From Pascal's Triangle to Sierpinski's Triangle

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Today’s journey

Q@A

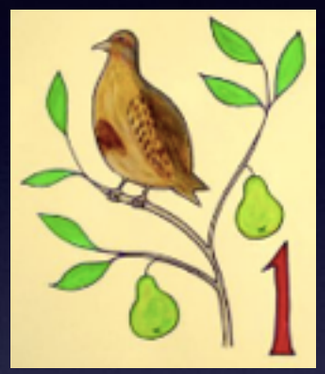
Conclusions

Intro

Sierpiński’s Triangle

Pascal’s Triangle

Do you remember "The 12 Days of Christmas" song?



A partridge in a pear tree

Two turtle doves

Three French hens

Four calling birds

Five gold rings

6 geese a-laying

7 swans a-swimming

8 maids a-milking

9 ladies dancing

10 lords a-leaping

11 pipers piping

12 drummers drumming

How many gifts were given in total over the 12 days?

- On the first day: 1
- On the second day: 1 + 2
- On the 3rd day: 1 + 2 + 3

.....

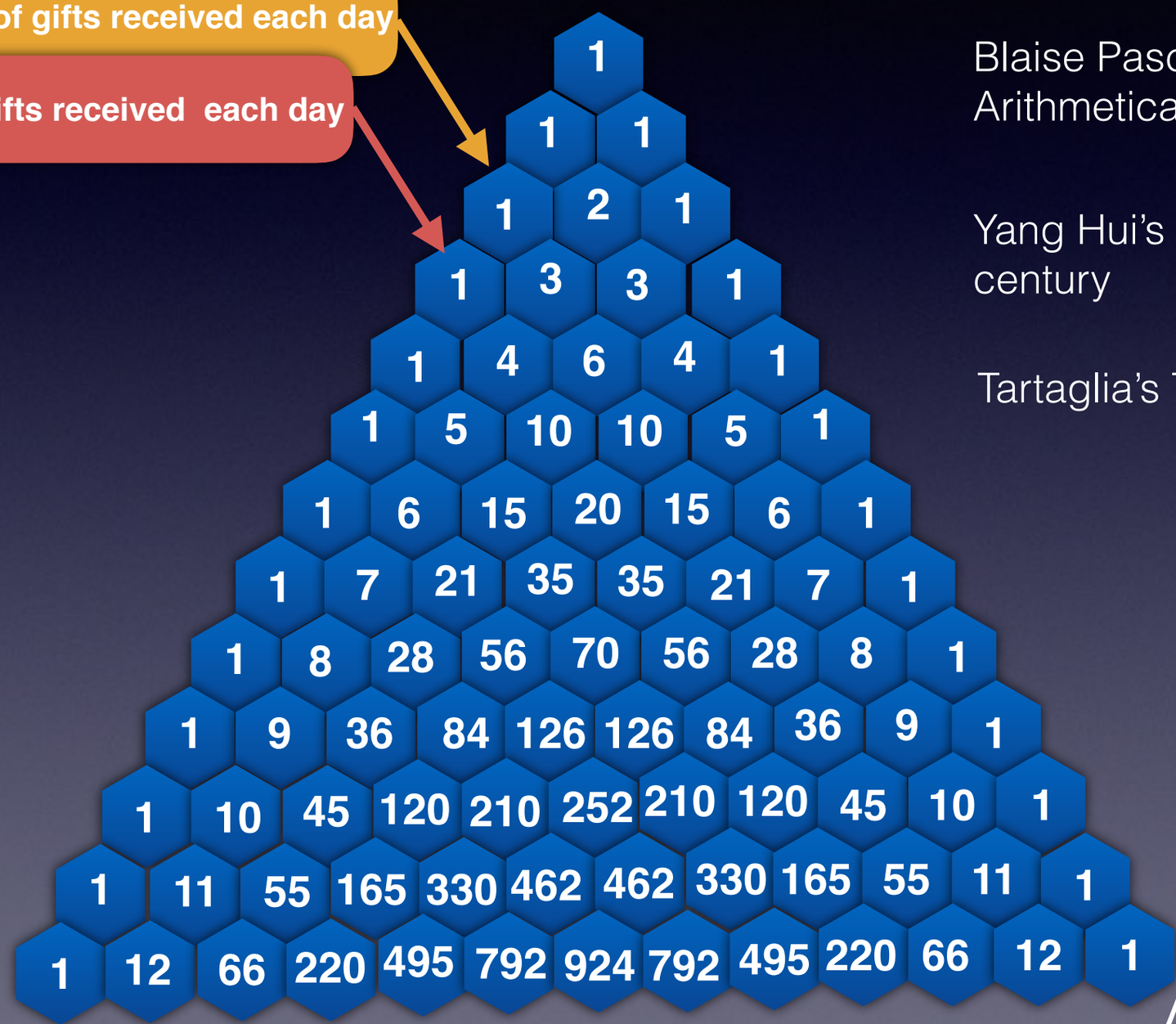
- On the 12th day: 1 + 2 + 3 +…+12

1+(1+2)+(1+2+3)+…+(1+2+3+…+12) = ?

Pascal's Triangle

Number of gifts received each day

Running total number of gifts received each day



Blaise Pascal - "Treatise on Arithmetical Triangle", 1655

Yang Hui's Triangle - the 13th century

Tartaglia's Triangle - in 1556

Answer: 364

Pascal's Method

In how many different paths can you spell SIERPINSKI if you start at the top and proceed to the next row by moving diagonally left or right?

Binomial Coefficients
Binomial Theorem

n = 0	1												
n = 1	1	1											
n = 2	1	2	1										
n = 3	1	3	3	1									
n = 4	1	4	6	4	1								
n = 5	1	5	10	10	5	1							
n = 6	1	6	15	20	15	6	1						

	${}_0C_0$						
	${}_1C_0$	${}_1C_1$					
	${}_2C_0$	${}_2C_1$	${}_2C_2$				
	${}_3C_0$	${}_3C_1$	${}_3C_2$	${}_3C_3$			
	${}_4C_0$	${}_4C_1$	${}_4C_2$	${}_4C_3$	${}_4C_4$		
	${}_5C_0$	${}_5C_1$	${}_5C_2$	${}_5C_3$	${}_5C_4$	${}_5C_5$	
	${}_6C_0$	${}_6C_1$	${}_6C_2$	${}_6C_3$	${}_6C_4$	${}_6C_5$	${}_6C_6$

n = 5

$$(a+b)^5 = {}_5C_0\, a^5 + {}_5C_1\, a^4\, b^1 + {}_5C_2\, a^3\, b^2 + {}_5C_3\, a^2 b^3 + {}_5C_4\, a^1 b^4 + {}_5C_5\, b^5$$

$$(a+b)^5 = 1 \cdot a^5 + 5 \cdot a^4\, b^1 + 10 \cdot a^3\, b^2 + 10 \cdot a^2 b^3 + 5 \cdot a^1 b^4 + 1 \cdot b^5$$

Particular cases of Binomial Theorem

$$(a+b)^n = {}_nC_0a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + {}_nC_3a^{n-3}b^3 + \ldots + {}_nC_nb^n$$

If a = b = 1

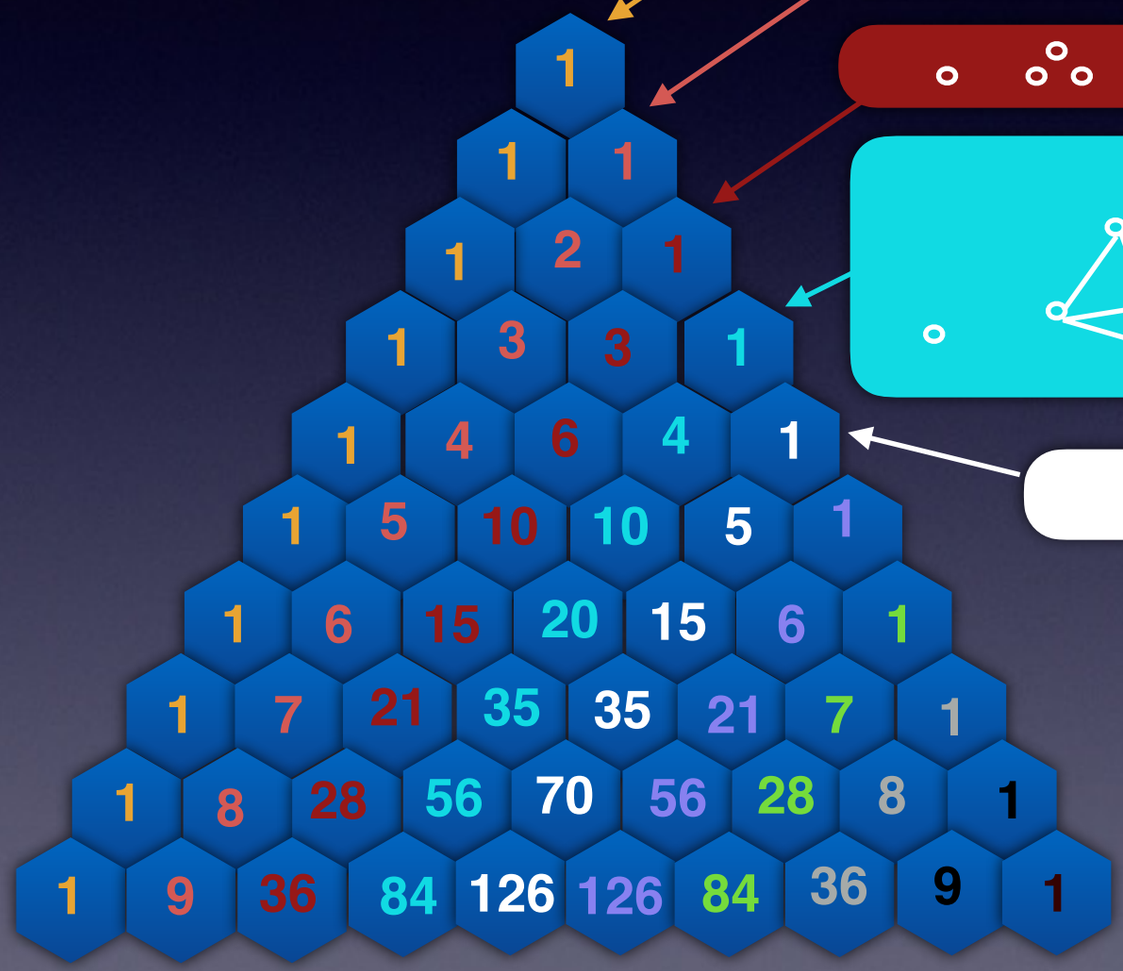
n = 0	1	$1 = 2^0$
n = 1	1 1	$1 + 1 = 2^1$
n = 2	1 2 1	$1 + 2 + 1 = 2^2$
n = 3	1 3 3 1	$1 + 3 + 3 + 1 = 2^3$
n = 4	1 4 6 4 1	$1 + 4 + 6 + 4 + 1 = 2^4$

If a =10, b = 1

n = 0	$(10+1)^0 = 1 \cdot 10^0$	$= 1$
n = 1	$(10+1)^1 = 1 \cdot 10^1 + 1 \cdot 10^0$	$= 11$
n = 2	$(10+1)^2 = 1 \cdot 10^2 + 2 \cdot 10^1 + 1 \cdot 10^0$	$= 121$
n = 3	$(10+1)^3 = 1 \cdot 10^3 + 3 \cdot 10^2 + 3 \cdot 10^1 + 1 \cdot 10^0$	$= 1331$

1
1 1
1 2 1
1 3 3 1

The diagonals in Pascal's Triangle



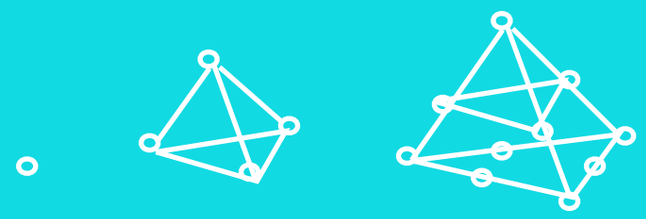
1, 1, 1, 1, 1 ,1, …

A constant sequence

1, 2, 3, 4, 5, 6 , …

The sequence of natural numbers

The sequence of triangular numbers



The sequence of tetrahedral numbers

1, 5, 15, 35, 70, …

The sequence of 4-simplex numbers

Henri Poincaré, about algebraic topology

Fibonacci sequence in Pascal's Triangle

$F_0 = 0 \qquad F_1 = 1 \qquad F_n = F_{n-1} + F_{n-2}$

0, 1, 1, 2, 3, 5, 8, …..

1

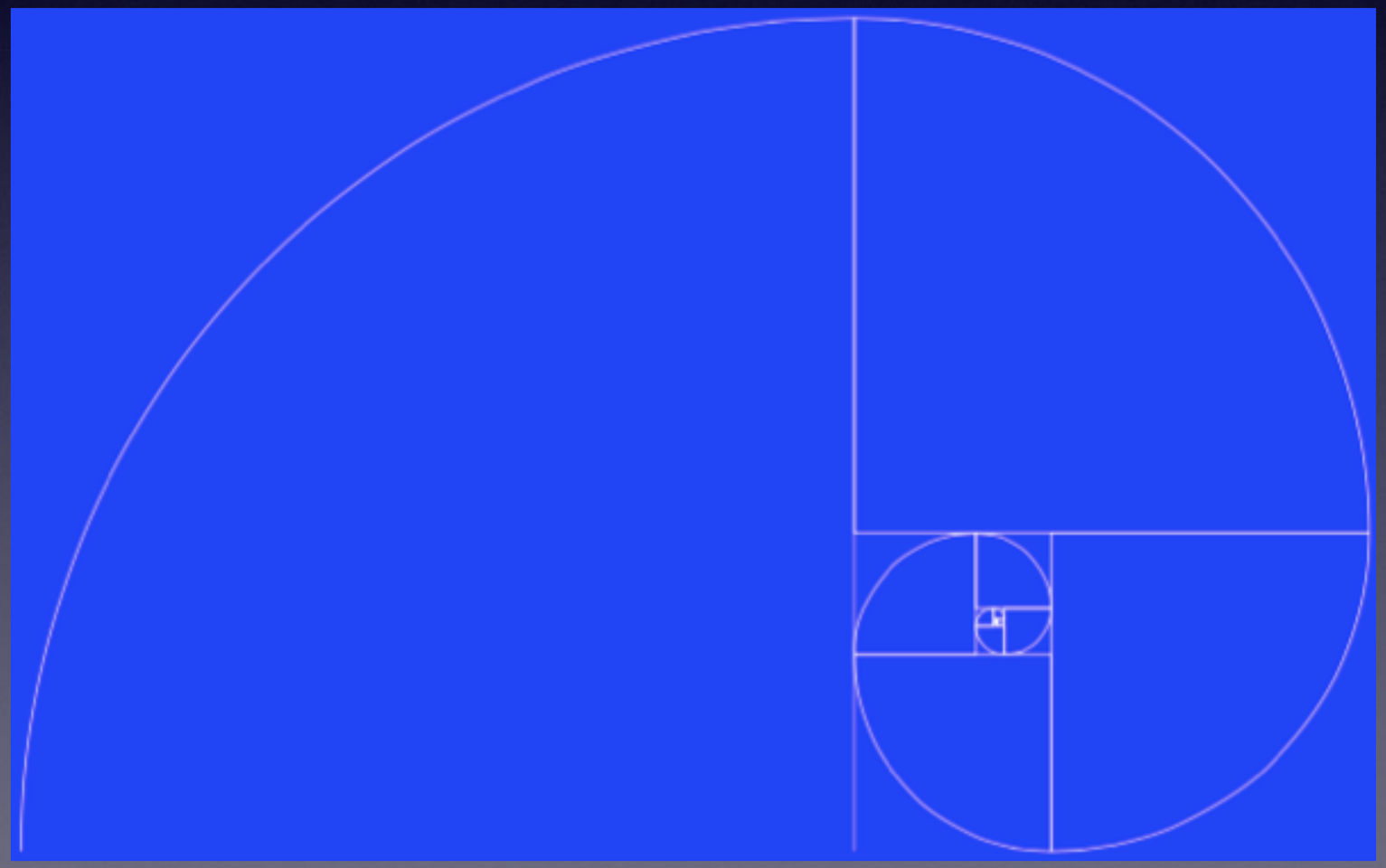
1 1

1 2 1

1 3 3 1

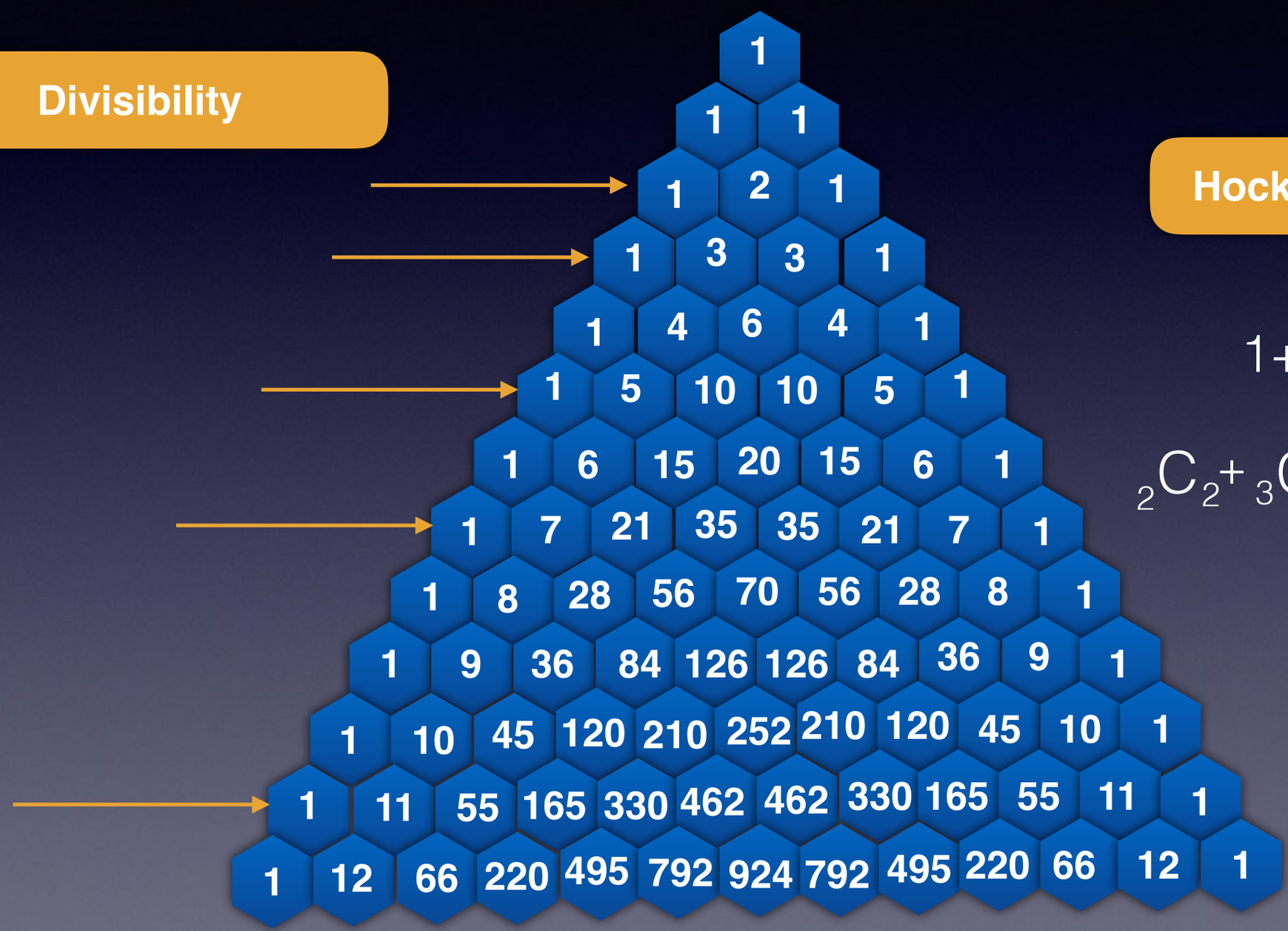
1 4 6 4 1

1 5 10 10 5 1



Other properties of Pascal's Triangle

Divisibility



Hockey-stick identity

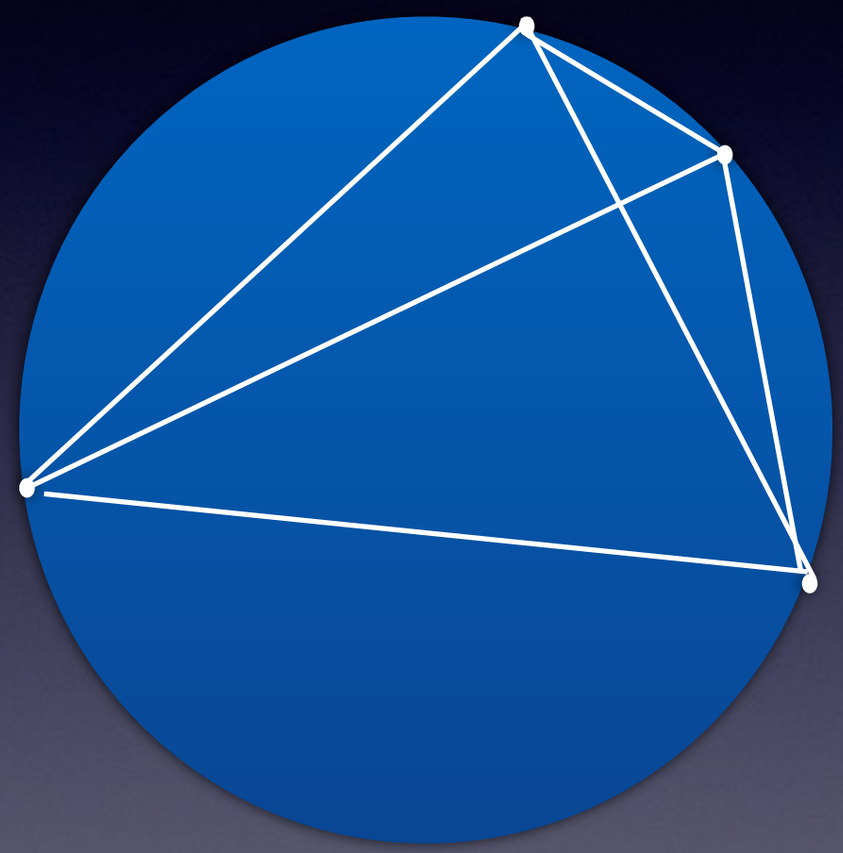
$1+3+6+10=20$

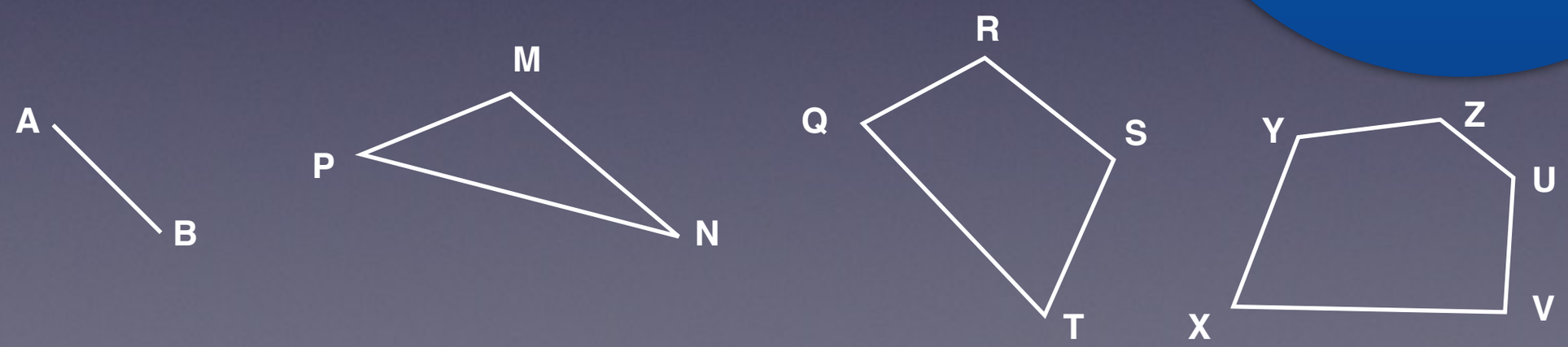
$${}_2C_2 + {}_3C_2 + {}_4C_2 + {}_5C_2 = {}_6C_3$$

$$\sum_{r=k}^{n} {}_rC_k = {}_{n+1}C_{k+1}$$

Is there some geometry in Pascal's Triangle?

	Number of points	Number of segments	Number of triangles	Number of quadri-laterals	Number of penta-gons	Number of hexa-gons
n=0	1					
n=1	1	1				
n=2	1	2	1			
n=3	1	3	3	1		
n=4	1	4	6	4	1	
n=5	1	5	10	10	5	1
n=6	1	6	15	20	15	6	1





Probabilities in Pascal's Triangle?

	Possible outcomes	Total number of outcomes
n=0		
n=1	H T	2^1
n=2	HH HT, TH TT	2^2
n=3	HHH HHT,HTH,THH TTH,THT,HTT TTT	2^3

*

1/2 1/2

1/4 2/4 1/4

1/8 3/8 3/8 1/8

n = number of toss

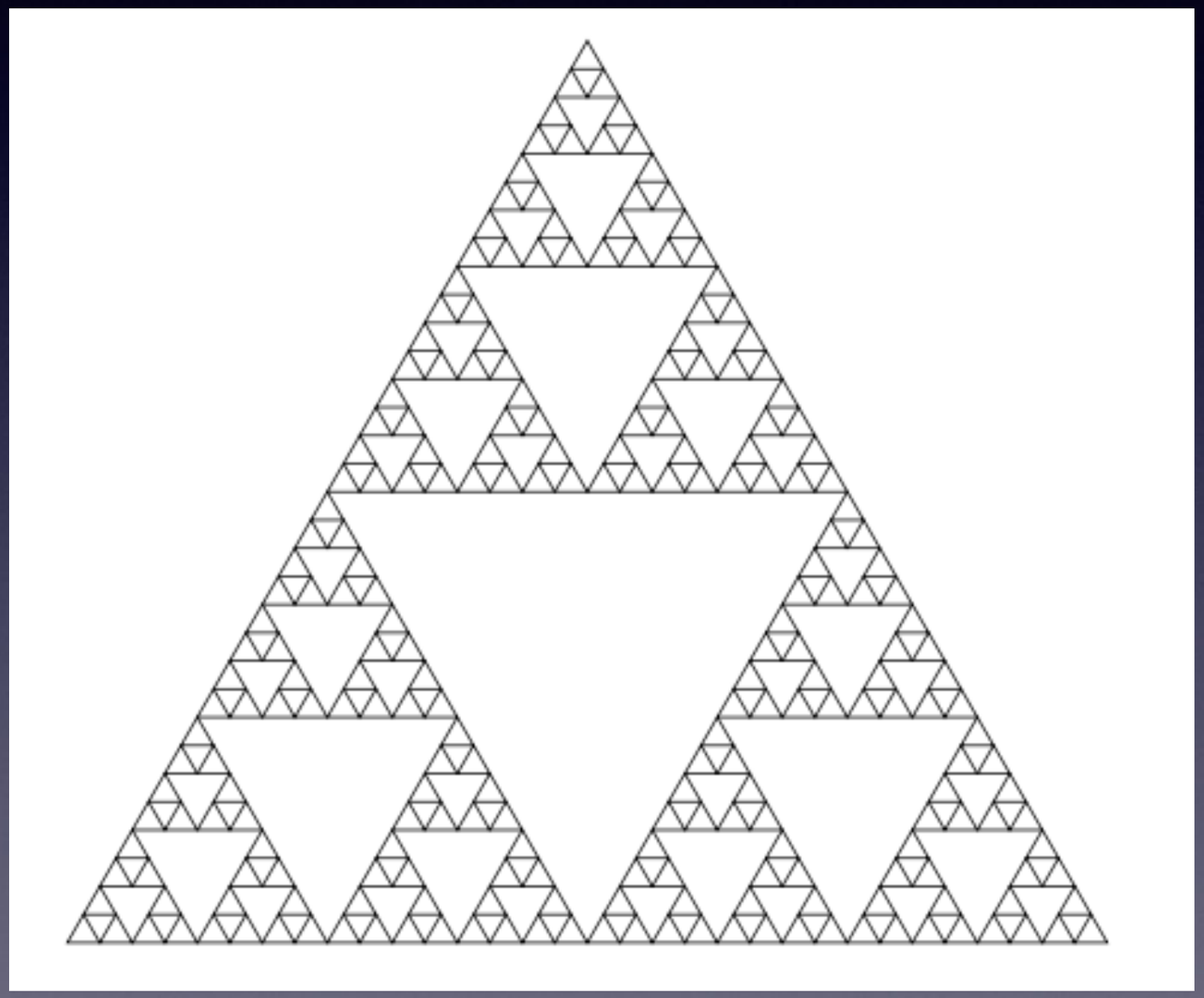
X= number of heads

$$P(X=k)= \frac{{}_nC_k}{2^n}$$

$$P(X=1)= \frac{{}_3C_1}{2^3} = \frac{3}{8}$$

Sierpinski's Triangle

Wacław Sierpiński described the Sierpinski Triangle in 1915. It is a fractal with the overall shape of an equilateral triangle, subdivided recursively into smaller equilateral triangles.



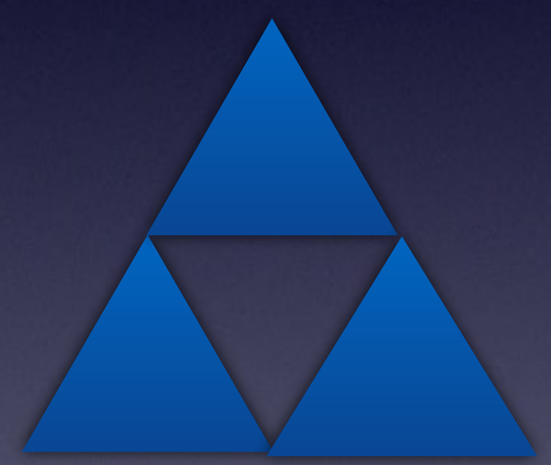
https://www.pinterest.ca/leterrae/italy-tour-cosmatesque-pavements/

However, similar patterns appear already in the 13th-century Cosmati mosaics in the cathedral of Anagni, Italy.

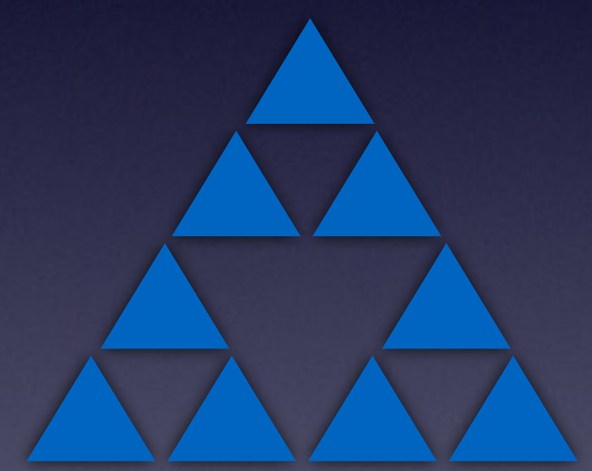
Constructing the Sierpinski Triangle

1. Shrinking and duplication

Step 1: Start with an equilateral triangle



Step 2: Shrink the triangle to 1/2 height and 1/2 width, make three copies

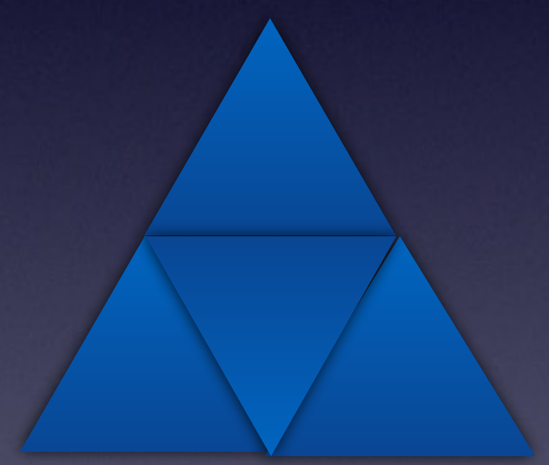


Step 3: Repeat step 2 with each of the smaller triangles

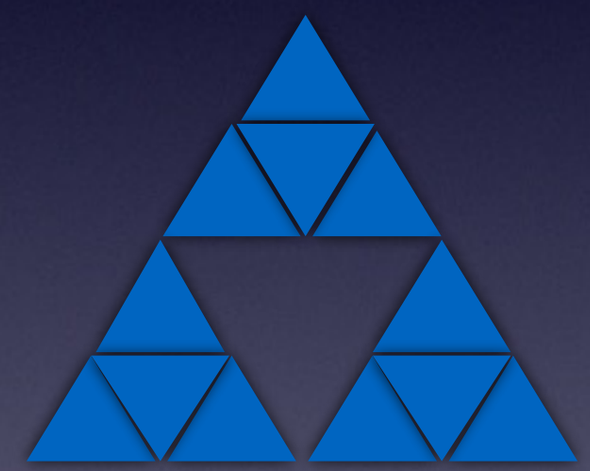
Constructing the Sierpinski Triangle

2. Removing triangles

Step 1: Start with an equilateral triangle



Step 2: Subdivide it into four smaller congruent equilateral triangles and remove the central triangle



Step 3: Repeat step 2 with each of the remaining smaller triangles infinitely

Constructing the Sierpinski Triangle

3. Pascal's Triangle

n=0							1								1	
n=1						1		1						1		1
n=2					1		2		1				1		0		1
n=3				1		3		3		1		1		1		1	
n=4			1		4		6		4		1		1		0		0		0		1
n=5		1		5		10		10		5		1		1		1		0		0		1		1
n=6	1		6		15		20		15		6		1	1		0		1		0		1		0		1
n=7	1	7		21		35		35		21		7		1	1		1		1		1		1		1		1

Pascal's Triangle

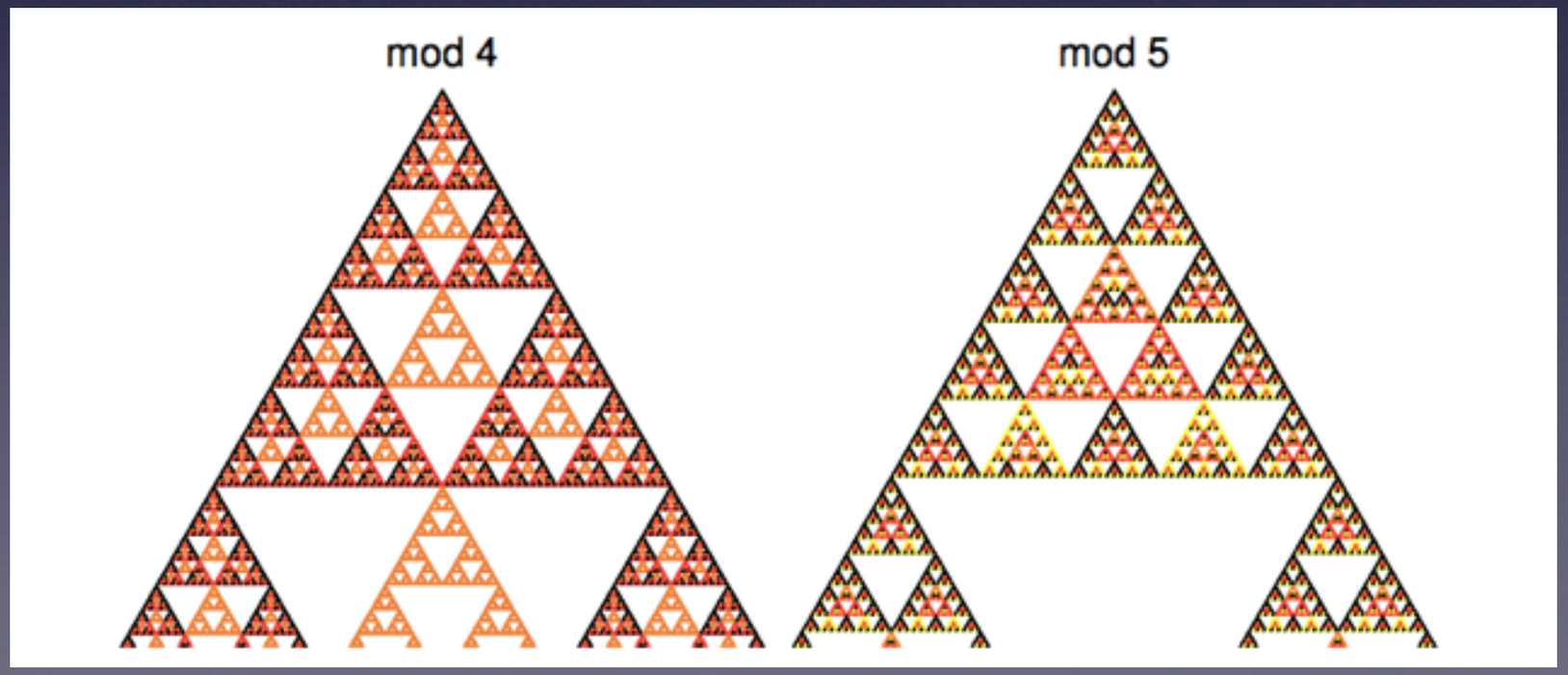
Pascal's Triangle (even/odd numbers)

n=0	1																	
n=1	1		1															
n=2	1		2		1													
n=3	1		0		0		1											
n=4	1		1		0		1		1									
n=5	1		2		1		1		2		1							
n=6	1		0		0		2		0		0		1					
n=7	1		1		0		2		2		0		1		1			
n=8	1		2		1		2		1		2		1		2		1	
n=9	1		0		0		0		0		0		0		0		0		1
n=10	1		1		0		0		0		0		0		0		1		1
n=11	1		2		1		0		0		0		0		0		1		2		1
n=12	1		0		0		1		0		0		0		0		1		0		0		1
n=13	1		1		0		1		1		0		0		0		0		1		1		0		1		1
n=14	1		2		1		1		2		1		0		0		0		1		2		1		1		2		1
n=15	1		0		0		2		0		0		1		0		0		1		0		0		2		0		0		1
n=16	1		1		0		2		2		0		1		1		0		1		1		0		2		2		0		1		1
n=17	1		2		1		2		1		2		1		2		1		1		2		1		2		1		2		1		2		1

Constructing the Sierpinski triangle

Definition If *a* and *b* are integers and $n > 0$, we write $a \equiv b \pmod{n}$ if and only if $n \mid (b - a)$.
We read this as *"a is congruent to b modulo n".*

Examples: $5 \equiv 2 \pmod{3}$, $9 \equiv 0 \pmod{3}$, $10 \equiv 1 \pmod{3}$
$15 \equiv 0 \pmod{5}$, $9 \equiv 4 \pmod{5}$, $11 \equiv 1 \pmod{5}$
$15 \equiv 1 \pmod{7}$, $9 \equiv 2 \pmod{7}$, $11 \equiv 4 \pmod{7}$



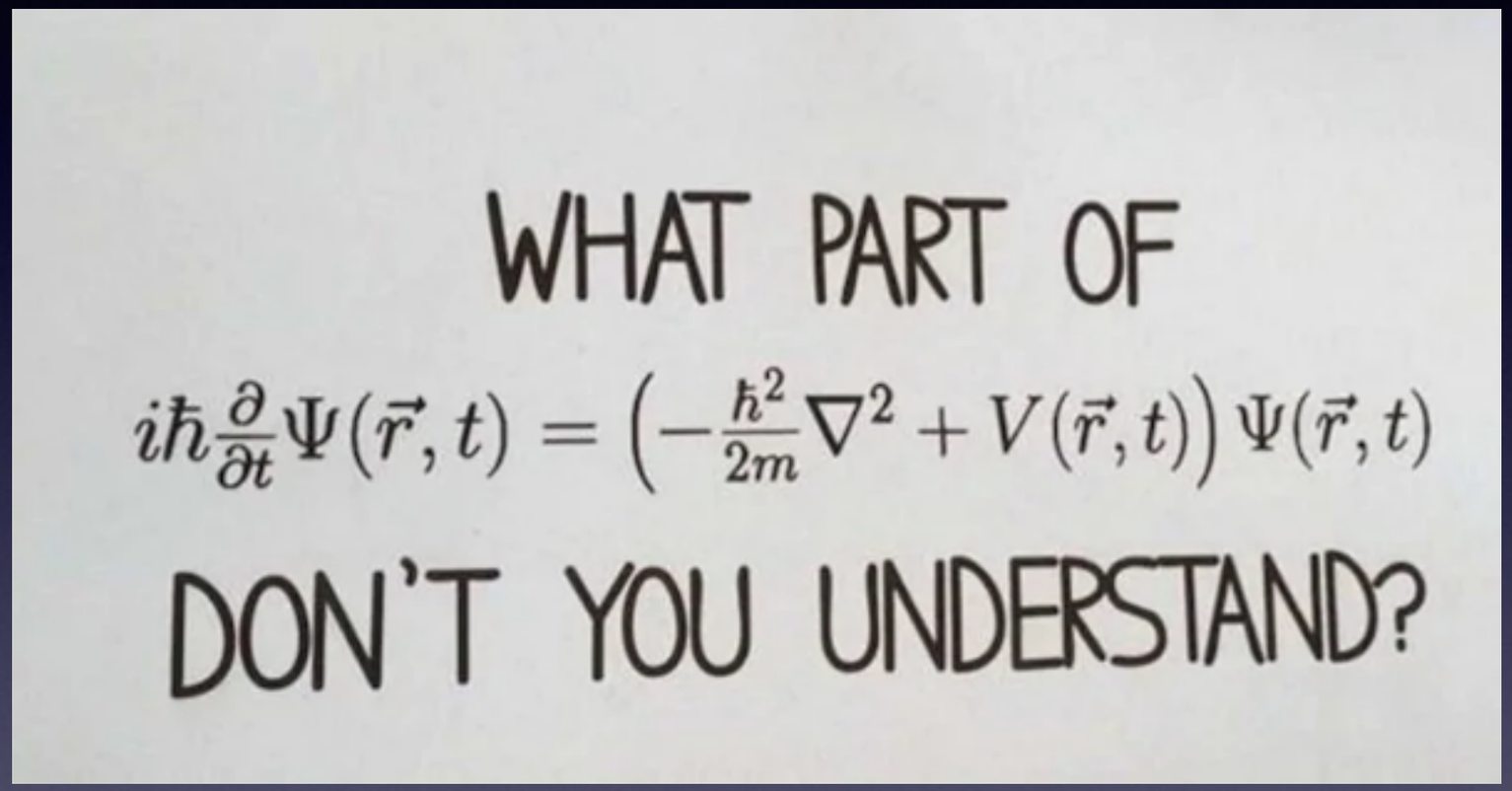
Quantum Pascal's Triangle and Sierpinski's carpet , 2017

Tom Bannink*, Harry Buhrman* ‡

Summary

- Pascal’s Triangle opens many different patterns. The more familiar students become with these patterns, the easier it will be for them to understand mathematics.
- Pascal’s triangle can be taught at different grade levels while showing the students that mathematics is all around us.
- The connections between Pascal's triangle and geometry, probabilities, fractals and arts grant us the opportunity to teach difficult topics in mathematics by using patterns in a more engaging way
- What is math about?

Q&A



Thank you!

Merci!

References

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