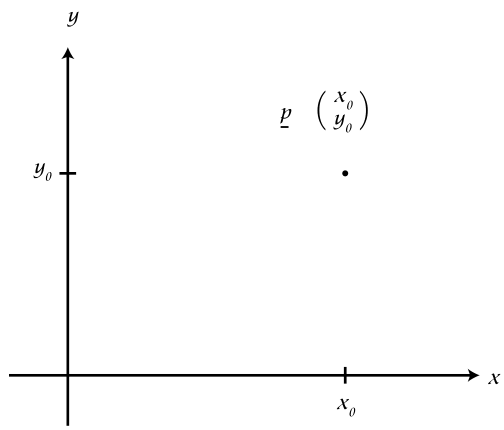
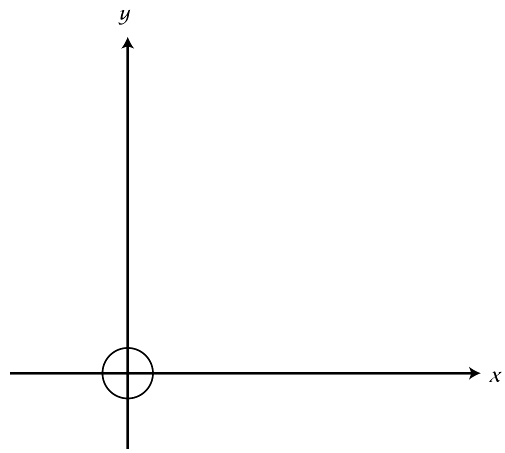
Senior Math Circles
Wednesday, February 8, 2017
Special Relativity I

1 Co-ordinates in the Plane

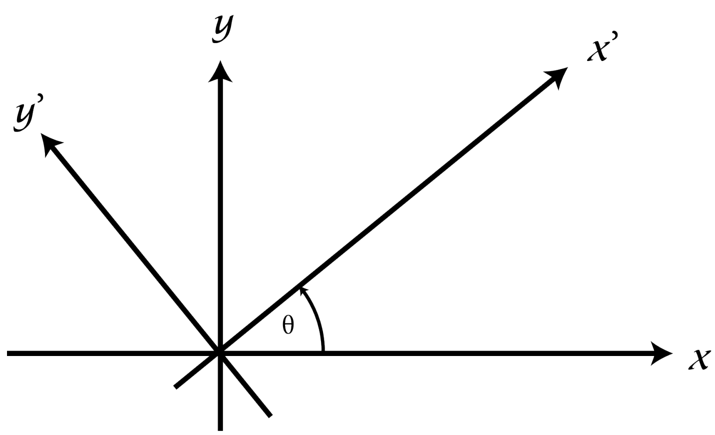
We are familiar with co-ordinates in $\mathbb{R}^2$ - the Cartesian plane.





Note that
(i) we usually draw the axes orthogonal,
(ii) lines $y =$ constant are parallel to $y = 0$:the x-axis (and symmetrically)
(iii) we often have a co-ordinate grid - that is why I have given you graph paper.

Changes of co-ordinates are important in these lectures:
we insist on having the same origin.



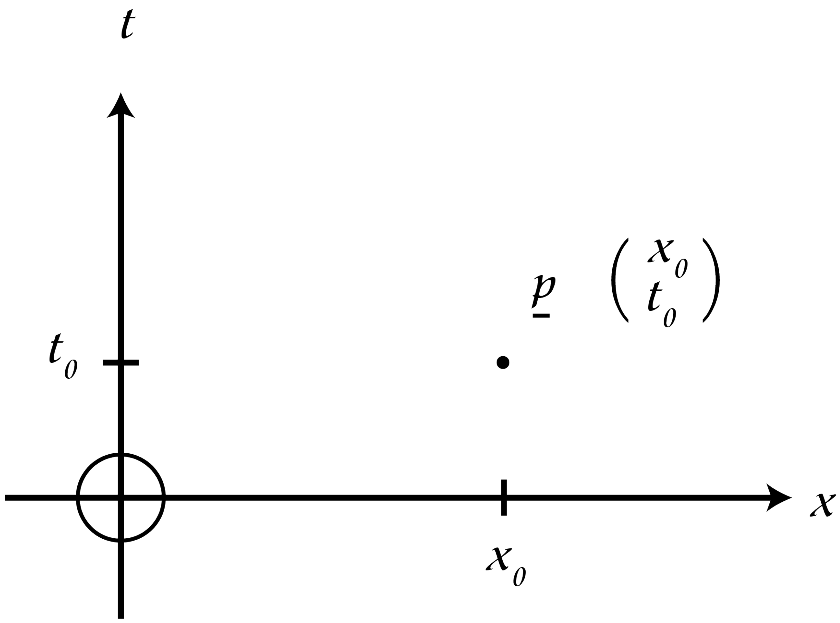
We assume that the new axes are also orthogonal. If we know the angle of rotation then we can relate the two sets of co-ordinates.

$$\begin{aligned} \acute{x} &= x\cos() + y\sin(\theta) \\ \acute{y} &= -x\sin(\theta) + y\cos(\theta) \end{aligned}$$

Note that we have no hesitation about mixing up the x and y co-ordinates.

2 Spacetime

To examine particles moving in our physical world we need a time co-ordinate too. Fortunately we will only need to consider one spatial co-ordinate for most of our work and so 2 dimensions is still appropriate.
We will refer to a point in spacetime as an **event** to remind us that there is a time component.

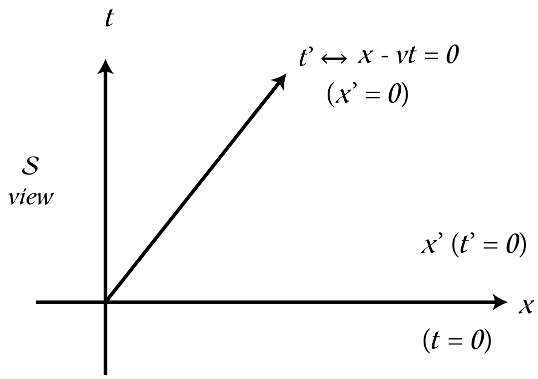


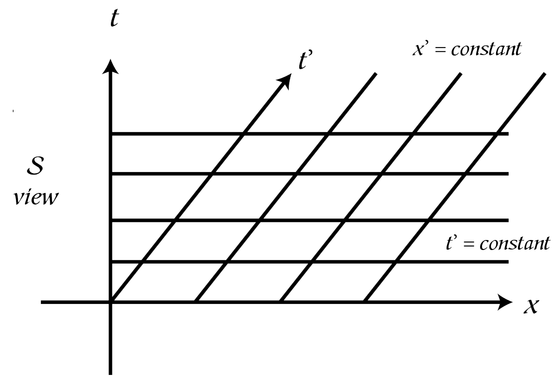
There now follows a talk about Galileo and his importance in our modern world.

2.1 The Galilean transformation

When we wish to relate two different co-ords we will assume that they have the same origin. We assume that we have two inertial frames S and $\acute{S}$ with the latter movind at speed v to the right with respect to the former.The Galilean transformation was widely believed true up to the late 1800's and it does not let time change from one system to another. The x co-ordinate changes in the natural way.

$$\begin{aligned} \acute{x} &= x - vt \\ \acute{t} &= t \end{aligned}$$





3 Inertial frames

A co-ordinate system in spacetime is an inertial co-ordinate system, equivalently an inertial frame and we will also use inertial observer when:
(I) It is locally Euclidean - looks like $\mathbb{R}^3$—spatially
(II) There exists a global time co-ordinate
(III) Free objects remain at rest or in a state of constant velocity

Thus there cannot be any acceleration and we are excluding gravity (see General Relativity).
Really we need a non accelerating set of axes a long way from anything else (which would be a source of gravity).
In practice a region of space and time on the earth is a reasonable approximation for small periods of both space and time.
We will only deal with inertial frames in Special Relativity.
If I am in an inertial frame, usually referred to as S then any other observer moving at constant velocity relative to me is also in an inertial frame.

3.1 Two fundamental Axioms

Axiom 1 The Principle of Relativity
The laws of physics are the same in all inertial frames.
The thinking behind this does go back to Galileo.

Thus there is no special (or preferred) inertial frame.

Axiom 2 The Law of Light Propagation

Light signals in a vacuum propagate in straight lines and the speed of propagation has the same value in all directions and relative to all inertial frames.

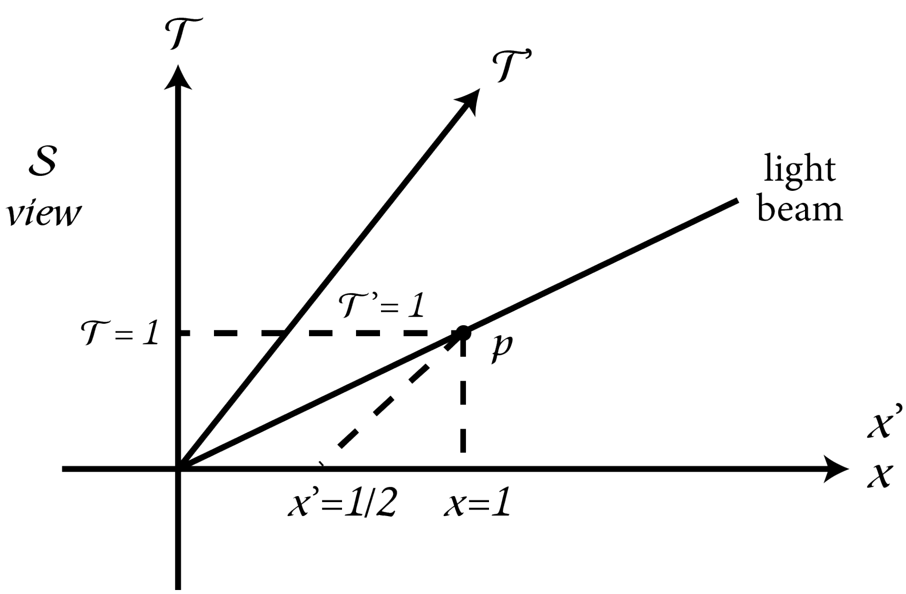
We refer to this value as c and it is approx. 3.10^8ms^{-1}

Thus if I shine a torch and measure the speed of the light, then I will get a value of c, and if you are on a spacecraft moving away from me at $\frac{c}{2}$, and if you also measure the velocity of the same beam of light then you will also get the same value of c.

There now follows a discussion about wave motion in general, the expectation that light needed a medium which was referred to as the aether, and the failure to detect it in the Michelson Morley experiment. The lack of the aether is strong evidence that the speed of light cannot be affected by relative motion: there is no aether to move relative too.

3.2 The end of the GalileanTransformation

We will use $ct = T$ and $c\acute{t} = \acute{T}$ axes in the following diagrams so that a light beam has slope of 45 degrees. Consider a light beam sent from the origin, and consider its path relative to two inertial frames S and $\acute{S}$ where we will assume that $\acute{S}$ is moving at $\frac{c}{2}$ to the right relative to S. See the figure.
If we consider the event P on the light path.
It has co-ordinates $x = 1$ $T = 1$ relative to S, to find its co-ordinates relative to $\acute{S}$ we use the Galilean transformation to get $\acute{x} = 1 - \frac{c}{2}\frac{1}{c} = \frac{1}{2}$ while $\acute{T} = T = 1$ so that $\acute{t} = \frac{1}{c}$, and thus the frame $\acute{S}$ measures the light pulse to have speed $\left(\frac{1}{2}\right) / \left(\frac{1}{c}\right) = \frac{c}{2}$. This is a violation of axiom 2, we conclude that the Galilean transformation is incorrect.



4 Derivation of the two dimensional Lorentz Transformation

Starting point:

$$x' = \gamma(x - vt) \ (1)$$

If we examine the motion of S relative to S' then we would obtain:

$$x = \acute{\gamma}(x + vt) \ (2)$$

Explain why $\gamma = \acute{\gamma}$ otherwise we have a way to distinguish inertial frames - violating axiom 1.

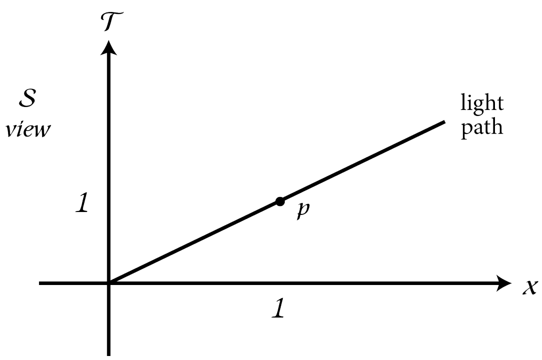
Plus sign because the motion is in the other direction

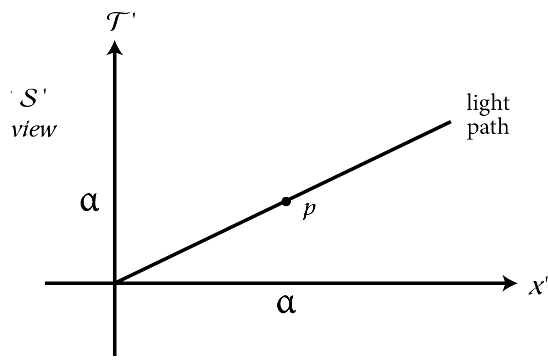
Substitute (1) into (2) to get:

$$x = \gamma(\gamma(x - vt) + v\acute{t}) \ (3)$$

which we solve for t' to get:

$$t' = \gamma t + \frac{x}{\gamma v}(1 - \gamma^2) \ (4)$$





Using (1) and (4) to transform the co-ords of **P** yields:

$$\begin{aligned} \alpha &= \gamma(1 - \frac{v}{c}) \ (5) \\ \alpha &= \gamma + \frac{c}{\gamma v}(1 - \gamma^2) \ (6) \end{aligned}$$

Equating these gives

$$\gamma(1 - \frac{v}{c}) = \gamma + \frac{c}{\gamma v}(1 - \gamma^2) \ (7)$$

and solving for γ produces the expression:

$$\gamma = \frac{1}{\sqrt{(1 - (\frac{v}{c})^2)}} = \left[(1 - \left(\frac{v}{c}\right)^2)\right]^{\frac{-1}{2}} \quad (8)$$

γ is called the Lorentz factor after the Dutch physicist H.A. Lorentz.

The derived transformation is called the Lorentz Transformation, it replaces the Galilean transformation and is significant at high velocities, i.e when the term $\frac{v}{c}$ is significant compared to unity.

4.1 The Lorentz Transformation

$$\begin{aligned} x' &= \gamma(x - vt) \quad (9) \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \quad (10) \end{aligned}$$

If we use the scaled variables T , $\acute{T}$ and V defined by

$$T = ct,\ \acute{T} = c\acute{t} \ \text{ and } V = \frac{v}{c}$$

then the Lorentz transformation takes on a much simpler and symmetric form:

$$\begin{aligned} x' &= \gamma(x - VT) \quad (11) \\ T' &= \gamma\,(T - Vx) \quad (12) \end{aligned}$$

Note that in the Lorentz transformation, space and time are allowed to be completely mixed up, and there is no longer the division of space and time that partially existed in the Galilean transformation (i.e. with $\acute{t} = t$).We now have a spacetime continuum.