Solution
The prime numbers between 10 and 20 are: 11, 13, 17, and 19.
And so, $Q = 11 + 13 + 17 + 19 = 60$.
Since $60 = 2 \times 2 \times 3 \times 5$, the largest prime divisor of Q is 5. ANSWER: (C)

13. The coordinates of the vertices of rectangle $PQRS$ are given in the diagram. The area of rectangle $PQRS$ is 120. The value of p is
(**A**) 10 (**B**) 12 (**C**) 13
(**D**) 14 (**E**) 15



Solution 1
$PS = 12 - 2 = 10$
Since the area of the rectangle is 120,

$$(PS)(PQ) = 120$$
$$(10)(PQ) = 120$$
$$PQ = 12.$$

Therefore, $p = 3 + 12 = 15$.

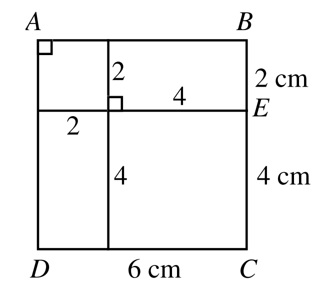
Solution 2
The dimensions of the rectangle are $(p - 3) \times 10$.
Since the area is 120, $10(p - 3) = 120$.
Thus, $p - 3 = 12$
or $p = 15$. ANSWER: (E)

14. A set of five different positive integers has an average (arithmetic mean) of 11. What is the largest possible number in this set?
(**A**) 45 (**B**) 40 (**C**) 35 (**D**) 44 (**E**) 46

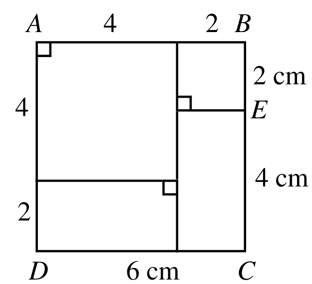
Solution
If the set of five different positive integers has an average of 11 the five integers must sum to 5×11 or 55. The four smallest possible integers are 1, 2, 3, and 4. The largest possible integer in the set is $55 - (1+2+3+4) = 45$. ANSWER: (A)

15. $ABCD$ is a square that is made up of two identical rectangles and two squares of area 4 cm^2 and 16 cm^2. What is the area, in cm^2, of the square $ABCD$?
(**A**) 64 (**B**) 49 (**C**) 25 (**D**) 36 (**E**) 20

Solution
One way to draw the required square is shown in the diagram. The smaller square has a side length of 2 cm and the larger a side length of 4 cm. This gives the side length of the larger square to be 6 cm and an area of 36 cm^2.



Note that it is also possible to divide the square up as follows:



ANSWER: (D)

16. Three tenths of our planet Earth is covered with land and the rest is covered with water. Ninety-seven percent of the water is salt water and the rest is fresh water. What percentage of the Earth is covered in fresh water?
(**A**) 20.1% (**B**) 79.9% (**C**) 32.1% (**D**) 2.1% (**E**) 9.6%

Solution
If three tenths of Earth is covered with land then seven tenths or 70% is covered with water. If 97% of this water is salt water then just 3% is fresh water. This implies that 3% of 70% or $(0.03)(0.7) = 0.021 = 2.1\%$ of the Earth is covered in fresh water. ANSWER: (D)

17. In a certain month, three of the Sundays have dates that are even numbers. The tenth day of this month is a
(**A**) Saturday (**B**) Sunday (**C**) Monday (**D**) Tuesday (**E**) Wednesday

Solution
A Sunday must occur during the first three days of any month with five Sundays. Since it is on an even day, it must be on the second day of the month. This implies that the ninth day of the month is also a Sunday, which makes the tenth day a Monday. ANSWER: (C)

18. Jim drives 60 km south, 40 km west, 20 km north, and 10 km east. What is the distance from his starting point to his finishing point?
(**A**) 30 km (**B**) 50 km (**C**) 40 km (**D**) 70 km (**E**) 35 km

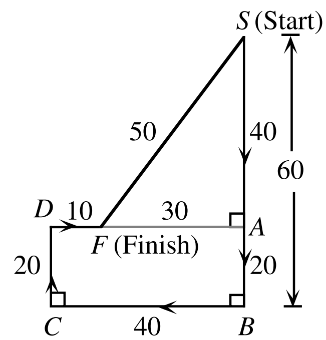
Solution
We can see that Jim's finishing point *F* is 40 km south and 30 km west of his starting point, *S*.

By Pythagoras, $AE^2 = 30^2 + 40^2$

$AE^2 = 2500$

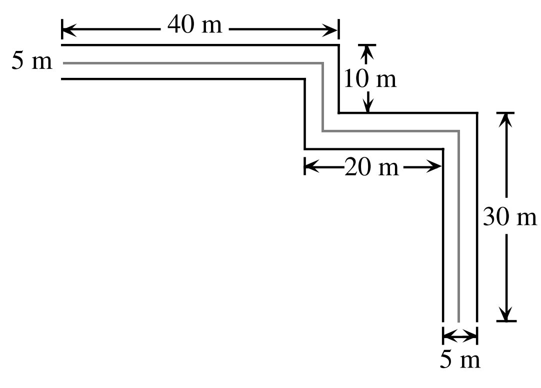
$AE = 50.$

The distance from his starting point to his end point is 50 km.



ANSWER: (B)

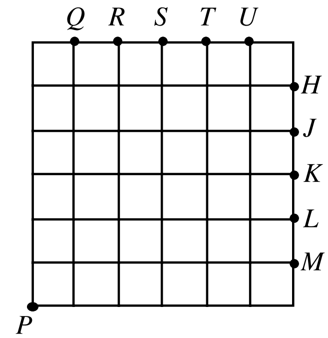
19. A paved pedestrian path is 5 metres wide. A yellow line is painted down the middle. If the edges of the path measure 40 m, 10 m, 20 m, and 30 m, as shown, what is the length of the yellow line?
(**A**) 100 m (**B**) 97.5 m (**C**) 95 m
(**D**) 92.5 m (**E**) 90 m



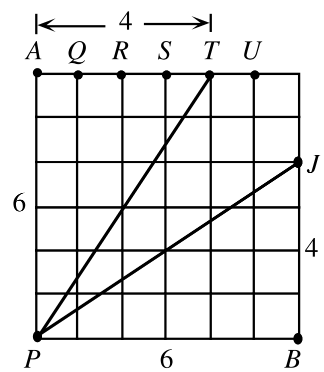
Solution
Since the path is 5 metres wide, a line in the middle is always 2.5 m from its edges.
Thus the total length is, $37.5 + 10 + 20 + 27.5$
$= 95$ m ANSWER: (C)

20. In the 6 by 6 grid shown, two lines are drawn through point P, dividing the grid into three regions of equal area. These lines will pass through the points

(**A**) M and Q (**B**) L and R (**C**) K and S
(**D**) H and U (**E**) J and T



Solution
Label points A and B as shown.
The area of the whole square is 36.
Since the square is divided into three equal areas, each area must be, $\frac{36}{3} = 12$.
The first required point must be one of the points from Q to U. It would have to be a part of a right triangle which would have AP as its height (or its base). Since $AP = 6$ then the base of the triangle would have to be 4 since $\frac{1}{2}(6)(4) = 12$, T is the only point that meets the requirement. In the same way, J also meets the requirement. The required points are thus J and T.



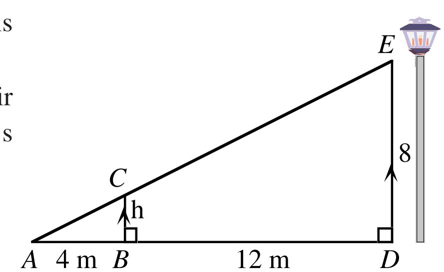
ANSWER: (E)

Part C

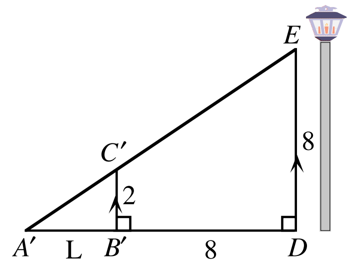
21. Sam is walking in a straight line towards a lamp post which is 8 m high. When he is 12 m away from the lamp post, his shadow is 4 m in length. When he is 8 m from the lamp post, what is the length of his shadow?

(**A**) $1\frac{1}{2}$ m (**B**) 2 m (**C**) $2\frac{1}{2}$ m (**D**) $2\frac{2}{3}$ m (**E**) 3 m

Solution
As Sam approaches the lamp post, we can visualize his position, as shown.
Since ΔABC and ΔADE are similar, the lengths of their corresponding sides are proportional. To determine Sam's height h, we solve $\frac{h}{4} = \frac{8}{16}$, and therefore h = 2 m.



As Sam moves to a position that is 8 m from the lamp post we now have the situation, as shown.
Using similar triangles as before, we can now calculate, L, the length of the shadow.



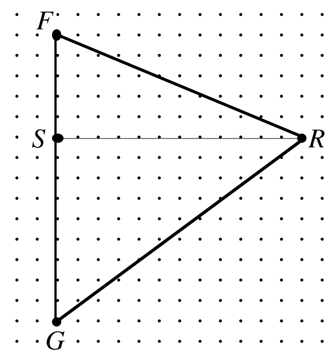
Thus, $\frac{L}{2} = \frac{L+8}{8}$.

Using the property of equivalent fractions, $\frac{L}{2} = \frac{4L}{8} = \frac{L+8}{8}$.

Thus, $4L = L + 8$
$3L = 8$
$L = 2\frac{2}{3}$ m

Answer: (D)

22. The homes of Fred (F), Sandy (S), Robert (R), and Guy (G) are marked on the rectangular grid with straight lines joining them. Fred is considering four routes to visit each of his friends:



(i) $F \to R \to S \to G$ (ii) $F \to S \to G \to R$
(iii) $F \to R \to G \to S$ (iv) $F \to S \to R \to G$

If $FS = 5$ km, $SG = 9$ km and $SR = 12$ km, the difference between the longest and the shortest trip (in km) is

(**A**) 8 (**B**) 13 (**C**) 15
(**D**) 2 (**E**) 0

Solution

$FS = 5,\ SR = 12 \Rightarrow FR = 13$. (By Pythagoras, $FR^2 = 5^2 + 12^2 = 169$)

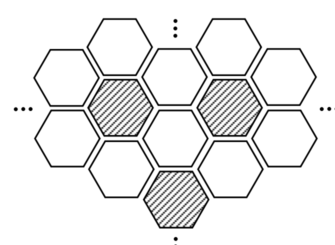
$SG = 9,\ SR = 12 \Rightarrow GR = 15$. (By Pythagoras, $GR^2 = 9^2 + 12^2 = 225$)

(i) $FR + RS + SG = 13 + 12 + 9 = 34$ km
(ii) $FS + SG + GR = 5 + 9 + 15 = 29$ km
(iii) $FR + RG + GS = 13 + 15 + 9 = 37$ km
(iv) $FS + SR + RG = 5 + 12 + 15 = 32$ km

$37 - 29 = 8$ km is the required distance.

Answer: (A)

23. A square floor is tiled, as partially shown, with a large number of regular hexagonal tiles. The tiles are coloured blue or white. Each blue tile is surrounded by 6 white tiles and each white tile is surrounded by 3 white and 3 blue tiles. Ignoring part tiles, the ratio of the number of blue tiles to the number of white tiles is closest to



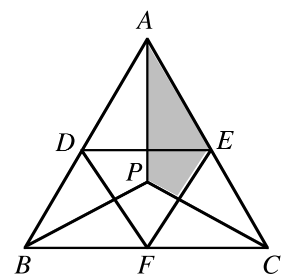
(**A**) 1:6 (**B**) 2:3 (**C**) 3:10
(**D**) 1:4 (**E**) 1:2

Solution

Let's start by considering seven tile configurations made up of one blue tile surrounded by six white tiles. If we look just at this tiling only in this way, it appears that there are six times as many white tiles as blue tiles. However, each white tile is adjacent to three different blue tiles. This means that every white tile is part of three different seven tile configurations. Thus, if we count white tiles as simply six times the number counted we will miss the fact that each white tile has been triple counted. Hence the number of white tiles is six times the number of blue tiles divided by three, or twice the number of blue tiles. The ratio of the number of blue tiles to the number of white tiles is 1:2.

ANSWER: (E)

24. In equilateral triangle ABC, line segments are drawn from a point P to the vertices A, B and C to form three identical triangles. The points D, E and F are the midpoints of the three sides and they are joined as shown in the diagram. What fraction of ΔABC is shaded?

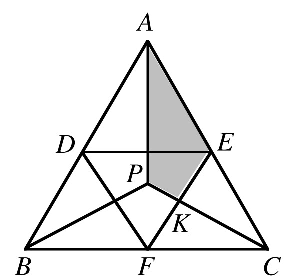


(**A**) $\frac{1}{5}$ (**B**) $\frac{5}{24}$ (**C**) $\frac{1}{4}$

(**D**) $\frac{2}{9}$ (**E**) $\frac{2}{7}$

Solution 1

Since P is a point of symmetry within ΔABC, the line segment CP divides ΔECF into 2 triangles of equal area. That is to say, the area of ΔEKC equals the area of ΔFKC. Since the area of ΔEFC is $\frac{1}{4}$ the area of ΔABC, the area of



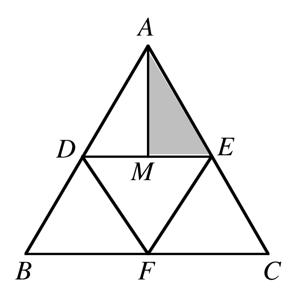
$$\Delta EKC = \left(\frac{1}{2} \times \frac{1}{4}\right) \text{ area of } \Delta ABC$$
$$= \frac{1}{8} \text{ (area of } \Delta ABC).$$

Again since P is a point of symmetry within ΔABC, the area of ΔAPC is $\frac{1}{3}$ the area of ΔABC.

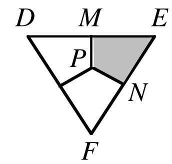
Since the shaded area is the area of ΔAPC – area of ΔKCE, it represents $\left(\frac{1}{3} - \frac{1}{8}\right) \times$ area of $\Delta ABC = \frac{5}{24} \times$ area of ΔABC.

Solution 2

Since D, E and F are the midpoints of the sides, we have four triangles of exactly the same area. That is to say, the areas of ΔADE, ΔDBF, ΔDEF, and ΔEFC are equal. Since ΔAME equals half the area of ΔADE, it represents $\frac{1}{8}$th the area of ΔABC.



Since the figure $MENP$ is one of three identical shapes making up ΔDEF it is one third its area. Since ΔDEF itself is one quarter the area of ΔABC, the figure $MENP$ is $\frac{1}{3}\times\frac{1}{4}$ or $\frac{1}{12}$th the area of ΔABC. Overall, the shaded area is $\frac{1}{8}+\frac{1}{12}=\frac{5}{24}$th the area of ΔABC.



ANSWER: (B)

25. The cookies in a jar contain a total of 1000 chocolate chips. All but one of these cookies contains the same number of chips; it contains one more chip than the others. The number of cookies in the jar is between one dozen and three dozen. What is the sum of the number of cookies in the jar and the number of chips in the cookie with the extra chocolate chip?

(**A**) 65 (**B**) 64 (**C**) 63 (**D**) 66 (**E**) 67

Solution

If we remove the extra chip from the special cookie, all cookies have the same number of chocolate chips for a total of 999 chips. We look at factorizations of 999.

The question states that the number of cookies in the jar is between 12 and 36 so this implies that the only factorization of 999 that works is $(3\times3\times3)(37)$.

Thus the only divisor of 999 between 12 and 36 is 27.

From this, we see that there are 27 cookies.

An ordinary cookie has $\frac{999}{27}=37$ chocolate chips, and the special cookie has 38 chocolate chips.

The required sum is $27+38=65$. ANSWER: (A)

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Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

1999 Solutions

Gauss Contest

(Grades 7 and 8)

GRADE 7

Part A

1. $1999 - 999 + 99$ equals

(**A**) 901 (**B**) 1099 (**C**) 1000 (**D**) 199 (**E**) 99

Solution

$1999 - 999 + 99$

$= 1000 + 99$

$= 1099$

ANSWER: (B)

2. The integer 287 is exactly divisible by

(**A**) 3 (**B**) 4 (**C**) 5 (**D**) 7 (**E**) 6

Solution 1

$\frac{287}{7} = 41$

Solution 2

If we think in terms of divisibility tests we see that:

287 is not divisible by 3 because $2 + 8 + 7 = 17$ is not a multiple of 3;

287 is not divisible by 4 because 87 is not divisible by 4;

287 is not divisible by 5 because it doesn't end in 0 or 5;

287 is divisible by 7 because $287 = 7 \times 41$;

287 is not divisible by 6 because it is not even and is not divisible by 3.

ANSWER: (D)

3. Susan wants to place 35.5 kg of sugar in small bags. If each bag holds 0.5 kg, how many bags are needed?

(**A**) 36 (**B**) 18 (**C**) 53 (**D**) 70 (**E**) 71

Solution

Number of bags = $\frac{35.5}{.5} = \frac{355}{5} = 71$.

ANSWER: (E)

4. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ is equal to

(**A**) $\frac{15}{8}$ (**B**) $1\frac{3}{14}$ (**C**) $\frac{11}{8}$ (**D**) $1\frac{3}{4}$ (**E**) $\frac{7}{8}$

Solution

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{8 + 4 + 2 + 1}{8} = \frac{15}{8}$$

ANSWER: (A)

5. Which one of the following gives an odd integer?

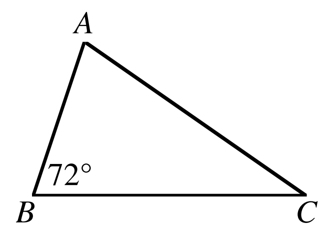
(**A**) 6^2 (**B**) $23 - 17$ (**C**) 9×24 (**D**) $96 \div 8$ (**E**) 9×41

Solution 1
$6^2 = 36$, $23 - 17 = 6$, $9 \times 24 = 216$, $96 \div 8 = 12$, $9 \times 41 = 369$

Solution 2
If we think in terms of even and odd integers we have the following:
(A) (even)(even) = even
(B) odd – odd = even
(C) (odd)(even) = even
(D) (even) ÷ (even) = even or odd (It is necessary to evaluate)
(E) (odd)(odd) = odd

ANSWER: (E)

6. In ΔABC, $\angle B = 72°$. What is the sum, in degrees, of the other two angles?
(**A**) 144 (**B**) 72 (**C**) 108
(**D**) 110 (**E**) 288



Solution
There are $180°$ in a triangle.
Therefore, $\angle A + \angle C + 72° = 180°$ ($\angle A$, $\angle B$ are in degrees.)
$\angle A + \angle C = 108°$.

ANSWER: (C)

7. If the numbers $\frac{4}{5}$, 81% and 0.801 are arranged from smallest to largest, the correct order is
(**A**) $\frac{4}{5}$, 81%, 0.801 (**B**) 81%, 0.801, $\frac{4}{5}$ (**C**) 0.801, $\frac{4}{5}$, 81%

(**D**) 81%, $\frac{4}{5}$, 0.801 (**E**) $\frac{4}{5}$, 0.801, 81%

Solution
In decimal form, $\frac{4}{5} = .80$ and 81% = .81.
Arranging the given numbers from smallest to largest, we have $\frac{4}{5}$, 0.801, .81.

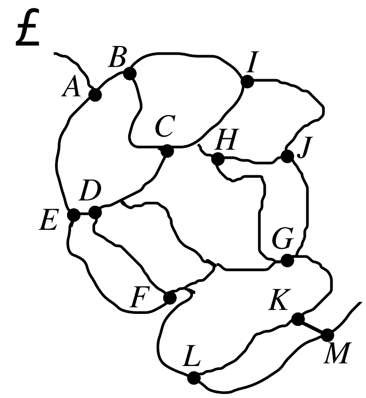
ANSWER: (E)

8. The average of 10, 4, 8, 7, and 6 is
(**A**) 33 (**B**)13 (**C**) 35 (**D**) 10 (**E**) 7

Solution
$\frac{10+4+8+7+6}{5} = \frac{35}{5} = 7$

ANSWER: (E)

9. André is hiking on the paths shown in the map. He is planning to visit sites A to M in alphabetical order. He can never retrace his steps and he must proceed directly from one site to the next. What is the largest number of labelled points he can visit before going out of alphabetical order?

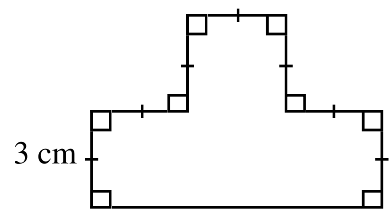


(**A**) 6 (**B**) 7 (**C**) 8
(**D**) 10 (**E**) 13

Solution
If we trace André's route we can see that he can travel to site J travelling in alphabetical order.
Once he reaches site J, it is not possible to reach K without passing through G or retracing his steps.
Since J is the tenth letter in the alphabet, he can visit ten sites before going out of order.

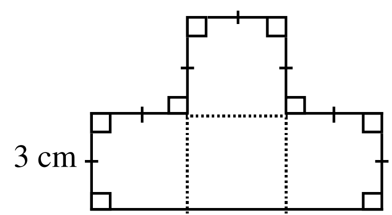
ANSWER: (D)

10. In the diagram, line segments meet at 90° as shown. If the short line segments are each 3 cm long, what is the area of the shape?



(**A**) 30 (**B**) 36 (**C**) 40
(**D**) 45 (**E**) 54

Solution
Each of the four squares are identical and each has an area of 3×3 or $9\ \text{cm}^2$.
The total area is thus 4×9 or $36\ \text{cm}^2$.

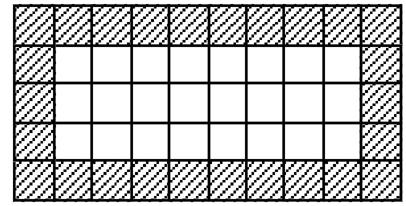


ANSWER: (B)

Part B

11. The floor of a rectangular room is covered with square tiles. The room is 10 tiles long and 5 tiles wide. The number of tiles that touch the walls of the room is
(**A**) 26 (**B**) 30 (**C**) 34 (**D**) 46 (**E**) 50

Solution



If we draw a grid that is 10×5, it is easy to count the number of tiles that touch the walls. From the diagram we can see that there are 26 tiles that touch the walls. Notice that if the question had said a length of l units (l an integer) and a width of w units (w an integer) we would arrive at the formula: $2w + 2l - 4$ where 4 represents the 4 corner tiles which would have been double counted. In this question, it would just be, $20 + 10 - 4 = 26$.

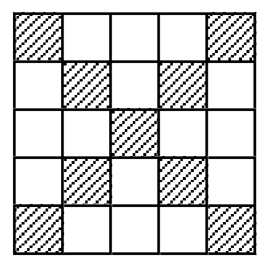
ANSWER: (A)

12. Five students named Fred, Gail, Henry, Iggy, and Joan are seated around a circular table in that order. To decide who goes first in a game, they play "countdown". Henry starts by saying '34', with Iggy saying '33'. If they continue to count down in their circular order, who will eventually say '1'?
(**A**) Fred (**B**) Gail (**C**) Henry (**D**) Iggy (**E**) Joan

Solution

This is an interesting question that mathematicians usually refer to as modular arithmetic. Henry starts by saying '34' and always says the number, $34 - 5n$, where n is a positive integer starting at 1. In other words, he says '34', '29', ..., '9', '4'. This implies Henry says '4', Iggy says '3', Joan says '2' and Fred says '1'. ANSWER: (A)

13. In the diagram, the percentage of small squares that are shaded is
(**A**) 9 (**B**) 33 (**C**) 36
(**D**) 56.25 (**E**) 64



Solution

There are 9 shaded squares out of a possible 25.
This represents, $\frac{9}{25}$ or 36%. ANSWER: (C)

14. Which of the following numbers is an odd integer, contains the digit 5, is divisible by 3, and lies between 12^2 and 13^2?
(**A**) 105 (**B**) 147 (**C**) 156 (**D**) 165 (**E**) 175

Solution

Since $12^2 = 144$ and $13^2 = 169$, we can immediately eliminate 105 and 175 as possibilities.
Since 156 is even it can also be eliminated. The only possibilities left are 147 and 165 but since 147 does not contain a 5 it can also be eliminated. The only candidate left is 165 and it can easily be checked that it meets the requirements of the question. ANSWER: (D)

15. A box contains 36 pink, 18 blue, 9 green, 6 red, and 3 purple cubes that are identical in size. If a cube

is selected at random, what is the probability that it is green?

(**A**) $\frac{1}{9}$ (**B**) $\frac{1}{8}$ (**C**) $\frac{1}{5}$ (**D**) $\frac{1}{4}$ (**E**) $\frac{9}{70}$

Solution

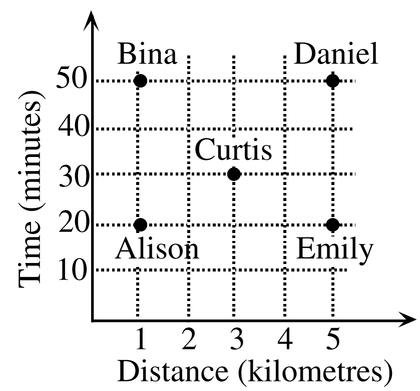
In total there are 72 cubes that have the same size.

Since there are 9 green cubes, the probability of selecting a green cube is $\frac{9}{72}$ or $\frac{1}{8}$.

ANSWER: (B)

16. The graph shown at the right indicates the time taken by five people to travel various distances. On average, which person travelled the fastest?

(**A**) Alison (**B**) Bina (**C**) Curtis
(**D**) Daniel (**E**) Emily



Solution

We summarize the results for the five people in the table.

We recall that average $\text{speed} = \frac{\text{distance}}{\text{time}}$.

	Distance	Time (minutes)	Speed (km/min.)
Alison	1	20	$\frac{1}{20} = 0.05$
Bina	1	50	$\frac{1}{50} = 0.02$
Curtis	3	30	$\frac{3}{30} = \frac{1}{10} = 0.1$
Daniel	5	50	$\frac{5}{50} = 0.1$
Emily	5	20	$\frac{5}{20} = 0.25$

Emily is the fastest.

ANSWER: (E)

17. In a “Fibonacci” sequence of numbers, each term beginning with the third, is the sum of the previous two terms. The first number in such a sequence is 2 and the third is 9. What is the eighth term in the sequence?

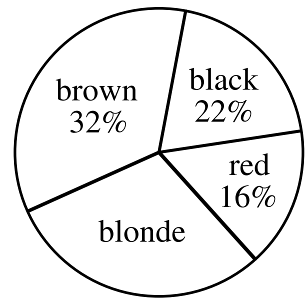
(**A**) 34 (**B**) 36 (**C**) 107 (**D**) 152 (**E**) 245

Solution
If the first number in the sequence is 2 and the third is 9, the second number in the sequence must be 7. The sequence is thus: 2, 7, 9, 16, 25, 41, 66, 107. The eighth term is 107.

ANSWER: (C)

18. The results of a survey of the hair colour of 600 people are shown in this circle graph. How many people have blonde hair?

(**A**) 30 (**B**) 160 (**C**) 180
(**D**) 200 (**E**) 420

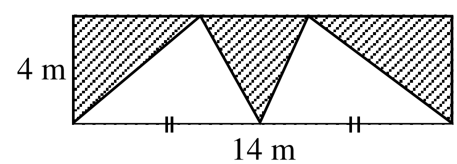


Hair Colour

Solution
From the diagram, blondes represent 30% of the 600 people.
Since 30% of $600 = (.3)(600) = 180$, there are 180 blondes in the survey. ANSWER: (C)

19. What is the area, in m^2, of the shaded part of the rectangle?

(**A**) 14 (**B**) 28 (**C**) 33.6
(**D**) 56 (**E**) 42



Solution
The two unshaded triangles each have a base of 7 m and a height of 4 m. This means that each of the triangles has an area of $\frac{7 \times 4}{2} = 14\ \mathrm{m}^2$. The two triangles thus have a total area of $28\ \mathrm{m}^2$.
The shaded triangles have an area of $56 - 28 = 28\ \mathrm{m}^2$. ANSWER: (B)

20. The first 9 positive odd integers are placed in the magic square so that the sum of the numbers in each row, column and diagonal are equal. Find the value of $A + E$.

(**A**) 32 (**B**) 28 (**C**) 26
(**D**) 24 (**E**) 16

A	1	B
5	C	13
D	E	3

Solution
The first nine odd positive integers sum to 81.
This implies that the sum of each column is $\frac{81}{3}$ or 27. From this we immediately see that $B = 11$ since $B + 13 + 3 = 27$. If we continue with the constraint that each row or column must add to 27 then $A = 15 \rightarrow D = 7 \rightarrow E = 17$. Therefore, $A + E = 15 + 17 = 32$. ANSWER: (A)

Part C

21. A game is played on the board shown. In this game, a player can move three places in any direction (up, down, right or left) and then can move two places in a direction perpendicular to the first move. If a player starts at S, which position on the board (P, Q, R, T, or W) cannot be reached through any sequence of moves?

		P		
	Q		R	
		T		
S				W

(**A**) P (**B**) Q (**C**) R
(**D**) T (**E**) W

Solution
If S is the starting position we can reach position R immediately. From S we can also reach P and then W and Q in sequence. To reach position T, it would have to be reached from the upper right or upper left square. There is no way for us to reach these two squares unless we are allowed to move outside the large square which is not permitted.

ANSWER: (D)

22. Forty-two cubes with 1 cm edges are glued together to form a solid rectangular block. If the perimeter of the base of the block is 18 cm, then the height, in cm, is

(**A**) 1 (**B**) 2 (**C**) $\frac{7}{3}$ (**D**) 3 (**E**) 4

Solution 1
Since we have a solid rectangular block with a volume of 42, its dimensions could be, $42\times1\times1$ or $6\times7\times1$ or $21\times2\times1$ or $2\times3\times7$ or $14\times3\times1$.
The only selection which has two factors adding to 9 is $2\times3\times7$, thus giving the base a perimeter of $2(2+7)=18$ which is required.
So the base is 2×7 and the height is 3.

Solution 2
Since the perimeter of the base is 18 cm, the length and width can only be one of the following:

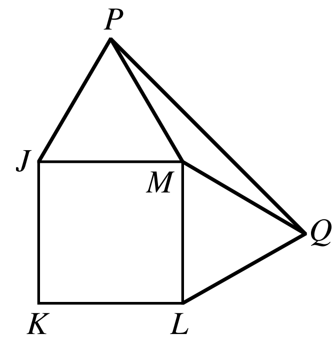
L	W
8	1
7	2
6	3
5	4

If the height is h, these cases lead to the following:
$8\times1\times h=42$, $7\times2\times h=42$, $6\times3\times h=42$ or $5\times4\times h=42$.
The only possible value of h which is an integer is $h=3$, with $L=7$ and $W=2$.

ANSWER: (D)

23. $JKLM$ is a square. Points P and Q are outside the square such that triangles JMP and MLQ are both equilateral. The size, in degrees, of angle PQM is

(**A**) 10 (**B**) 15 (**C**) 25
(**D**) 30 (**E**) 150



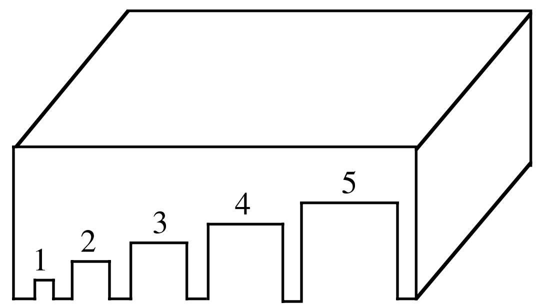
Solution

$\angle PMQ = 360° - 90° - 60° - 60° = 150°$

Since ΔPQM is isosceles, $\angle PQM = \dfrac{180° - 150°}{2} = 15°$.

ANSWER: (B)

24. Five holes of increasing size are cut along the edge of one face of a box as shown. The number of points scored when a marble is rolled through that hole is the number above the hole. There are three sizes of marbles: small, medium and large. The small marbles fit through any of the holes, the medium fit only through holes 3, 4 and 5 and the large fit only through hole 5. You may choose up to 10 marbles of each size to roll and every rolled marble goes through a hole. For a score of 23, what is the maximum number of marbles that could have been rolled?



(**A**) 12 (**B**) 13 (**C**) 14
(**D**) 15 (**E**) 16

Solution

We are looking for a ***maximum*** so we want to use lots of marbles. Let's start with 10 small ones. If they all go through hole #1, we have $23 - 10 = 13$ points to be divided between medium and large marbles. We could use 2 large and 1 medium $(5+5+3=13)$ and thus use $10+3=13$ marbles or we could use 4 medium and have one of these go through hole #4 $(3+3+3+4=13)$ which gives 14 marbles. Alternatively, of the 10 small marbles, if 9 go through hole #1 and 1 goes through hole #2, we have scored 11 points. The 4 medium marbles can now go through hole #3 giving a score of $11 + 3 \times 4 = 23$. This again gives a total of 14 marbles. ANSWER: (C)

25. In a softball league, after each team has played every other team 4 times, the total accumulated points are: Lions 22, Tigers 19, Mounties 14, and Royals 12. If each team received 3 points for a win, 1 point for a tie and no points for a loss, how many games ended in a tie?

(**A**) 3 (**B**) 4 (**C**) 5 (**D**) 7 (**E**) 10

Solution

When every team plays every other team there are $3+2+1=6$ games. Since each team plays each of the other teams 4 times, there are $4(6)=24$ games.

When there is a winner 3 points are awarded. If each of the 24 games had winners there would be $24\times 3=72$ points awarded. The actual point total is $22+19+14+12=67$.

When there are ties, only $1+1=2$ points are awarded and so every point below 72 represents a tie.

Thus, the number of ties is $72-67=5$. ANSWER: (C)

GRADE 8

Part A

1. $10^3 + 10^2 + 10$ equals

(**A**) 1110 (**B**) 101 010 (**C**) 111 (**D**) 100 010 010 (**E**) 11 010

Solution

$10^3 + 10^2 + 10$

$= 1000 + 100 + 10$

$= 1110$

ANSWER: (A)

2. $\frac{1}{2} + \frac{1}{3}$ is equal to

(**A**) $\frac{2}{5}$ (**B**) $\frac{1}{6}$ (**C**) $\frac{1}{5}$ (**D**) $\frac{3}{2}$ (**E**) $\frac{5}{6}$

Solution 1

$\frac{1}{2} + \frac{1}{3}$

$= \frac{3}{6} + \frac{2}{6}$

$= \frac{5}{6}$

ANSWER: (E)

3. Which one of the following gives an odd integer?

(**A**) 6^2 (**B**) $23 - 17$ (**C**) 9×24 (**D**) 9×41 (**E**) $96 \div 8$

Solution 1

We can calculate each of the answers directly.

(**A**) $6^2 = 36$ (**B**) $23 - 17 = 6$ (**C**) $9 \times 24 = 216$ (**D**) $9 \times 41 = 369$ (**E**) $96 \div 8 = 12$

Solution 2

If we think in terms of even and odd integers we have the following:

(**A**) (even)(even) = even

(**B**) odd – odd = even

(**C**) (odd)(even) = even

(**D**) (odd)(odd) = odd

(**E**) (even) ÷ (even) = even or odd - the result must be calculated.

ANSWER: (D)

4. What is the remainder when 82 460 is divided by 8?

(**A**) 0 (**B**) 5 (**C**) 4 (**D**) 7 (**E**) 2

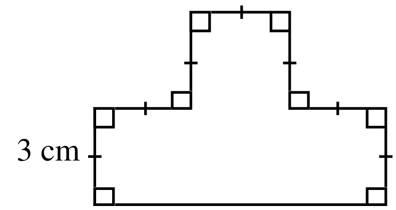
Solution

When considering division by 8, it is only necessary to consider division using the last 3 digits.

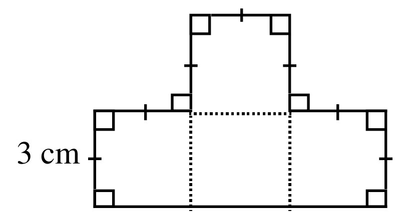
In essence, then, we are asking for the remainder when 460 is divided by 8.

Since $460 = 8 \times 57 + 4$, the remainder is 4. ANSWER: (C)

5. In the diagram, line segments meet at $90°$ as shown. If the short line segments are each 3 cm long, what is the area of the shape?
(**A**) 30 (**B**) 36 (**C**) 40
(**D**) 45 (**E**) 54



Solution
Each of the four squares are identical and each has an area of 3×3 or $9\ \text{cm}^2$.
The total area is thus 4×9 or $36\ \text{cm}^2$.



ANSWER: (B)

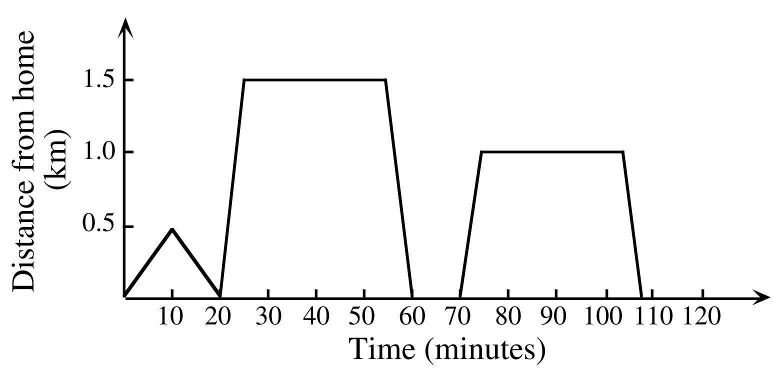
6. The average of -5, -2, 0, 4, and 8 is
(**A**) 1 (**B**) 0 (**C**) $\frac{19}{5}$ (**D**) $\frac{5}{4}$ (**E**) $\frac{9}{4}$

Solution
The average is, $\dfrac{(-5)+(-2)+(0)+(4)+(8)}{5} = 1$. ANSWER: (A)

7. If the sales tax rate were to increase from 7% to 7.5%, then the tax on a \$1000 item would go up by
(**A**) \$75.00 (**B**) \$5.00 (**C**) \$0.5 (**D**) \$0.05 (**E**) \$7.50

Solution
If the sales tax rate increases by .5%, this would represent an increase of \$.50 on each \$100.
Thus the increase in tax would be $(10)(\$.50) = \$5.00$. ANSWER: (B)

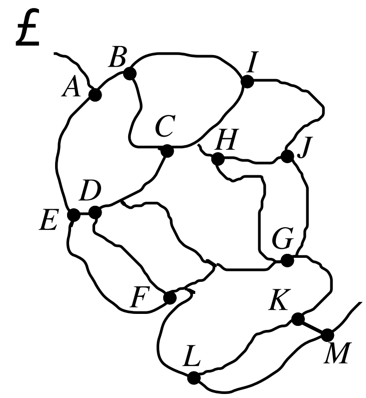
8. Tom spent part of his morning visiting and playing with friends. The graph shows his travels. He went to his friends' houses and stopped to play if they were at home. The number of houses at which he stopped to play is
(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5



Solution

From the graph, we can see that Tom stops at two houses in his travels. Notice that his first visit to a house, illustrated by the 'triangular' shape implies that his friend was not at home.
During his other two visits, the horizontal line indicates the fact that Tom stayed.
In both instances, Tom stayed for about 30 minutes. ANSWER: (B)

9. André is hiking on the paths shown in the map. He is planning to visit sites A to M in alphabetical order. He can never retrace his steps and he must proceed directly from one site to the next. What is the largest number of labelled points he can visit before going out of alphabetical order?



(**A**) 6 (**B**) 7 (**C**) 8
(**D**) 10 (**E**) 13

Solution

If we trace André's route we can see that he can travel to site J travelling in alphabetical order.
Once he reaches site J, it is not possible to reach K without passing through G or retracing his steps.
Since J is the tenth letter in the alphabet, he can visit ten sites before going out of order.
ANSWER: (D)

10. The area of a rectangular shaped garden is 28 m^2. It has a length of 7 m. Its perimeter, in metres, is
(**A**) 22 (**B**) 11 (**C**) 24 (**D**) 36 (**E**) 48

Solution

If the garden has a length of 7 m then its width will be 4 m.
Its perimeter is $2(4+7)=22\text{ m}$. ANSWER: (A)

Part B

11. Which of the following numbers is an odd integer, contains the digit 5, is divisible by 3, and lies between 12^2 and 13^2?
(**A**) 105 (**B**) 147 (**C**) 156 (**D**) 165 (**E**) 175

Solution

Since $12^2=144$ and $13^2=169$, we can immediately eliminate 105 and 175 as possibilities.
Since 156 is even it can also be eliminated. The only possibilities left are 147 and 165 but since 147 does not contain a 5 it can also be eliminated. The only candidate left is 165 and it can easily be checked that it meets the requirements of the question. ANSWER: (D)

12. If $\dfrac{n+1999}{2} = -1$, then the value of n is

(**A**) -2001 (**B**) -2000 (**C**) -1999 (**D**) -1997 (**E**) 1999

Solution

By inspection or by multiplying each side by 2, we arrive at $n+1999=-2$ or $n=-2001$.

ANSWER: (A)

13. The expression $n!$ means the product of the positive integers from 1 to n. For example, $5! = 1\times2\times3\times4\times5$. The value of $6!-4!$ is

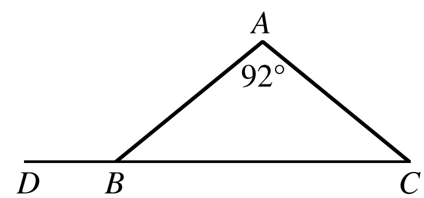
(**A**) 2 (**B**) 18 (**C**) 30 (**D**) 716 (**E**) 696

Solution

According to the given definition, $6! = 6\times5\times4\times3\times2\times1 = 720$.
Similarly, $4! = 24$. As a result, $6!-4! = 720-24 = 696$. ANSWER: (E)

14. ABC is an isosceles triangle in which $\angle A = 92°$. CB is extended to a point D. What is the size of $\angle ABD$?

(**A**) 88° (**B**) 44° (**C**) 92°
(**D**) 136° (**E**) 158°



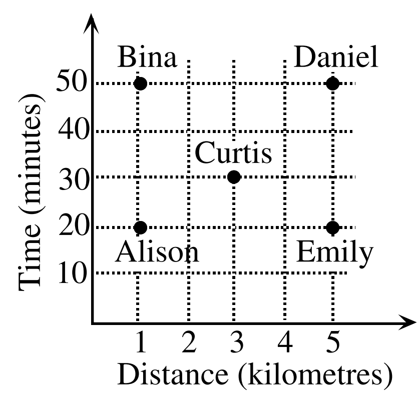
Solution

Since $\angle A = 92°$ then $\angle ABC = \angle ACB = \dfrac{180°-92°}{2} = 44°$.

Therefore, $\angle ABD = 180° - 44° = 136°$. ANSWER: (D)

15. The graph shown at the right indicates the time taken by five people to travel various distances. On average, which person travelled the fastest?

(**A**) Alison (**B**) Bina (**C**) Curtis
(**D**) Daniel (**E**) Emily



Solution

We summarize the results for the five people in the table.

We recall that average speed $= \dfrac{\text{distance}}{\text{time}}$.

	Distance	Time (minutes)	Speed (km/min.)
Alison	1	20	$\frac{1}{20}=0.05$
Bina	1	50	$\frac{1}{50}=0.02$
Curtis	3	30	$\frac{3}{30}=\frac{1}{10}=0.1$
Daniel	5	50	$\frac{5}{50}=0.1$
Emily	5	20	$\frac{5}{20}=0.25$

Emily is the fastest.

ANSWER: (E)

16. In a set of five numbers, the average of two of the numbers is 12 and the average of the other three numbers is 7. The average of all five numbers is

(**A**) $8\frac{1}{3}$ (**B**) $8\frac{1}{2}$ (**C**) 9 (**D**) $8\frac{3}{4}$ (**E**) $9\frac{1}{2}$

Solution

In order that two numbers have an average of 12, the sum of the two numbers must have been 24.
Similarly, the three numbers must have had a sum of 21.
Thus the average of the five numbers is, $\frac{21+24}{5}=9$. ANSWER: (C)

17. In the subtraction question, $\begin{array}{r} 1957 \\ a9 \\ \hline 18b8 \end{array}$, the sum of the digits a and b is

(**A**) 15 (**B**) 14 (**C**) 10 (**D**) 5 (**E**) 4

Solution 1

If we treat the question as an ordinary subtraction question we get the following:

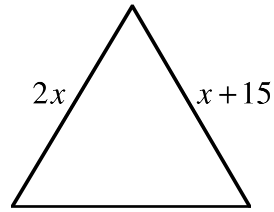
$$\begin{array}{cccc} & 8 & 14 & \\ 1 & \cancel{9} & \cancel{5} & 17 \\ & & a & 9 \\ \hline 1 & 8 & b & 8 \end{array}$$

From this, $14-a=b$ or $a+b=14$.

Solution 2

Using trial and error, we can try different possibilities for a and b.
A good starting point is the possibility that $a+b=15$ and $a=8$, $b=7$, for example. If we simply do the arithmetic this does not work. Of (**A**), (**B**), and (**C**), the only one that works is $a+b=14$.
From observation, it should be clear that if $a+b=4$ or 5 that the digits a and b would be far too small for the subtraction to work. ANSWER: (B)

18. The equilateral triangle has sides of $2x$ and $x+15$ as shown. The perimeter of the triangle is
(**A**) 15 (**B**) 30 (**C**) 90
(**D**) 45 (**E**) 60



Solution
Since we are told the triangle is equilateral, $2x = x + 15$, or $x = 15$.
This makes the side length 30 and the perimeter 90. ANSWER: (C)

19. In a traffic study, a survey of 50 moving cars is done and it is found that 20% of these contain more than one person. Of the cars containing only one person, 60% of these are driven by women. Of the cars containing just one person, how many were driven by men?
(**A**) 10 (**B**) 16 (**C**) 20 (**D**) 30 (**E**) 40

Solution
The number of cars containing one person is 80% of 50, which is 40. Since 40% of these 40 cars are driven by men, the number driven by men is $.4 \times 40$ or 16. ANSWER: (B)

20. A game is played on the board shown. In this game, a player can move three places in any direction (up, down, right or left) and then can move two places in a direction perpendicular to the first move. If a player starts at *S*, which position on the board (*P*, *Q*, *R*, *T*, or *W*) cannot be reached through any sequence of moves?
(**A**) *P* (**B**) *Q* (**C**) *R*
(**D**) *T* (**E**) *W*

		P		
	Q		*R*	
		T		
S				*W*

Solution
If *S* is the starting position we can reach position *R* immediately. From *S* we can also reach *P* and then *W* and *Q* in sequence. To reach position *T*, it would have to be reached from the upper right or upper left square. There is no way for us to reach these two squares unless we are allowed to move outside the large square which is not permitted.
ANSWER: (D)

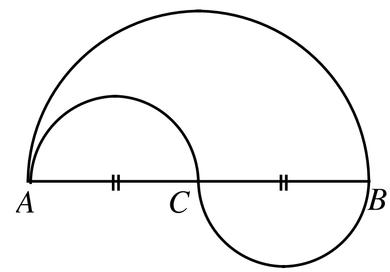
Part C

21. The sum of seven consecutive positive integers is always
(**A**) odd (**B**) a multiple of 7 (**C**) even
(**D**) a multiple of 4 (**E**) a multiple of 3

Solution
The easiest way to do this is to start with $1+2+3+4+5+6+7=28$. If we consider the next possibility, $2+3+4+5+6+7+8=35$, we notice that the acceptable sums are one of,

$\{28, 35, 42, 49, ...\}$. Each one of these numbers is a multiple of 7. ANSWER: (B)

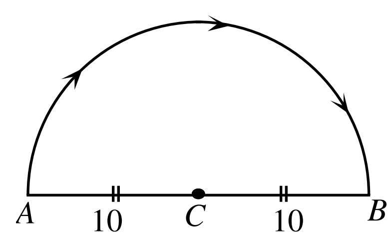
22. In the diagram, $AC = CB = 10$ m, where AC and CB are each the diameter of the small equal semi-circles. The diameter of the larger semi-circle is AB. In travelling from A to B, it is possible to take one of two paths. One path goes along the semi-circular arc from A to B. A second path goes along the semi-circular arcs from A to C and then along the semi-circular arc from C to B. The difference in the lengths of these two paths is



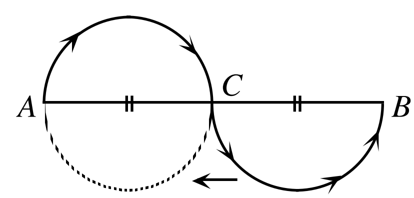
(**A**) 12π (**B**) 6π (**C**) 3π
(**D**) 2π (**E**) 0

Solution
Consider the two calculations.

Calculation 1
The distance travelled here would be one-half the circumference of the circle with radius 10.
This distance would be $\frac{1}{2}[2\pi(10)] = 10\pi$.



Calculation 2
The distance travelled would be the equivalent to the circumference of a circle with radius 5. The distance would be $2\pi(5) = 10\pi$.
Since these distances are equal, their difference would be $10\pi - 10\pi = 0$.



ANSWER: (E)

23. Kalyn writes down all of the integers from 1 to 1000 that have 4 as the sum of their digits. If $\frac{a}{b}$ (in lowest terms) is the fraction of these numbers that are prime, then $a + b$ is
(**A**) 5 (**B**) 4 (**C**) 15 (**D**) 26 (**E**) 19

Solution
The numbers between 1 and 1000 that have 4 as the sum of their digits are 4, 13, 22, 31, 40, 103, 112, 121, 130, 202, 211, 220, 301, 310, 400.
The circled numbers are prime which means that 4 out of the 15 are prime and $a + b = 19$.
ANSWER: (E)

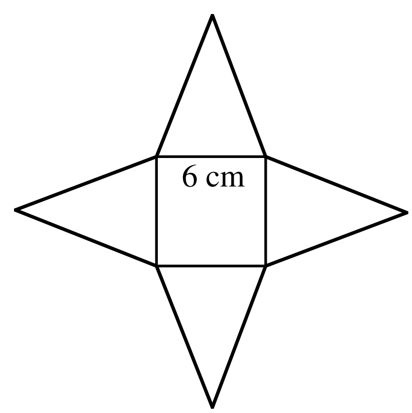
24. Raymonde's financial institution publishes a list of service charges as shown in the table. For her first twenty five transactions, she uses Autodebit three times as often as she writes cheques. She also writes as many cheques as she makes cash withdrawals. After her twenty-fifth transaction, she begins to make single transactions. What is the smallest number of transactions she needs to make so that her monthly service charges will exceed the $15.95 'all-in-one' fee?

	Service Fee per Item
Cheque	\$0.50
Autodebit	\$0.60
Cash Withdrawal	\$0.45
'All-in-one' fee is \$15.95	

(**A**) 29 (**B**) 30 (**C**) 27
(**D**) 28 (**E**) 31

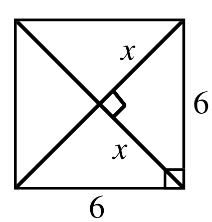
Solution
For Raymonde's first twenty five transactions, each set of five would cost $.50+.45+3(.60)=2.75$. After 25 transactions, her total cost would be \$13.75. In order to exceed \$15.95, she would have to spend \$2.20. In order to minimize the number of transactions, she would use Autodebit four times. In total, the number of transactions would be $25+4=29$. ANSWER: (A)

25. Four identical isosceles triangles border a square of side 6 cm, as shown. When the four triangles are folded up they meet at a point to form a pyramid with a square base. If the height of this pyramid is 4 cm, the total area of the four triangles and the square is



(**A**) 84 cm^2 (**B**) 98 cm^2 (**C**) 96 cm^2
(**D**) 108 cm^2 (**E**) 90 cm^2

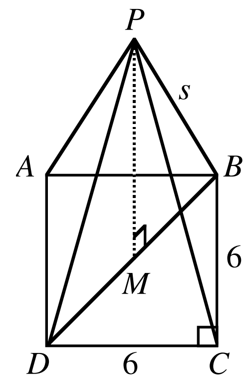
Solution
We draw in the two diagonals of the base square and label as shown. We can now say,



$$x^2+x^2=36$$
$$2x^2=36$$
$$x^2=18$$
$$x=\sqrt{18}.$$

Note: This question would work out very nicely if we had used $\sqrt{18} \doteq 4.24$ instead of the exact form, i.e. $\sqrt{18}$.

In this part of the solution, we have drawn the completed pyramid and labelled it as shown. We draw a line perpendicular to the square base from P. By using the fact that the pyramid has a square base and its sides are equal we conclude that this perpendicular line will pass through the mid-point of diagonal DB at the point M.



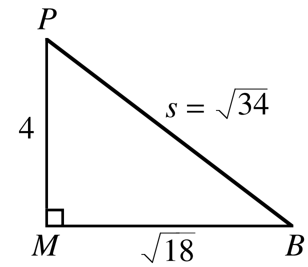
Using ΔPMB, we can now calculate the side length, s, of the pyramid.

$s^2 = 4^2 + \left(\sqrt{18}\right)^2$; Note that $x = MB = \sqrt{18}$

$s^2 = 16 + 18$

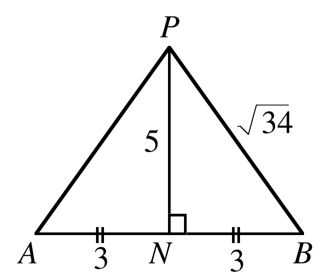
$s^2 = 34$

Therefore $s = \sqrt{34}$.



If we wish to calculate the height of the side triangles, which are each isosceles, we once again draw a perpendicular from P to the mid-point of one side of the square. We use ΔPAB and label the mid-point of AB point N. (Since ΔPAB is isosceles, the point N is the mid-point of AB.) Once again, we use pythagoras to calculate the heights of the isosceles triangles.

$$
\begin{aligned}
PB^2 &= PN^2 + NB^2 \\
\left(\sqrt{34}\right)^2 &= PN^2 + 3^2 \\
PN^2 &= 34 - 9 \\
PN^2 &= 25 \\
PN &= 5.
\end{aligned}
$$



We thus conclude that the height of each triangle is 5 and the area of each side triangle is $\dfrac{6 \times 5}{2} = 15$.

Thus, the total area is $4 \times 15 + 6 \times 6 = 96$. ANSWER: (C)

Canadian Mathematics Competition
An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

1998 Solutions

Gauss Contest
(Grade 7)

Part A

1. The value of $\frac{1998-998}{1000}$ is

(**A**) 1 (**B**) 1000 (**C**) 0.1 (**D**) 10 (**E**) 0.001

Solution

$\frac{1998-998}{1000}=\frac{1000}{1000}=1$ ANSWER: (A)

2. The number 4567 is tripled. The ones digit (units digit) in the resulting number is

(**A**) 5 (**B**) 6 (**C**) 7 (**D**) 3 (**E**) 1

Solution

If we wish to determine the units digit when we triple 4567, it is only necessary to triple 7 and take the units digit of the number 21.

The required digit is 1. ANSWER: (E)

3. If $S=6\times 10\ 000+5\times 1000+4\times 10+3\times 1$, what is S?

(**A**) 6543 (**B**) 65 043 (**C**) 65 431 (**D**) 65 403 (**E**) 60 541

Solution

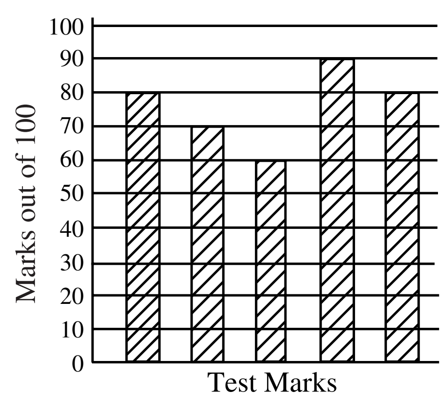
$S=60\ 000+5000+40+3$

$=65\ 043$ ANSWER: (B)

4. Jean writes five tests and achieves the marks shown on the graph. What is her average mark on these five tests?

(**A**) 74 (**B**) 76 (**C**) 70

(**D**) 64 (**E**) 79



Solution

Jean's average is $\frac{80+70+60+90+80}{5}=\frac{380}{5}=76$. ANSWER: (B)

5. If a machine produces 150 items in one minute, how many would it produce in 10 seconds?

(**A**) 10 (**B**) 15 (**C**) 20 (**D**) 25 (**E**) 30

Solution

Since 10 seconds represents one-sixth of a minute, the machine will produce $\frac{1}{6} \times 150$ or 25 items.

ANSWER: (D)

6. In the multiplication question, the sum of the digits in the four boxes is

(**A**) 13 (**B**) 12 (**C**) 27
(**D**) 9 (**E**) 22

$$\begin{array}{r} 879 \\ \times\, 492 \\ \hline \square 758 \\ 7\square\;11 \\ 35\square 6 \\ \hline 43\square 468 \end{array}$$

Solution

Multiplying out,

$$\begin{array}{r} 879 \\ \times\, 492 \\ \hline 1758 \\ 7911 \\ 3516 \\ \hline 432468 \end{array}$$

The sum is $1+9+1+2=13$.

ANSWER: (A)

7. A rectangular field is 80 m long and 60 m wide. If fence posts are placed at the corners and are 10 m apart along the four sides of the field, how many posts are needed to completely fence the field?

(**A**) 24 (**B**) 26 (**C**) 28 (**D**) 30 (**E**) 32

Solution

There is 1 post on each corner making a total of 4 plus 7 along each of the two lengths and 5 along each of the two widths.

This gives a total of 28 posts.

ANSWER: (C)

8. Tuesday's high temperature was $4°$C warmer than that of Monday's. Wednesday's high temperature was $6°$C cooler than that of Monday's. If Tuesday's high temperature was $22°$C, what was Wednesday's high temperature?

(**A**) $20°$C (**B**) $24°$C (**C**) $12°$C (**D**) $32°$C (**E**) $16°$C

Solution

If Tuesday's temperature was $22°$C then Monday's high temperature was $18°$C.

Wednesday's temperature was $12°$C since it was $6°$C cooler than that of Monday's high temperature.

ANSWER: (C)

9. Two numbers have a sum of 32. If one of the numbers is -36, what is the other number?

(**A**) 68 (**B**) -4 (**C**) 4 (**D**) 72 (**E**) -68

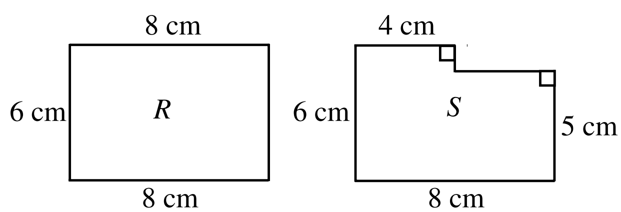
Solution
$68 + (-36) = 32$ ANSWER: (A)

10. At the waterpark, Bonnie and Wendy decided to race each other down a waterslide. Wendy won by 0.25 seconds. If Bonnie's time was exactly 7.80 seconds, how long did it take for Wendy to go down the slide?
(**A**) 7.80 seconds (**B**) 8.05 seconds (**C**) 7.55 seconds (**D**) 7.15 seconds (**E**) 7.50 seconds

Solution
If Wendy finished 0.25 seconds ahead of Bonnie and Bonnie took 7.80 seconds then Wendy took $7.80 - 0.25$ or 7.55 seconds. ANSWER: (C)

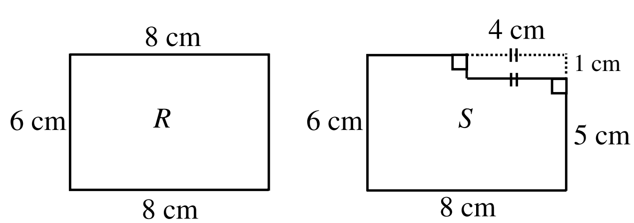
Part B

11. Kalyn cut rectangle *R* from a sheet of paper. A smaller rectangle is then cur from the large rectangle *R* to produce figure *S*. In comparing *R* to *S*



(**A**) the area and perimeter both decrease
(**B**) the area decreases and the perimeter increases
(**C**) the area and perimeter both increase
(**D**) the area increases and the perimeter decreases
(**E**) the area decreases and the perimeter stays the same

Solution
If figure *S* is cut out of rectangle *R* then *S* must be smaller in area.
If we compare perimeters, however, we find that the perimeter of figure *S* is identical to that of rectangle *R*.
The comparison of perimeter is not too difficult to see if we complete figure *S* as shown and compare lengths.
The perimeters of *R* and *S* are equal.



ANSWER: (E)

12. Steve plants ten trees every three minutes. If he continues planting at the same rate, how long will it take him to plant 2500 trees?

(**A**) $1\frac{1}{4}$ h (**B**) 3 h (**C**) 5 h (**D**) 10 h (**E**) $12\frac{1}{2}$ h

Solution 1

Since Steve plants ten trees every three minutes, he plants one tree every $\frac{3}{10}$ minute.

In order to plant 2500 trees, he will need $\frac{3}{10} \times 2500 = 750$ minutes or $\frac{750}{60} = 12\frac{1}{2}$ hours.

Solution 2

Since Steve plants ten trees every three minutes, he plants 200 trees per hour.

In order to plant 2500 trees, he will need $\frac{2500}{200} = 12\frac{1}{2}$ hours. ANSWER: (E)

13. The pattern of figures △●□▲○ is repeated in the sequence

△,●,□,▲,○,△,●,□,▲,○, ...

The 214th figure in the sequence is

(**A**) △ (**B**) ● (**C**) □ (**D**) ▲ (**E**) ○

Solution

Since the pattern repeats itself after every five figures, it begins again after 210 figures have been completed.

The 214th figure would be the fourth element in the sequence or ▲. ANSWER: (D)

14. A cube has a volume of 125 cm^3. What is the area of one face of the cube?

(**A**) 20 cm^2 (**B**) 25 cm^2 (**C**) $41\frac{2}{3}\text{ cm}^2$ (**D**) 5 cm^2 (**E**) 75 cm^2

Solution

If the volume of the cube is 125 cm^3, then the length, width and height are each 5 cm.

The area of one face is 5×5 or 25 cm^2. ANSWER: (B)

15. The diagram shows a magic square in which the sums of the numbers in any row, column or diagonal are equal. What is the value of n?

8		
9		5
4	n	

(**A**) 3 (**B**) 6 (**C**) 7

(**D**) 10 (**E**) 11

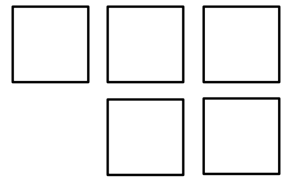
Solution

The 'magic' sum is $8+9+4=21$, so the centre square is 7.

If the centre square is 7, then the square on the lower right has 6 in it giving $4+n+6=21$.

Therefore $n=11$. ANSWER: (E)

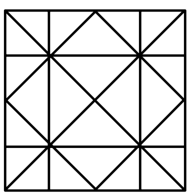
16. Each of the digits 3, 5, 6, 7, and 8 is placed one to a box in the diagram. If the two digit number is subtracted from the three digit number, what is the smallest difference?

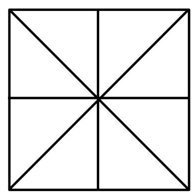


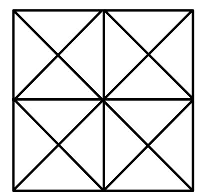
(**A**) 269 (**B**) 278 (**C**) 484
(**D**) 271 (**E**) 261

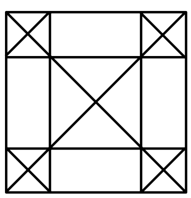
Solution
The smallest difference will be produced when the three digit number is as small as possible, that is 356, and the two digit number is as large as possible, that is 87.
The smallest difference is $356 - 87 = 269$. ANSWER: (A)

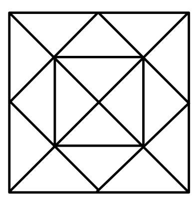
17. Claire takes a square piece of paper and folds it in half four times without unfolding, making an isosceles right triangle each time. After unfolding the paper to form a square again, the creases on the paper would look like

(**A**)


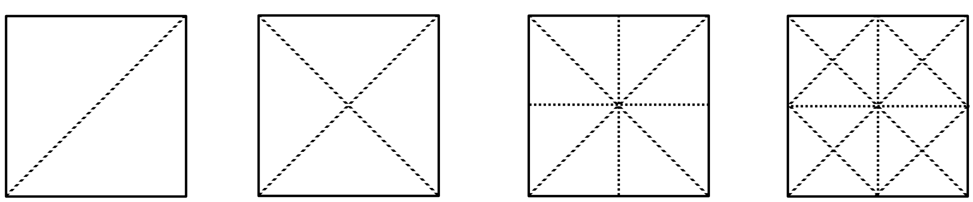
(**B**)


(**C**)


(**D**)


(**E**)


Solution



Fold 1 Fold 2 Fold 3 Fold 4

ANSWER: (C)

18. The letters of the word 'GAUSS' and the digits in the number '1998' are each cycled separately and then numbered as shown.

1. AUSSG 9981
2. USSGA 9819
3. SSGAU 8199

etc.

If the pattern continues in this way, what number will appear in front of GAUSS 1998?

(**A**) 4 (**B**) 5 (**C**) 9 (**D**) 16 (**E**) 20

Solution

Because the word 'GAUSS' has five letters in it, the numbers 5, 10, 15, 20, ... will appear beside this word. Similarly, the four digits of '1998' will have the numbers 4, 8, 12, 16, 20, 24, ... beside this number.

From this listing, we can see that the correct number is 20 which is the l.c.m. of 5 and 4.

ANSWER: (E)

19. Juan and Mary play a two-person game in which the winner gains 2 points and the loser loses 1 point. If Juan won exactly 3 games and Mary had a final score of 5 points, how many games did they play?

(**A**) 7 (**B**) 8 (**C**) 4 (**D**) 5 (**E**) 11

Solution

If Juan won 3 games then Mary lost 3 points so that she must have had 8 points before losing in order to have a final total of 5.

If Mary had 8 points before losing then she must have won 4 games.

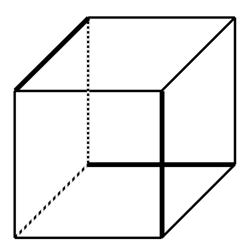
If Mary won 4 games and Juan won 3 games there was a total of 7 games. ANSWER: (A)

20. Each of the 12 edges of a cube is coloured either red or green. Every face of the cube has at least one red edge. What is the smallest number of red edges?

(**A**) 2 (**B**) 3 (**C**) 4 (**D**) 5 (**E**) 6

Solution

If the heavy black lines represent the colour red, every face will have exactly one red edge. So the smallest number of red edges is 3.



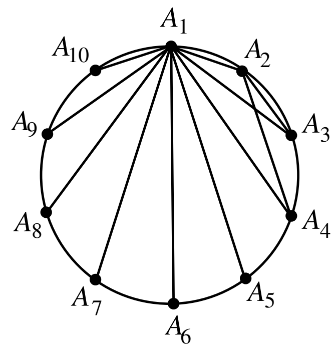
ANSWER: (B)

Part C

21. Ten points are spaced equally around a circle. How many different chords can be formed by joining any 2 of these points? (A chord is a straight line joining two points on the circumference of a circle.)
(**A**) 9 (**B**) 45 (**C**) 17 (**D**) 66 (**E**) 55

Solution

Space the ten points equally around the circle and label them $A_1, A_2, ..., A_{10}$ for convenience.



If we start with A_1 and join it to each of the other nine points, we will have 9 chords.

Similarly, we can join A_2 to each of the other 8 points. If we continue this process until we join A_9 to A_{10} we will have $9+8+7+6+5+4+3+2+1=45$ chords.

ANSWER: (B)

22. Each time a bar of soap is used, its volume decreases by 10%. What is the minimum number of times a new bar would have to be used so that less than one-half its volume remains?
(**A**) 5 (**B**) 6 (**C**) 7 (**D**) 8 (**E**) 9

Solution

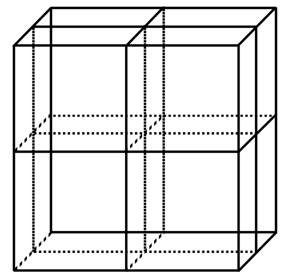
Number of Times Soap Used	Approximate Volume Remaining (as %)
1	0.9 or 90%
2	$(0.9)^2$ or 81%
3	$(0.9)^3$ or 72.9%
4	$(0.9)^4$ or 65.61%
5	$(0.9)^5$ or 59.1%
6	$(0.9)^6$ or 53.1%
7	$(0.9)^7$ or 47.8%

So if the soap is used 7 times the volume will be less than $\frac{1}{2}$ of the original volume.

NOTE: In essence, we are trying to find a positive integer x so that $(0.9)^x < 0.5$. The value of x can be found by using the $\boxed{y^x}$ button on your calculator where $y = 0.9$ and experimenting to find a value for x.

ANSWER: (C)

23. A cube measures $10\text{ cm} \times 10\text{ cm} \times 10\text{ cm}$. Three cuts are made parallel to the faces of the cube as shown creating eight separate solids which are then separated. What is the increase in the total surface area?

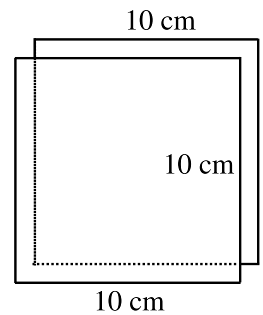


(**A**) 300 cm^2 (**B**) 800 cm^2 (**C**) 1200 cm^2
(**D**) 600 cm^2 (**E**) 0 cm^2

Solution

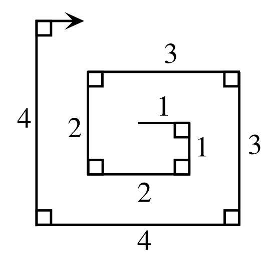
One cut increases the surface area by the equivalent of two $10\text{ cm} \times 10\text{ cm}$ squares or 200 cm^2.

Then the three cuts produce an increase in area of $3 \times 200\text{ cm}^2$ or 600 cm^2.



ANSWER: (D)

24. On a large piece of paper, Dana creates a "rectangular spiral" by drawing line segments of lengths, in cm, of 1, 1, 2, 2, 3, 3, 4, 4, ... as shown. Dana's pen runs out of ink after the total of all the lengths he has drawn is 3000 cm. What is the length of the longest line segment that Dana draws?



(**A**) 38 (**B**) 39 (**C**) 54
(**D**) 55 (**E**) 30

Solution

The formula for the sum of the natural numbers from 1 to n is $\frac{(n)(n+1)}{2}$.

That is, $1+2+3+...+n = \frac{(n)(n+1)}{2}$.

(For example, $1+2+3+...+10 = \frac{(10)(11)}{2} = 55$.)

We can find the sum of a double series, like the one given, by doubling each side of the given formula.

We know $1+2+3+...+n = \frac{(n)(n+1)}{2}$.

If we double each side we get $2(1+2+...+n) = n(n+1)$.

So, $(1+1)+(2+2)+(3+3)+...+(n+n) = n(n+1)$.

In this question we want the value of n so that the following is true:

$(1+1)+(2+2)+(3+3)+...+(n+n) \le 3000$.

Or, if we use the formula $(n)(n+1) \le 3000$.

We would now like to find the largest value of n for which this is true.

The best way to start is by taking $\sqrt{3000} \doteq 54.7$ as a beginning point.

If we try $n = 54$, we find $(54)(55) = 2970 < 3000$ which is a correct estimate.

(If we try $n = 55$ we find $55(56) = 3080 > 3000$. So $n = 55$ is not acceptable.)

This means that $(1+1)+(2+2)+(3+3)+...+(54+54) = 2970$ so that the longest length that Dana completed was 54 cm. (If we had included the length 55 then we would have had a sum of 3 025 which is too large.) ANSWER: (C)

25. Two natural numbers, p and q, do not end in zero. The product of any pair, p and q, is a power of 10 (that is, 10, 100, 1000, 10 000, ...). If $p > q$, the last digit of $p - q$ cannot be

(**A**) 1 (**B**) 3 (**C**) 5 (**D**) 7 (**E**) 9

Solution

If the two natural numbers p and q do not end in zero themselves and if their product is a power of 10 then p must be of the form 5^n and q must be of the form 2^n.

This is true because $10 = 2 \times 5$ and $10^n = (2 \times 5)^n = 2^n \times 5^n$.

The possibilities for powers of two are 2, 4, 8, 16, 32, ... and for corresponding powers of five are 5, 25, 125, 625, 3125, ...

If we take their differences and look at the last digit of $p - q$ we find the following,

p	q	last digit of $p - q$
5	2	3
25	4	1
125	8	7
625	16	9
3125	32	3
15 625	64	1
⋮	⋮	7
		9
		⋮

and the pattern continues in groups of 4.

Thus, the last digit of $p - q$ cannot be 5. ANSWER: (C)

Anniversary
1963 – 1998

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

1998 Solutions

Gauss Contest

(Grade 8)

Part A

1. The number 4567 is tripled. The ones digit (units digit) in the resulting number is

   (**A**) 5 (**B**) 6 (**C**) 7 (**D**) 3 (**E**) 1

   Solution

   If we wish to determine the units digit when we triple 4567, it is only necessary to triple 7 and take the units digit of the number 21.

   The required number is 1. ANSWER: (E)

2. The smallest number in the set $\{0, -17, 4, 3, -2\}$ is

   (**A**) -17 (**B**) 4 (**C**) -2 (**D**) 0 (**E**) 3

   Solution

   By inspection, the smallest number is -17. ANSWER: (A)

3. The average of -5, -2, 0, 4, and 8 is

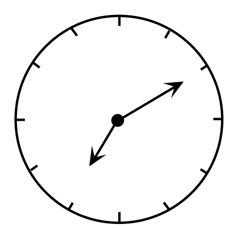
   (**A**) $\frac{5}{4}$ (**B**) 0 (**C**) $\frac{19}{5}$ (**D**) 1 (**E**) $\frac{9}{4}$

   Solution

   The sum of the integers is 5.

   They have an average of 1. ANSWER: (D)

4. Emily sits on a chair in a room. Behind her is a clock. In front of her is a mirror. In the mirror, she sees the image of the clock as shown. The actual time is closest to

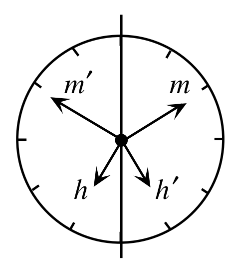
   

   (**A**) 4:10 (**B**) 7:10 (**C**) 5:10
   (**D**) 6:50 (**E**) 4:50

   Solution

   Draw a mirror line to run from 12 o'clock to 6 o'clock and then reflect both the minute and hour hand in this mirror line. The minute hand (m) is reflected to 10 o'clock (which is labelled m') and the hour hand (h) is reflected to just before 5 o'clock (h').

   The required time is 4:50.

   ANSWER: (E)

5. If 1.2×10^6 is doubled, what is the result?

(**A**) 2.4×10^6 (**B**) 2.4×10^{12} (**C**) 2.4×10^3 (**D**) 1.2×10^{12} (**E**) 0.6×10^{12}

Solution

Doubling the given number means that 1.2 must be doubled.

The required number is 2.4×10^6. ANSWER: (A)

6. Tuesday's high temperature was 4°C warmer than that of Monday's. Wednesday's high temperature was 6°C cooler than that of Monday's. If Tuesday's high temperature was 22°C, what was Wednesday's high temperature?

(**A**) 20°C (**B**) 24°C (**C**) 12°C (**D**) 32°C (**E**) 16°C

Solution

If Tuesday's temperature was 22°C then Monday's high temperature was 18°C.

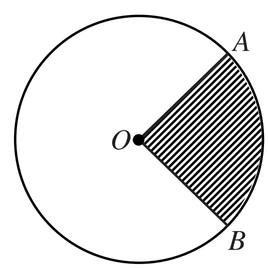
Wednesday's temperature was 12°C since it was 6°C cooler than that of Monday's high temperature.

ANSWER: (C)

7. In the circle with centre O, the shaded sector represents 20% of the area of the circle. What is the size of angle AOB?

(**A**) 36° (**B**) 72° (**C**) 90°

(**D**) 80° (**E**) 70°



Solution

If the area of the sector represents 20% of the area of the circle then angle AOB is 20% of 360° or 72°.

ANSWER: (B)

8. The pattern of figures △●□▲○ is repeated in the sequence

△,●,□,▲,○,△,●,□,▲,○, ...

The 214th figure in the sequence is

(**A**) △ (**B**) ● (**C**) □ (**D**) ▲ (**E**) ○

Solution

Since the pattern repeats itself after every five figures, it begins again after 210 figures have been completed.

The 214th figure would be the fourth element in the sequence or ▲. ANSWER: (D)

9. When a pitcher is $\frac{1}{2}$ full it contains exactly enough water to fill three identical glasses. How full would the pitcher be if it had exactly enough water to fill four of the same glasses?

(A) $\frac{2}{3}$ **(B)** $\frac{7}{12}$ **(C)** $\frac{4}{7}$ **(D)** $\frac{6}{7}$ **(E)** $\frac{3}{4}$

Solution

If three glasses of water are the same as $\frac{1}{2}$ a pitcher then one glass is the same as $\frac{1}{6}$ of the pitcher.

If there were four glasses of water in the pitcher, this would be the same as $\frac{4}{6}=\frac{2}{3}$ of a pitcher.

ANSWER: (A)

10. A bank employee is filling an empty cash machine with bundles of \$5.00, \$10.00 and \$20.00 bills. Each bundle has 100 bills in it and the machine holds 10 bundles of each type. What amount of money is required to fill the machine?

(A) \$30 000 **(B)** \$25 000 **(C)** \$35 000 **(D)** \$40 000 **(E)** \$45 000

Solution

Since there are three bundles, each with 100 bills in them, the three bundles would be worth \$500, \$1000 and \$2000 respectively.

Since there are 10 bundles of each type of bill, their overall value would be

$10(\$500+\$1000+\$2000)=\$35\,000$. ANSWER: (C)

Part B

11. The weight limit for an elevator is 1500 kilograms. The average weight of the people in the elevator is 80 kilograms. If the combined weight of the people is 100 kilograms over the limit, how many people are in the elevator?

(A) 14 **(B)** 17 **(C)** 16 **(D)** 20 **(E)** 13

Solution

The combined weight of the people on the elevator is 100 kilograms over the limit which implies that their total weight is 1600 kilograms.

If the average weight is 80 kilograms there must be $\frac{1600}{80}$ or 20 people on the elevator.

ANSWER: (D)

12. In the 4×4 square shown, each row, column and diagonal should contain each of the numbers 1, 2, 3, and 4. Find the value of $K+N$.

(A) 4 **(B)** 3 **(C)** 5
(D) 6 **(E)** 7

1	F	G	H
T	2	J	K
L	M	3	N
P	Q	1	R

Solution

Since R is on a main diagonal and the numbers 1, 2 and 3 have already been used on this diagonal, then $R = 4$.
The easiest way to look at how to arrange the numbers is to look at boxes P and Q.
Q must be either 2 or 3 but since there is already a 2 in the same column as Q, we conclude that $Q = 3$ and $P = 2$.

1			
	2		
		3	
2	3	1	4

'cannot be 2'

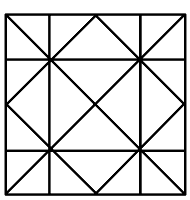
From this point we simply fill in the boxes according to the rule that each row, column and diagonal contains each of the numbers 1, 2, 3, and 4.
Doing this, we arrive at the following arrangement of numbers.

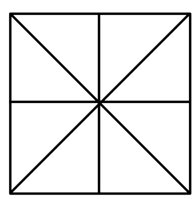
1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

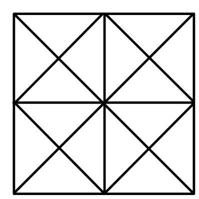
We see that $K + N = 3$.
NOTE 1: It is not necessary that we complete all the boxes but it is a useful way to verify the overall correctness of our work.
NOTE 2: We could have started by considering H, K and N but this takes a little longer to complete.

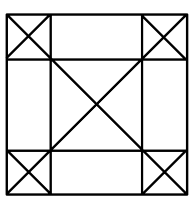
ANSWER: (B)

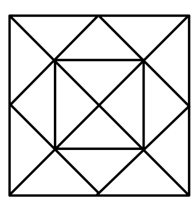
13. Claire takes a square piece of paper and folds it in half four times without unfolding, making an isosceles right triangle each time. After unfolding the paper to form a square again, the creases on the paper would look like

(A) 

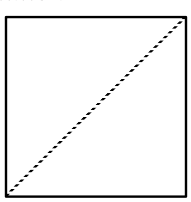
(B) 

(C) 

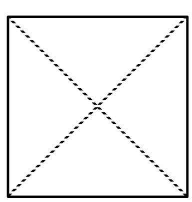
(D) 

(E) 

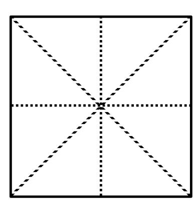
Solution



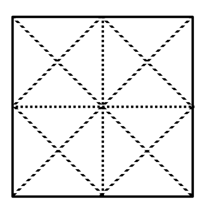
Fold 1



Fold 2



Fold 3



Fold 4

ANSWER: (C)

14. Stephen had a 10:00 a.m. appointment 60 km from his home. He averaged 80 km/h for the trip and arrived 20 minutes late for the appointment. At what time did he leave his home?

(**A**) 9:35 a.m. (**B**) 9:15 a.m. (**C**) 8:40 a.m. (**D**) 9:00 a.m. (**E**) 9:20 a.m.

Solution

If Stephen averaged 80 km/h for a trip which was 60 km in length, it must have taken him 45 minutes to make the trip.

If the entire trip took 45 minutes and he arrived 20 minutes late he must have left home at 9:35 a.m.

ANSWER: (A)

15. Michael picks three *different* digits from the set $\{1, 2, 3, 4, 5\}$ and forms a mixed number by placing the digits in the spaces of $\square\frac{\square}{\square}$. The fractional part of the mixed number must be less than 1. (For example, $4\frac{2}{3}$). What is the difference between the largest and smallest possible mixed number that can be formed?

(**A**) $4\frac{3}{5}$ (**B**) $4\frac{9}{20}$ (**C**) $4\frac{3}{10}$ (**D**) $4\frac{4}{15}$ (**E**) $4\frac{7}{20}$

Solution

The largest possible number that Michael can form is $5\frac{3}{4}$ while the smallest is $1\frac{2}{5}$.

The difference is $5\frac{3}{4} - 1\frac{2}{5} = 4\frac{7}{20}$.

ANSWER: (E)

16. Suppose that x^* means $\frac{1}{x}$, the reciprocal of x. For example, $5^* = \frac{1}{5}$. How many of the following statements are true?

(i) $2^* + 4^* = 6^*$
(ii) $3^* \times 5^* = 15^*$
(iii) $7^* - 3^* = 4^*$
(iv) $12^* \div 3^* = 4^*$

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 4

Solution

(i) $\frac{1}{2} + \frac{1}{4} = \frac{3}{4} \neq \frac{1}{6}$, (i) is not true

(ii) $\frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$, (ii) is true

(iii) $\frac{1}{7} - \frac{1}{3} = \frac{3}{21} - \frac{7}{21} = \frac{4}{21} \neq \frac{1}{4}$, (iii) is not true

(iv) $\dfrac{\frac{1}{12}}{\frac{1}{3}} = \frac{1}{12} \times \frac{3}{1} = \frac{1}{4}$, (iv) is true

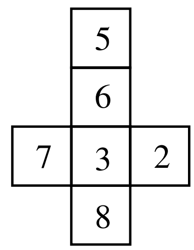
Only two of these statements is correct.

ANSWER: (C)

17. In a ring toss game at a carnival, three rings are tossed over any of three pegs. A ring over peg A is worth *one* point, over peg B *three* points and over peg C *five* points. If all three rings land on pegs, how many different point totals are possible? (It is possible to have more than one ring on a peg.)
(**A**) 12 (**B**) 7 (**C**) 10 (**D**) 13 (**E**) 6

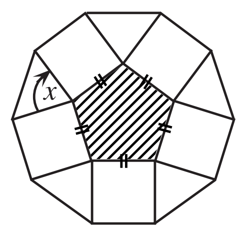
Solution
The lowest possible score is 3 and the highest is 15.
It is not possible to get an even score because this would require 3 odd numbers to add to an even number. Starting at a score of 3, it is possible to achieve every odd score between 3 and 15. This implies that 3, 5, 7, 9, 11, 13, and 15 are possible scores.
There are 7 possible scores. ANSWER: (B)

18. The figure shown is folded to form a cube. Three faces meet at each corner. If the numbers on the three faces at a corner are multiplied, what is the largest possible product?
(**A**) 144 (**B**) 168 (**C**) 240
(**D**) 280 (**E**) 336



Solution
When the figure is folded to make a cube, the numbers 8 and 6 are on opposite faces so that it is NOT possible to achieve $8 \times 7 \times 6$ or 336. It is possible, however, to have the sides with 5, 7 and 8 meet at a corner which gives the answer $5 \times 7 \times 8$ or 280. ANSWER: (D)

19. A regular pentagon has all sides and angles equal. If the shaded pentagon is enclosed by squares and triangles, as shown, what is the size of angle x?
(**A**) 75° (**B**) 108° (**C**) 90°
(**D**) 60° (**E**) 72°



Solution
Since a pentagon can be divided into three triangles, the sum of the angles in the pentagon is $3 \times 180° = 540°$. Since the angles in a regular pentagon are all equal, each one is $540° \div 5 = 108°$. The sum of the angles of each vertex of the pentagon is $x + 90° + 90° + 108° = 360°$.
Therefore $x = 72°$. ANSWER: (E)

20. Three playing cards are placed in a row. The club is to the right of the heart and the diamond. The 5 is to the left of the heart. The 8 is to the right of the 4. From left to right, the cards are
(**A**) 4 of hearts, 5 of diamonds, 8 of clubs
(**B**) 5 of diamonds, 4 of hearts, 8 of clubs
(**C**) 8 of clubs, 4 of hearts, 5 of diamonds
(**D**) 4 of diamonds, 5 of clubs, 8 of hearts
(**E**) 5 of hearts, 4 of diamonds, 8 of clubs

Solution

Since the club is to the right of the heart and diamond we know that the order is either heart, diamond, club or diamond, heart, club.

It is given that the 5 is to the left of the heart so this card must be the 5 of diamonds.

The order is 5 of diamonds, heart, club. Since the 8 is to the right of the 4, the heart must be a 4 and the club an 8.

The correct choice is B. ANSWER: (B)

Part C

21. The number 315 can be written as the product of two odd integers each greater than 1. In how many ways can this be done?

(**A**) 0 (**B**) 1 (**C**) 3 (**D**) 4 (**E**) 5

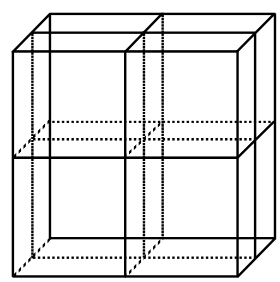
Solution

Factoring 315 into primes, we find that $315 = 3\times3\times5\times7$.

The factorization of 315 as the product of 2 odd integers is 3×105, 5×63, 7×45, 9×35, and 15×21.

There are 5 possible factorizations. ANSWER: (E)

22. A cube measures $10\text{ cm}\times10\text{ cm}\times10\text{ cm}$. Three cuts are made parallel to the faces of the cube as shown creating eight separate solids which are then separated. What is the increase in the total surface area?



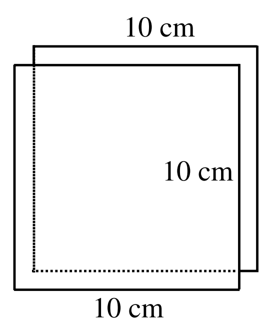
(**A**) 300 cm^2 (**B**) 800 cm^2 (**C**) 1200 cm^2

(**D**) 600 cm^2 (**E**) 0 cm^2

Solution

One cut increases the surface area by the equivalent of two $10\text{ cm}\times10\text{ cm}$ squares or 200 cm^2.

Then the three cuts produce an increase in area of $3\times200\text{ cm}^2$ or 600 cm^2.



ANSWER: (D)

23. If the sides of a triangle have lengths 30, 40 and 50, what is the length of the shortest altitude?

(**A**) 20 (**B**) 24 (**C**) 25 (**D**) 30 (**E**) 40

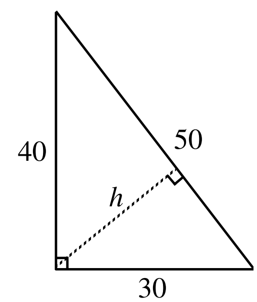
Solution

Since $30^2 + 40^2 = 50^2$ this is a right-angled triangle with a hypotenuse of 50 units.

The area of the triangle is $\frac{30 \times 40}{2}$ or 600 sq. units.

If we draw a perpendicular from the right angle and call this height, h, an expression for the area is $\frac{1}{2}(h)(50) = 25h$.

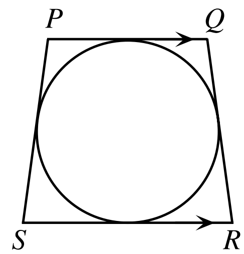
Equating the two, we have $25h = 600$ or $h = 24$ which is the length of the shortest altitude.



ANSWER: (B)

24. A circle is inscribed in trapezoid $PQRS$.
If $PS = QR = 25$ cm, $PQ = 18$ cm and $SR = 32$ cm, what is the length of the diameter of the circle?

(**A**) 14 (**B**) 25 (**C**) 24
(**D**) $\sqrt{544}$ (**E**) $\sqrt{674}$



Solution

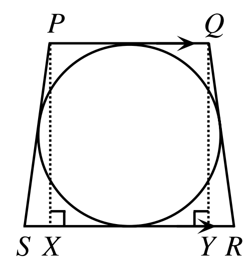
We start by drawing perpendiculars from P and Q to meet SR at X and Y respectively.

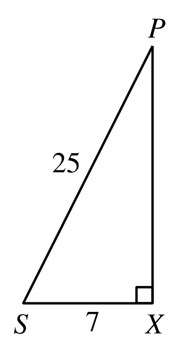
By symmetry, we see that $XY = PQ = 18$. We also note that $SX = YR$ which means that $SX = YR = \frac{32-18}{2} = 7$.

By applying the Pythagorean Theorem in ΔPXS we find,

$$(PX)^2 + 7^2 = 25^2$$
$$(PX)^2 = 576$$
$$PX = 24$$

The diameter of the circle is thus 24 cm.





ANSWER: (C)

25. A sum of money is to be divided among Allan, Bill and Carol. Allan receives \$1 plus one-third of what is left. Bill then receives \$6 plus one-third of what remains. Carol receives the rest, which amounts to \$40. How much did Bill receive?

(**A**) \$26 (**B**) \$28 (**C**) \$30 (**D**) \$32 (**E**) \$34

Solution
After Allan had received his share, Bill received \$6 plus one-third of the remainder.
Since Carol gets the rest, she received two-thirds of the remainder, which is \$40.
Thus, one-third of the remainder is \$20.
The amount Bill receives is \$26. ANSWER: (A)