Canadian Intermediate Mathematics Contest
CIMC
Sample Contest Solutions

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Solution A1

Calculating, $\dfrac{\sqrt{25-16}}{\sqrt{25}-\sqrt{16}} = \dfrac{\sqrt{9}}{5-4} = \dfrac{3}{1} = 3$.

Solution A2

Since $PQ = QR$, then $\angle QPR = \angle QRP$.
Since $\angle PQR + \angle QPR + \angle QRP = 180°$, then $40° + 2(\angle QRP) = 180°$, so $2(\angle QRP) = 140°$ or $\angle QRP = 70°$.
Since $\angle PRQ$ and $\angle SRT$ are opposite angles, then $\angle SRT = \angle PRQ = 70°$.
Since $RS = RT$, then $\angle RST = \angle RTS = x°$.
Since $\angle SRT + \angle RST + \angle RTS = 180°$, then $70° + 2x° = 180°$ or $2x = 110$ or $x = 55$.

Solution A3

$$\begin{array}{rrcll}
\text{We are given that} & x+y+z & = & 25 & (1) \\
 & x+y & = & 19 & (2) \\
 & y+z & = & 18 & (3) \\
(2)+(3) & x+2y+z & = & 37 & (4) \\
(4)-(1) & y & = & 12 &
\end{array}$$

Solution A4

The sum of the odd numbers from 5 to 21 is

$$5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 = 117$$

Therefore, the sum of the numbers in any row is one-third of this total, or 39.
This means as well that the sum of the numbers in any column or diagonal is also 39.
Since the numbers in the middle row add to 39, then the number in the centre square is $39 - 9 - 17 = 13$.
Since the numbers in the middle column add to 39, then the number in the middle square in the bottom row is $39 - 5 - 13 = 21$.

	5	
9	13	17
x	21	

Since the numbers in the bottom row add to 39, then the number in the bottom right square is $39 - 21 - x = 18 - x$.
Since the numbers in the bottom left to top right diagonal add to 39, then the number in the top right square is $39 - 13 - x = 26 - x$.
Since the numbers in the rightmost column add to 39, then $(26 - x) + 17 + (18 - x) = 39$ or $61 - 2x = 39$ or $2x = 22$, and so $x = 11$.
We can complete the magic square as follows:

19	5	15
9	13	17
11	21	7

Solution A5

Solution 1
Note that $n^{200} = (n^2)^{100}$ and $3^{500} = (3^5)^{100}$.
Since n is a positive integer, then $n^{200} < 3^{500}$ is equivalent to $n^2 < 3^5 = 243$.
Note that $15^2 = 225$, $16^2 = 256$ and if $n \geq 16$, then $n^2 \geq 256$.
Therefore, the largest possible value of n is 15.

Solution 2
Since n is a positive integer and $500 = 200(2.5)$, then $n^{200} < 3^{500}$ is equivalent to $n^{200} < (3^{2.5})^{200}$, which is equivalent to $n < 3^{2.5} = 3^2 3^{0.5} = 9\sqrt{3}$.
Since $9\sqrt{3} \approx 15.59$ and n is an integer, the largest possible value of n is 15.

Solution A6

Since the grid is a 5 by 5 grid of squares and each square has side length 10 cm, then the whole grid is 50 cm by 50 cm.

Since the diameter of the coin is 8 cm, then the radius of the coin is 4 cm.
We consider where the centre of the coin lands when the coin is tossed, since the location of the centre determines the position of the coin.
Since the coin lands so that no part of it is off of the grid, then the centre of the coin must land at least 4 cm (1 radius) away from each of the outer edges of the grid.
This means that the centre of the coin lands anywhere in the region extending from 4 cm from the left edge to 4 cm from the right edge (a width of $50 - 4 - 4 = 42$ cm) and from 4 cm from the top edge to 4 cm to the bottom edge (a height of $50 - 4 - 4 = 42$ cm).
Thus, the centre of the coin must land in a square that is 42 cm by 42 cm in order to land so that no part of the coin is off the grid.
Therefore, the total admissible area in which the centre can land is $42 \times 42 = 1764$ cm^2.
Consider one of the 25 squares. For the coin to lie completely inside the square, its centre must land at least 4 cm from each edge of the square.
As above, it must land in a region of width $10 - 4 - 4 = 2$ cm and of height $10 - 4 - 4 = 2$ cm.
There are 25 possible such regions (one for each square) so the area in which the centre of the coin can land to create a winning position is $25 \times 2 \times 2 = 100$ cm^2.
Thus, the probability that the coin lands in a winning position is equal to the area of the region in which the centre lands giving a winning position, divided by the area of the region in which the coin may land, or $\frac{100}{1764} = \frac{25}{441}$.

Solution B1

(a) The average of any set of integers is equal to the sum of the integers divided by the number of integers.
Thus, the average of the integers is $\frac{71+72+73+74+75}{5} = \frac{365}{5} = 73$.

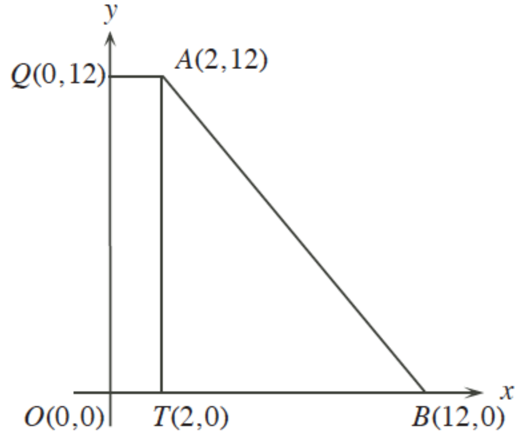
(b) (i) Simplifying, $n + (n+1) + (n+2) + (n+3) + (n+4) = 5n + 10$.

(ii) Since the sum of the 5 consecutive integers is $5n + 10$, the average of these integers is $\dfrac{5n+10}{5} = n + 2$.
If n is an even integer, then $n + 2$ is an even integer.
If n is an odd integer, then $n + 2$ is an odd integer.
Thus for the average $n + 2$ to be odd, the integer n must be odd.

(c) Simplifying, $n + (n+1) + (n+2) + (n+3) + (n+4) + (n+5) = 6n + 15$.
Since the sum of the 6 consecutive integers is $6n + 15$, the average of these integers is $\dfrac{6n+15}{6} = n + \frac{15}{6} = n + \frac{5}{2}$.
For every integer n, $n + \frac{5}{2}$ is never an integer.
Therefore, the average of six consecutive integers is never an integer.

Solution B2

(a) *Solution 1*



If point T is placed at $(2,0)$, then T is on OB and AT is perpendicular to OB.
Since QO is perpendicular to OB, then QO is parallel to AT.
Both QA and OT are horizontal, so then QA is parallel to OT.
Therefore, $QATO$ is a rectangle.
The area of rectangle $QATO$ is $QA \times QO$ or $(2-0) \times (12-0) = 24$.
Since AT is perpendicular to TB, we can treat AT as the height of $\triangle ATB$ and TB as the base.
The area of $\triangle ATB$ is $\frac{1}{2} \times TB \times AT$ or $\frac{1}{2} \times (12-2) \times (12-0) = \frac{1}{2} \times 10 \times 12 = 60$.
The area of $QABO$ is the sum of the areas of rectangle $QATO$ and $\triangle ATB$, or $24 + 60 = 84$.

Solution 2

Both QA and OB are horizontal, so then QA is parallel to OB.
Thus, $QABO$ is a trapezoid.
Since QO is perpendicular to OB, we can treat QO as the height of the trapezoid.
Then, $QABO$ has area $\frac{1}{2} \times QO \times (QA + OB) = \frac{1}{2} \times 12 \times (2+12) = \frac{1}{2} \times 12 \times 14 = 84$.

(b) Since CO is perpendicular to OB, we can treat CO as the height of $\triangle COB$ and OB as the base. The area of $\triangle COB$ is $\frac{1}{2} \times OB \times CO$ or $\frac{1}{2} \times (12-0) \times (p-0) = \frac{1}{2} \times 12 \times p = 6p$.

(c) Since QA is perpendicular to QC, we can treat QC as the height of $\triangle QCA$ and QA as the base. The area of $\triangle QCA$ is $\frac{1}{2} \times QA \times QC$ or $\frac{1}{2} \times (2-0) \times (12-p) = \frac{1}{2} \times 2 \times (12-p) = 12-p$.

(d) The area of $\triangle ABC$ can be found by subtracting the area of $\triangle COB$ and the area of $\triangle QCA$ from the area of quadrilateral $QABO$.
From parts (a), (b) and (c), the area of $\triangle ABC$ is thus $84 - 6p - (12-p) = 72 - 5p$.
Since the area of $\triangle ABC$ is 27, then $72 - 5p = 27$ or $5p = 45$, so $p = 9$.

Solution B3

(a) The largest positive integer N that can be written in this form is obtained by maximizing the values of the integers a, b, c, d, and e. Thus, $a = 1$, $b = 2$, $c = 3$, $d = 4$, and $e = 5$, which gives $N = 1(1!)+2(2!)+3(3!)+4(4!)+5(5!) = 1+2(2)+3(6)+4(24)+5(120) = 719$.

Solution B3 (continued)

(b) For any two positive integers n and m, it is always possible to write a division statement of the form,

$$n = qm + r,$$

where the *quotient* q and *remainder* r are non-negative integers and $0 \leq r < m$.
The following table shows some examples of this.

n	m	q	r	$n = qm + r$
20	6	3	2	$20 = 3(6) + 2$
12	13	0	12	$12 = 0(13) + 12$
9	7	1	2	$9 = 1(7) + 2$
36	9	4	0	$36 = 4(9) + 0$

Notice that in each of the 4 examples, the inequality $0 \leq r < m$ has been satisfied.
We can always satisfy this inequality by beginning with n and then subtracting multiples of m from it until we get a number in the range 0 to $m - 1$. We let r be this number, or $r = n - qm$, so that $n = qm + r$.
Further, this process is repeatable. For example, beginning with $n = 653$ and $m = 5! = 120$, we get $653 = 5(120) + 53$. We can now repeat the process using remainder $r = 53$ as our next n, and $4! = 24$ as our next m. This process is shown in the table below with each new remainder becoming our next n and m taking the successive values of $5!$, $4!$, $3!$, $2!$, and $1!$.

n	m	q	r	$n = qm + r$
653	120	5	53	$653 = 5(120) + 53$
53	24	2	5	$53 = 2(24) + 5$
5	6	0	5	$5 = 0(6) + 5$
5	4	1	1	$5 = 2(2) + 1$
1	1	1	0	$1 = 1(1) + 0$

From the 5th column of the table above,

$$\begin{aligned}
653 &= 5(120) + 53 \\
&= 5(120) + 2(24) + 5 \\
&= 5(120) + 2(24) + 0(6) + 5 \\
&= 5(120) + 2(24) + 0(6) + 2(2) + 1 \\
&= 5(120) + 2(24) + 0(6) + 2(2) + 1(1) + 0 \\
&= 5(5!) + 2(4!) + 0(3!) + 2(2!) + 1(1!)
\end{aligned}$$

Thus, $n = 653$ is written in the required form with $a = 1$, $b = 2$, $c = 0$, $d = 2$, and $e = 5$.

Solution B3 (continued)

(c) Following the process used in (b) above, we obtain the more general result shown here.

n	m	q	r	$n = qm + r$	restriction on r
n	120	e	r_1	$n = e(120) + r_1$	$0 \leq r_1 < 120$
r_1	24	d	r_2	$r_1 = d(24) + r_2$	$0 \leq r_2 < 24$
r_2	6	c	r_3	$r_2 = c(6) + r_3$	$0 \leq r_3 < 6$
r_3	2	b	r_4	$r_3 = b(2) + r_4$	$0 \leq r_4 < 2$
r_4	1	a	r_5	$r_4 = a(1) + r_5$	$0 \leq r_5 < 1$

From the 5th column of this table,

$$
\begin{aligned}
n &= e(120) + r_1 \\
&= e(120) + d(24) + r_2 \\
&= e(120) + d(24) + c(6) + r_3 \\
&= e(120) + d(24) + c(6) + b(2) + r_4 \\
&= e(120) + d(24) + c(6) + b(2) + a(1) + r_5 \\
&= e(5!) + d(4!) + c(3!) + b(2!) + a(1!) \text{ (since } r_5 = 0)
\end{aligned}
$$

We must justify that the integers a, b, c, d, and e satisfy their required inequality.
From part (b), each of these quotients is a non-negative integer. Therefore, it remains to show that $a \leq 1$, $b \leq 2$, $c \leq 3$, $d \leq 4$, and $e \leq 5$.
From part (a), $N = 719$, therefore $0 \leq n < 720$.
From the table above, we have $n = e(120)+r_1$. Therefore $e(120)+r_1 < 720$ or $e(120) < 720$ (since $r_1 \geq 0$), and so $e < 6$. Thus, $e \leq 5$, as required.
Also from the table above, $r_1 < 120$, so $d(24) + r_2 < 120$ or $d(24) < 120$ (since $r_2 \geq 0$), and therefore $d < 5$. Thus, $d \leq 4$, as required.
Also, $r_2 < 24$, so $c(6) + r_3 < 24$ or $c(6) < 24$ (since $r_3 \geq 0$), and therefore $c < 4$.
Thus, $c \leq 3$, as required.
Continuing, $r_3 < 6$, so $b(2) + r_4 < 6$ or $b(2) < 6$ (since $r_4 \geq 0$), and therefore $b < 3$.
Thus, $b \leq 2$, as required.
Finally, $r_4 < 2$, so $a(1) + r_5 < 2$ or $a(1) < 2$ (since $r_5 = 0$), and therefore $a < 2$.
Thus, $a \leq 1$, as required.
Therefore, all integers n, with $0 \leq n \leq N$, can be written in the required form.

Solution B3 (continued)

(d) Since $c = 0$, we are required to find the sum of all integers n of the form $n = a + 2b + 24d + 120e$, with the stated restrictions on the integers a, b, d, and e.
Since $n = a + 2b + 24d + 120e = (a + 2b) + 24(d + 5e)$, let $n_1 = a + 2b$ and $n_2 = d + 5e$ so that $n = n_1 + 24n_2$. First, consider all possible values of n_1.
Since $0 \le a \le 1$ and $0 \le b \le 2$ and $n_1 = a + 2b$, we have that n_1 can equal any of the numbers in the set $\{0, 1, 2, 3, 4, 5\}$. Each of these comes from exactly one pair (a, b).
Next, find all possible values for $n_2 = d + 5e$. Since $0 \le d \le 4$ and $0 \le e \le 5$, we have that $d + 5e$ can equal any of the numbers in the set $\{0, 1, 2, 3, 4, 5, 6, 7, \ldots, 29\}$.
Each of these comes from exactly one pair (d, e).
Therefore, $24n_2$ can equal any of the numbers in the set $\{24 \times 0, 24 \times 1, 24 \times 2, \ldots, 24 \times 29\} = \{0, 24, 48, \ldots, 696\}$, the multiples of 24 from 0 to 696.
Adding each of these possible values of $24n_2$ in turn to each of the 6 possible values of n_1, we get the set of all possible $n = n_1 + 24n_2$:

$$\{0, 1, 2, 3, 4, 5, 24, 25, 26, 27, 28, 29, 48, 49, 50, 51, 52, 53, \ldots, 696, 697, 698, 699, 700, 701\}$$

Because each of the 6 possible values of n_1 comes from exactly one pair (a, b) and each of the 30 possible values of n_2 comes from exactly one pair (d, e), then each of these integers above occurs exactly once as a, b, d, and e move through their possible values.
It remains to find the sum of these possible values for n:

$$\begin{aligned}
& 0+1+2+3+4+5+24+25+26+27+28+29+48+49+\cdots+699+700+701 \\
= \ & 0+1+2+3+4+5+(24+0)+(24+1)+(24+2)+(24+3)+(24+4)+(24+5) \\
& +(48+0)+(48+1)+\cdots+(696+3)+(696+4)+(696+5) \\
= \ & (0+1+2+3+4+5)+24\times 6+(0+1+2+3+4+5)+48\times 6 \\
& +(0+1+2+3+4+5)+\cdots+696\times 6+(0+1+2+3+4+5) \\
= \ & 30(0+1+2+3+4+5)+24\times 6+48\times 6+\cdots+696\times 6 \\
= \ & 30(15)+24(6)[1+2+3+\cdots+29] \\
= \ & 30(15)+24(6)\left[\frac{29\times 30}{2}\right] \\
= \ & 63090
\end{aligned}$$