Population Models Part III

Marek Stastna and Kris Rowe

University of Waterloo

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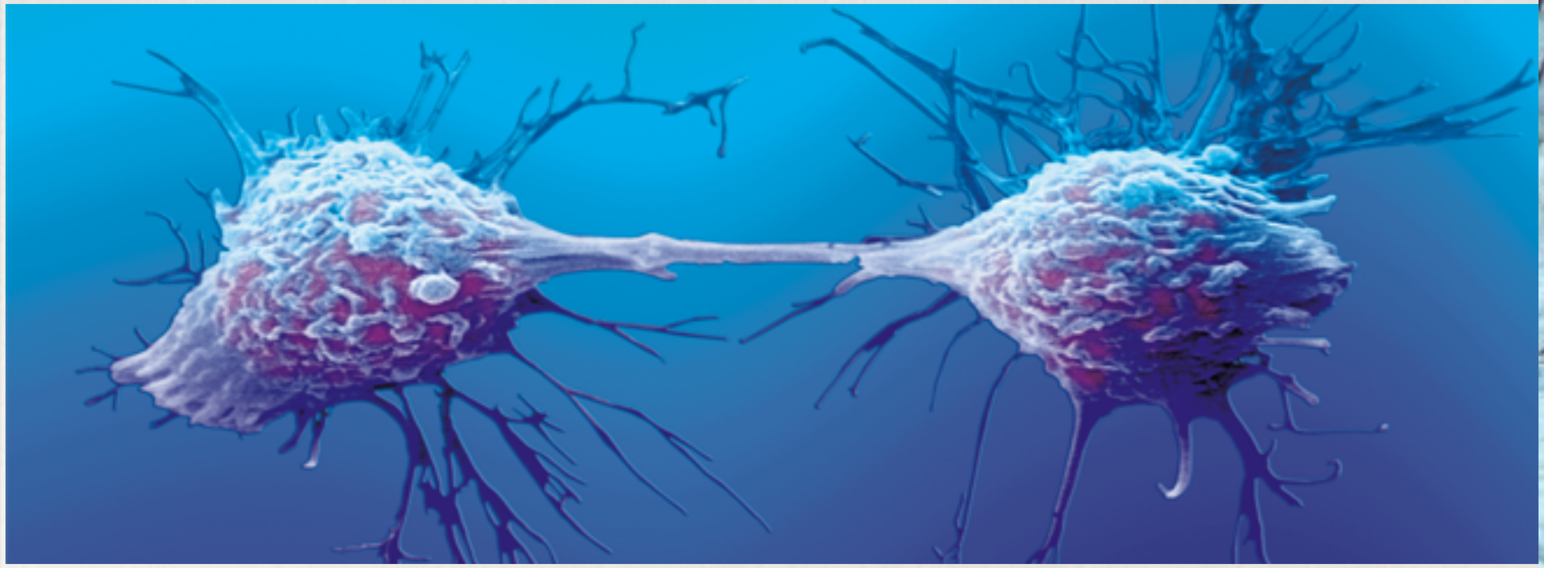
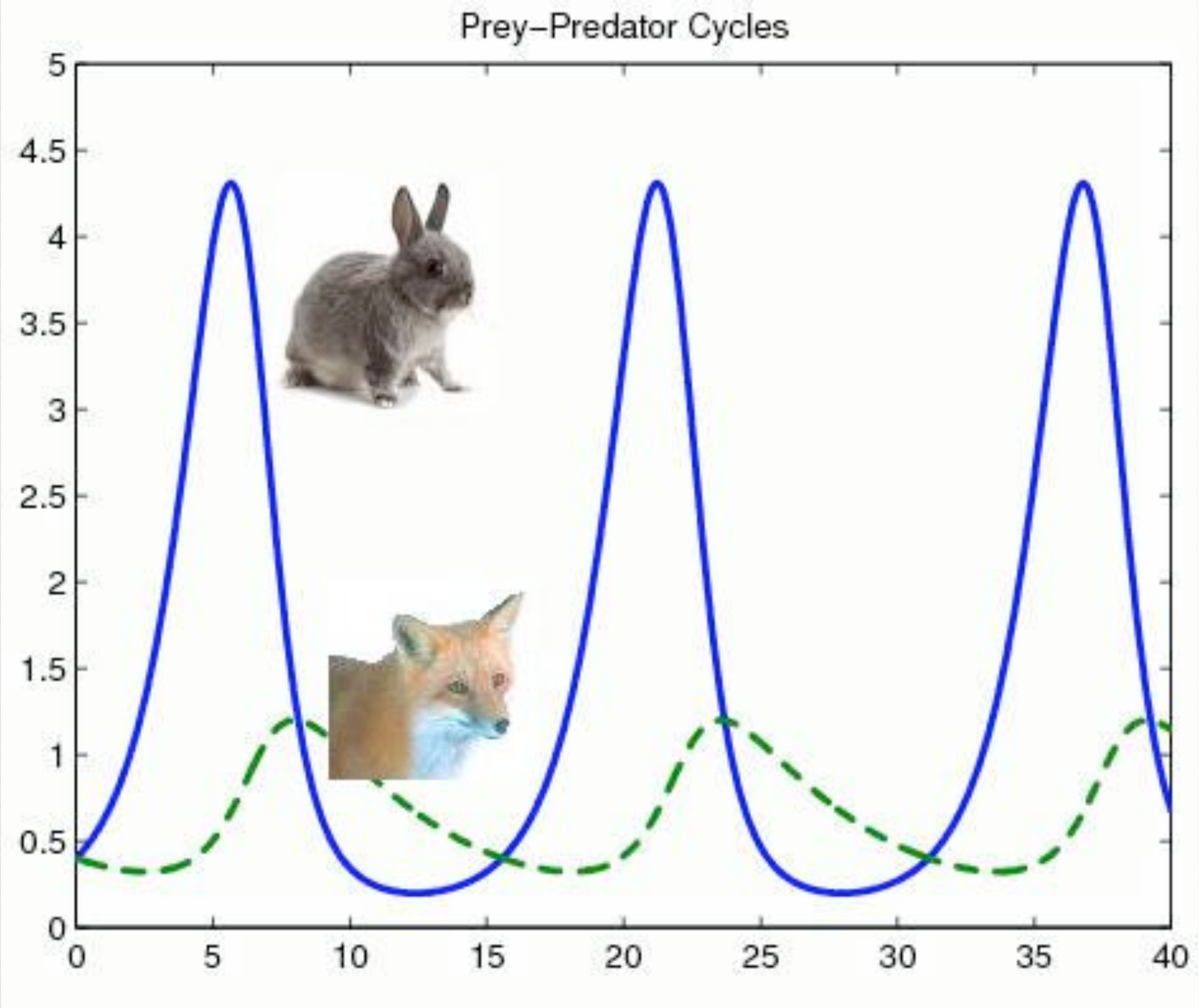
- *Last time you got a quick introduction to Population Models.*
- *We discussed both the math side of things (which may have been new) and the modeling side of things (which was new for sure).*




- *Let's recall some of the main take away messages:*
- *The exponential growth model is the simplest type of population model and gives a population that keeps growing forever when the birth rate is larger than the death rate.*
- *When the death rate is bigger than the birth rate the population decreases, but as it gets smaller the decrease also gets smaller.*
- *Calculating the population step by step is a lot easier than getting a solution to the recursion relation as a formula.*
- *It is easy to break the mathematical requirements of our model (recall 1.4 individuals...)*

- *The predator-prey model whose results are on the right is very different from the "real" predator prey interaction shown on the left.*
- *This is because a real fox has to actually catch a real rabbit in order to eat it and that is not given by a mathematical formula or an abstract variable like "population".*



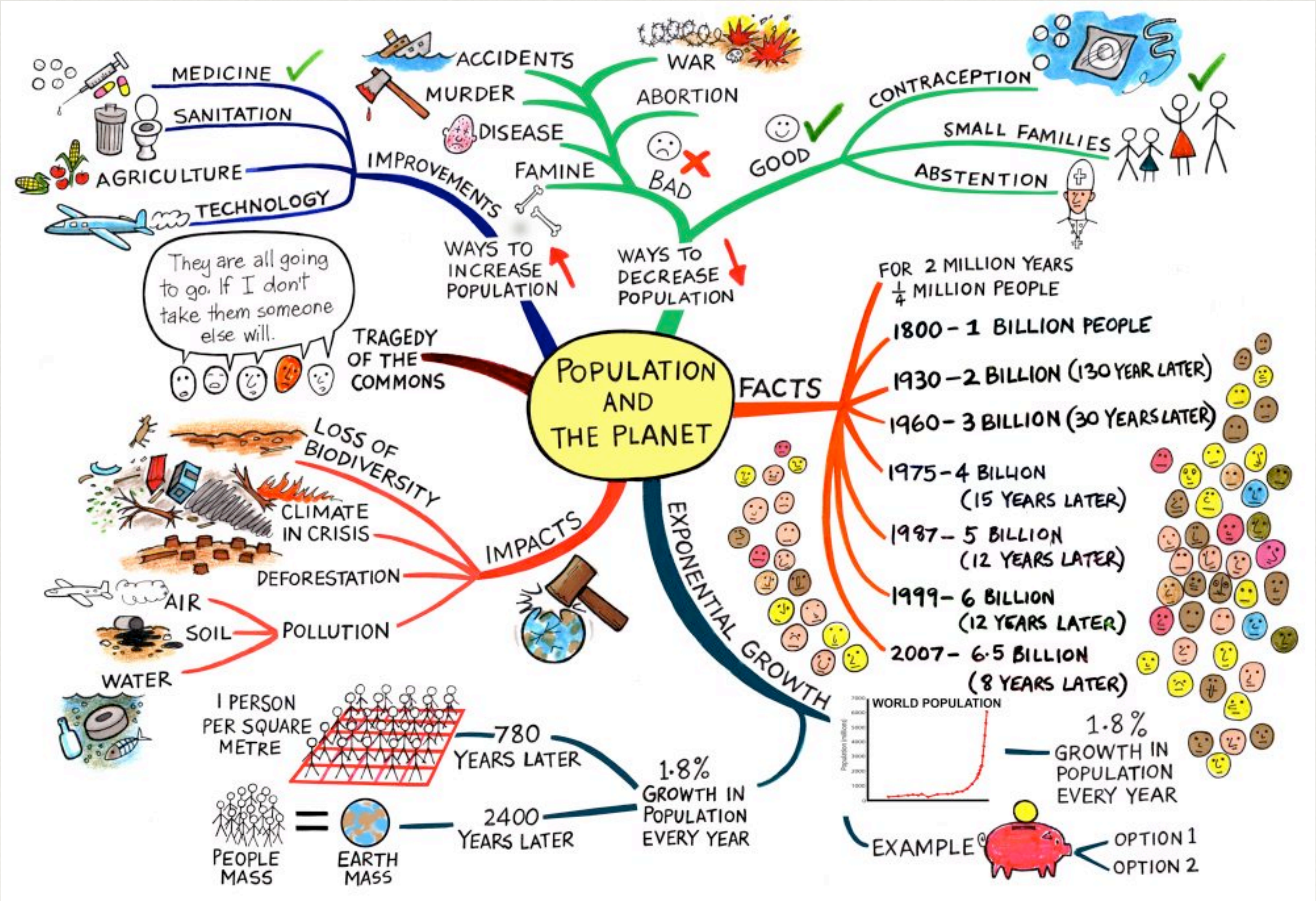
- *This lecture and its problem sets are all about the idea of an uncertain outcome and how to include it in our model.*
- *Start with a simple game: Two players (A and B) start with 3 poker chips each and one coin to flip.*
- *The coin is flipped. Heads means A pays B, tails means B pays A.*
- *When does the game stop?*

Game	Turn 1	Turn 2	Turn 3	Turn 4	Turn 5	Turn 6	Turn 7
1	2-4	3-3	2-4	1-5	2-4	3-3	2-4
2	4-2	5-1	6-0				
3	4-2	3-3	4-2	5-1	6-0		

- *2 of the 3 games ended. But you can see if we were to flip heads then tails then heads then tails, and so on, the game would never end.*

- *In games of chance, like the simple coin flip game, or casino gambling like roulette, or blackjack the randomness is a big part of the attraction of the game.*
- *Of course it isn't the only attraction of the game. For real gamblers it is the mix of strategy and randomness that keeps them at the table.*
- *For population models the randomness is a way to acknowledge that our model is likely a "cartoon" of the real world.*

- *Even this very complicated population model is an oversimplification of the real world.*

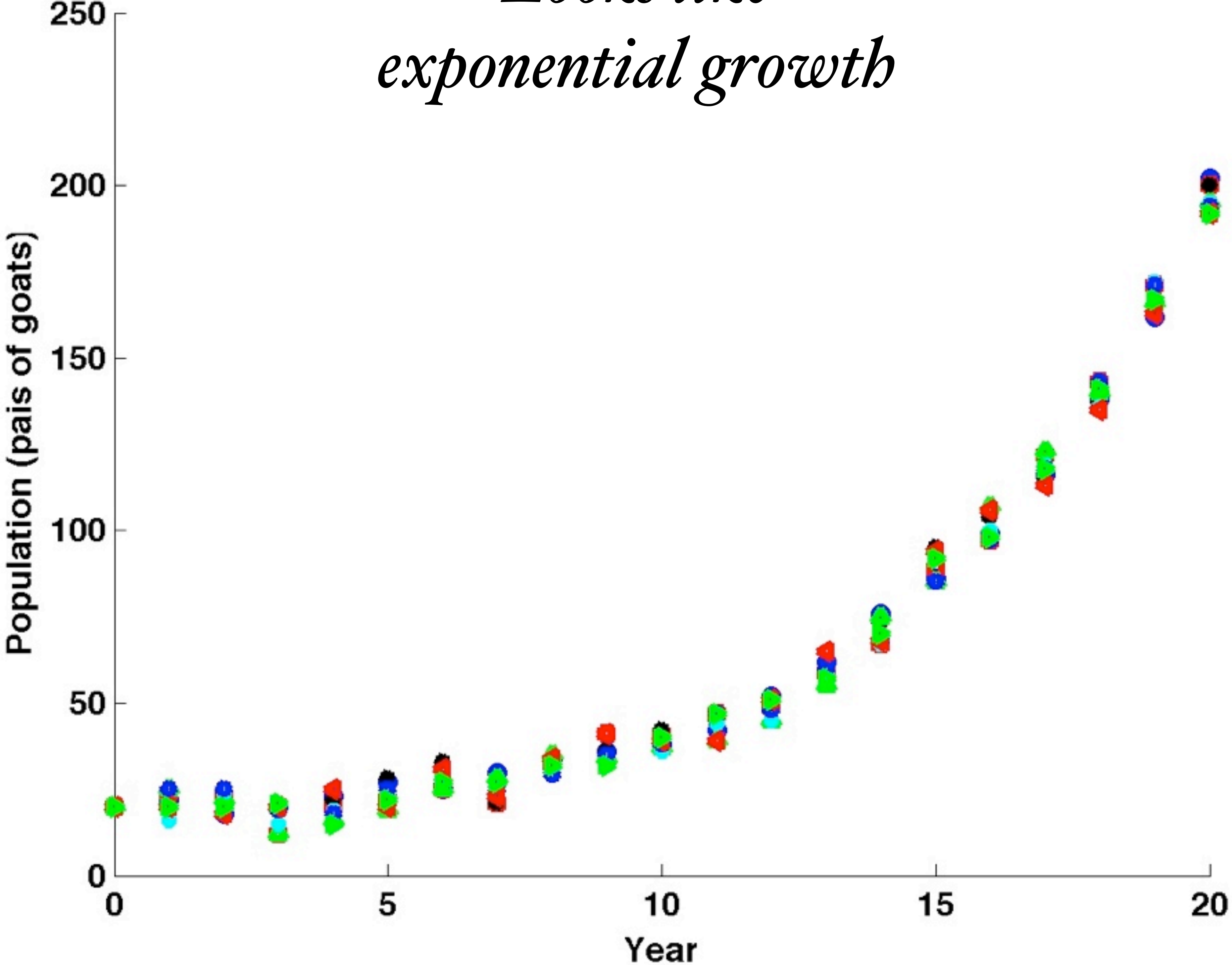


- *Let's consider the example of an isolated population of goats on an island with plenty for them to eat.*
- *Let's assume the exponential model works well enough (try to recall what this means about how the population compares to the carrying capacity).*
- *Now let's assume that once a year 10 hunters are allowed on the island and each has a chance to hunt precisely 1 pair of goats.*
- *They may or may not catch the goats, so we will assume a coin flip of heads means "yes" and tails means "no".*

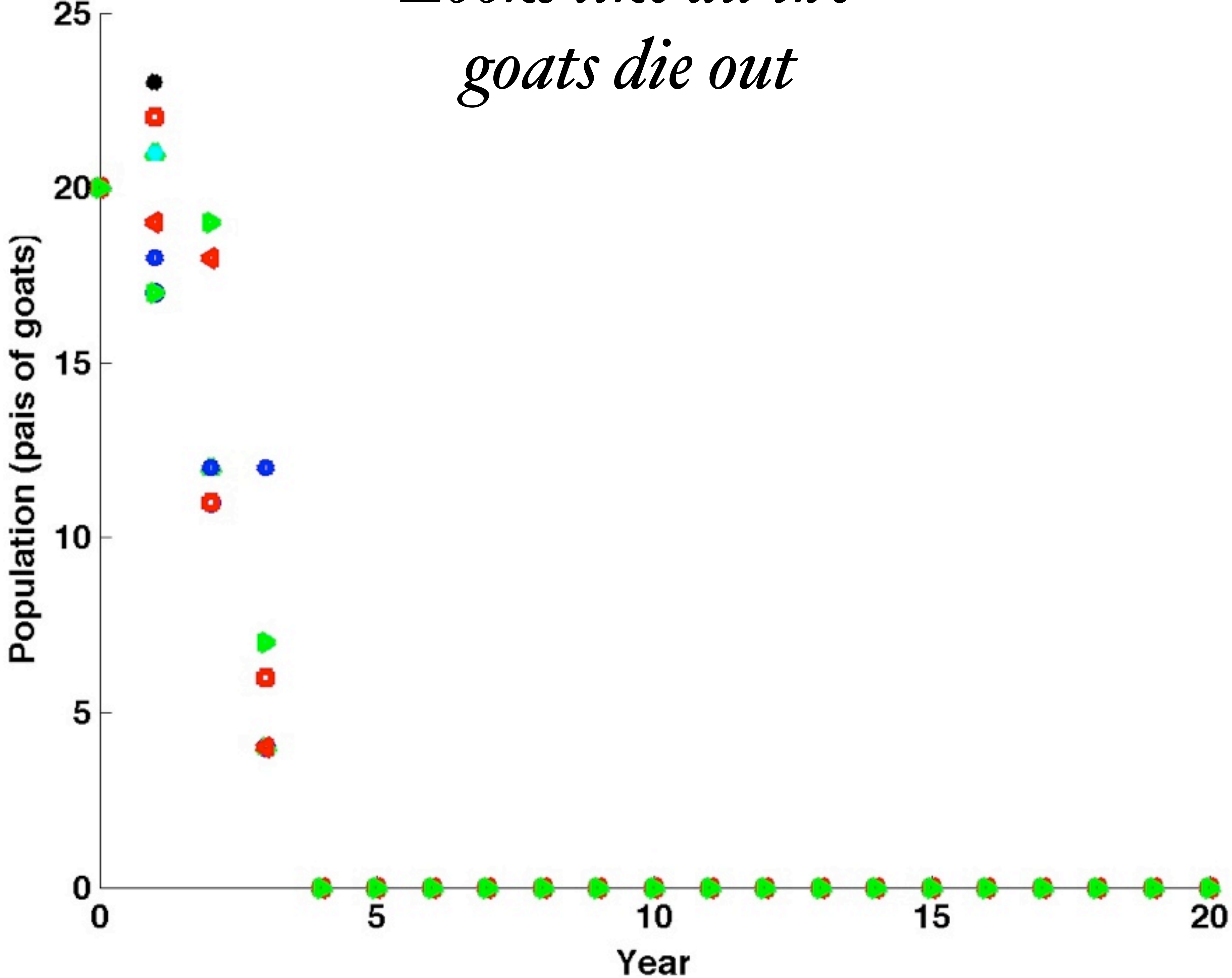
$$P^{(n+1)} = \lfloor (1+\alpha-\beta)P^{(n)} \rfloor - H^{(n)}$$

- *This is how the model is written mathematically.*
- *The H term is a random number between 0 (all tails) and 10 (all heads).*
- *If we start with 20 pairs of goats, and alpha=0.5, beta=0.25 we can play the game (I will let you try it later) and graph the results.*
- *I play 8 games at a time on my computer so the graphs don't get too cluttered.*

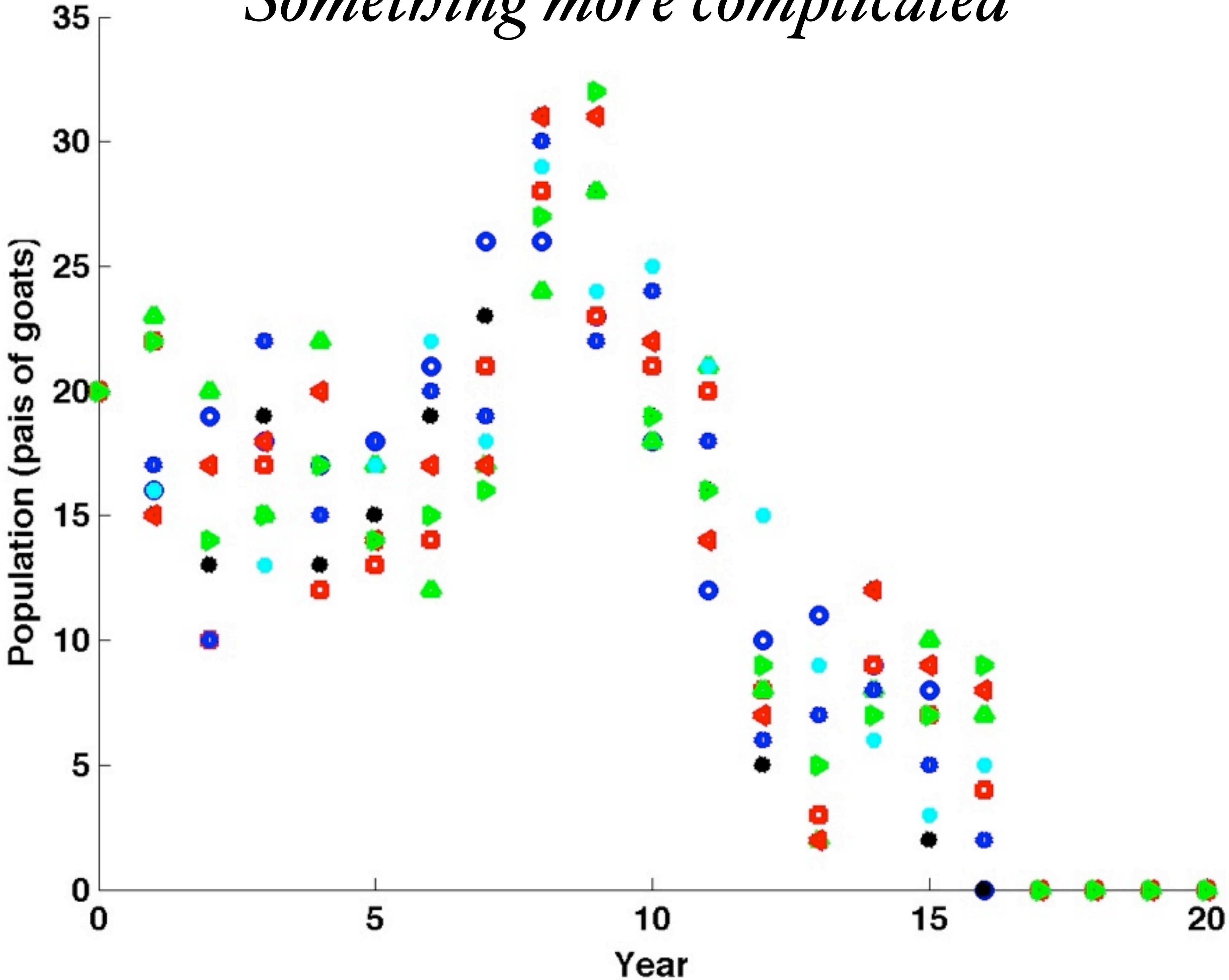
Looks like exponential growth



Looks like all the goats die out

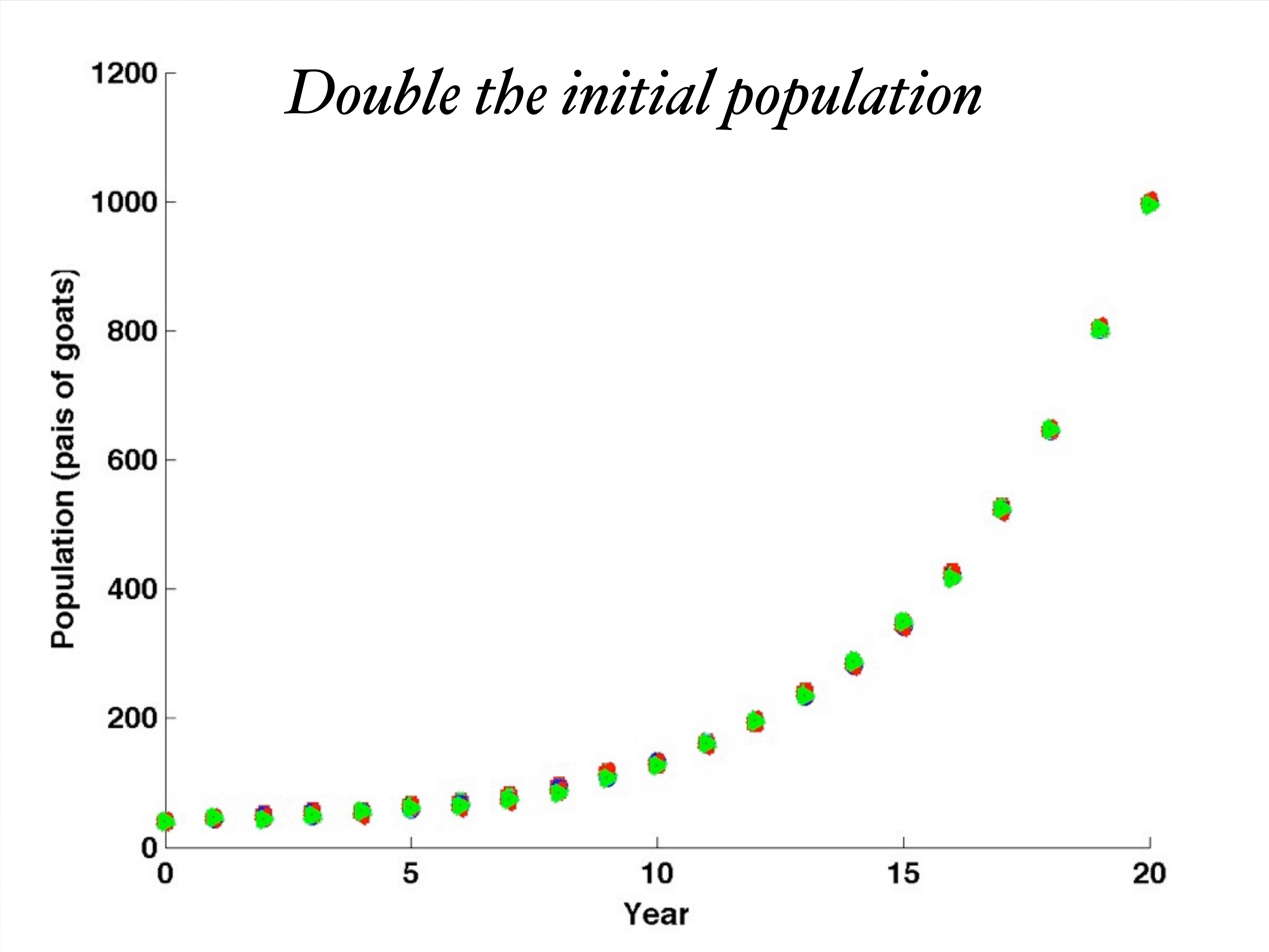


Something more complicated



$$P^{(n+1)} = \lfloor (1+\alpha-\beta)P^{(n)} \rfloor - H^{(n)}$$

- *What do the results say?*
- *Well at a very basic level they say the randomness plays a really strong role.*
- *In some cases the population grows pretty much like the exponential model says it should.*
- *In other cases the goats die out completely!*
- *What could help us ensure the goats don't die out?*

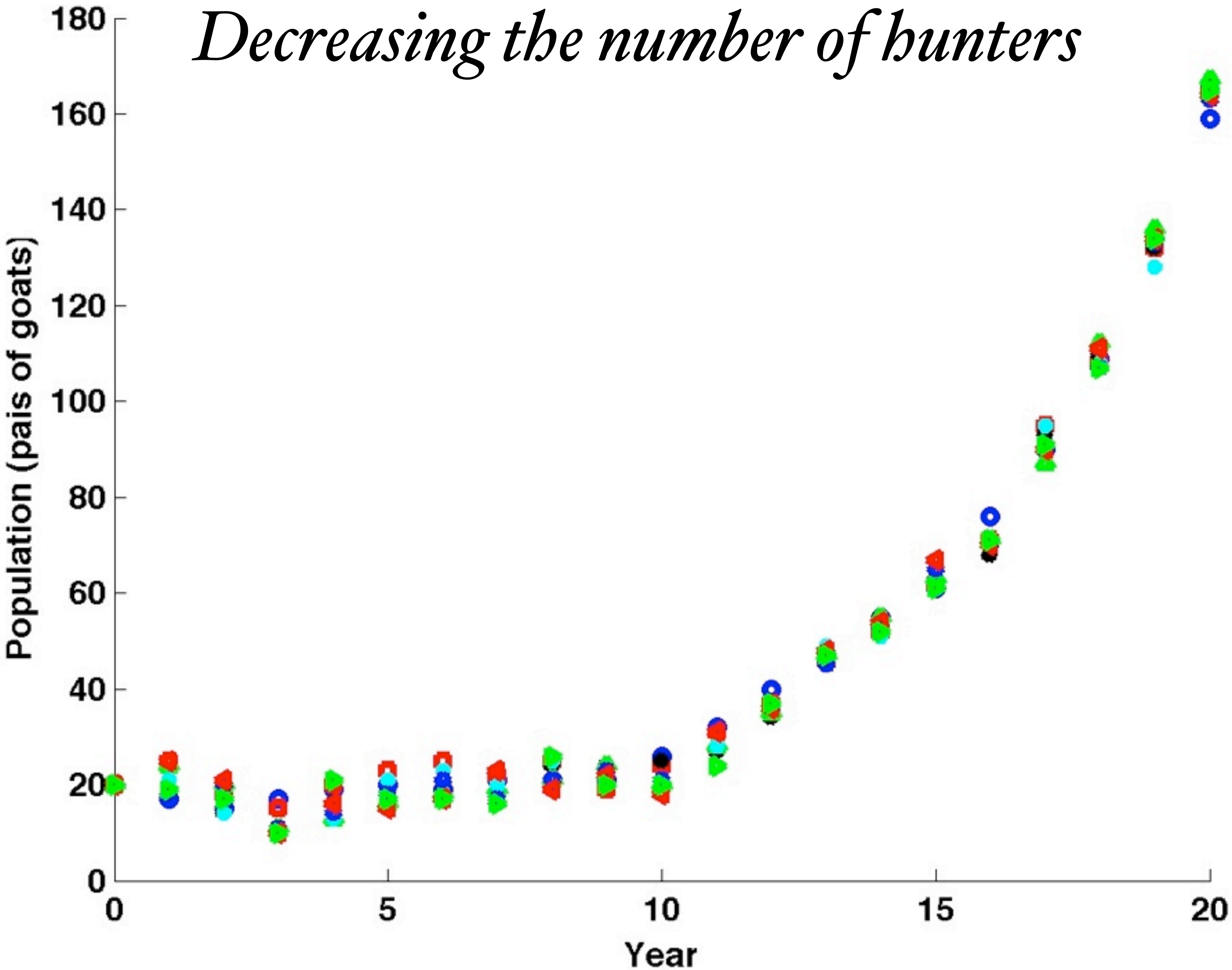


Double the initial population
1200
1000
800
600
400
200
0
Population (pais of goats)
0
5
10
15
20
Year

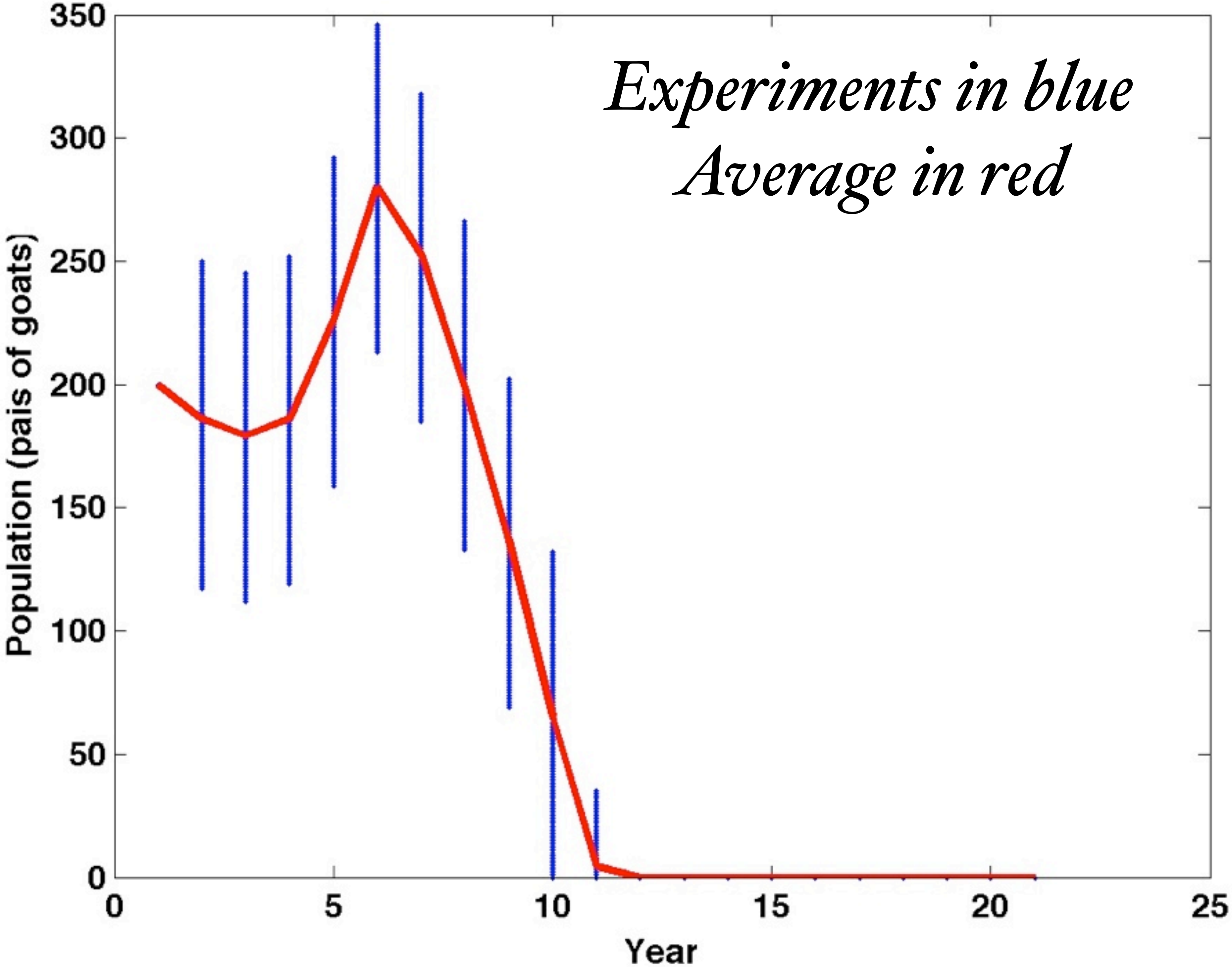
$$P^{(n+1)} = \lfloor (1 + \alpha - \beta) P^{(n)} \rfloor - H^{(n)}$$

- *Why does doubling the initial population work?*
- *If we think of our island as a national park would changing the initial population be something the park warden could do?*
- *What could they do instead? How about lowering the number of hunters?*

Decreasing the number of hunters



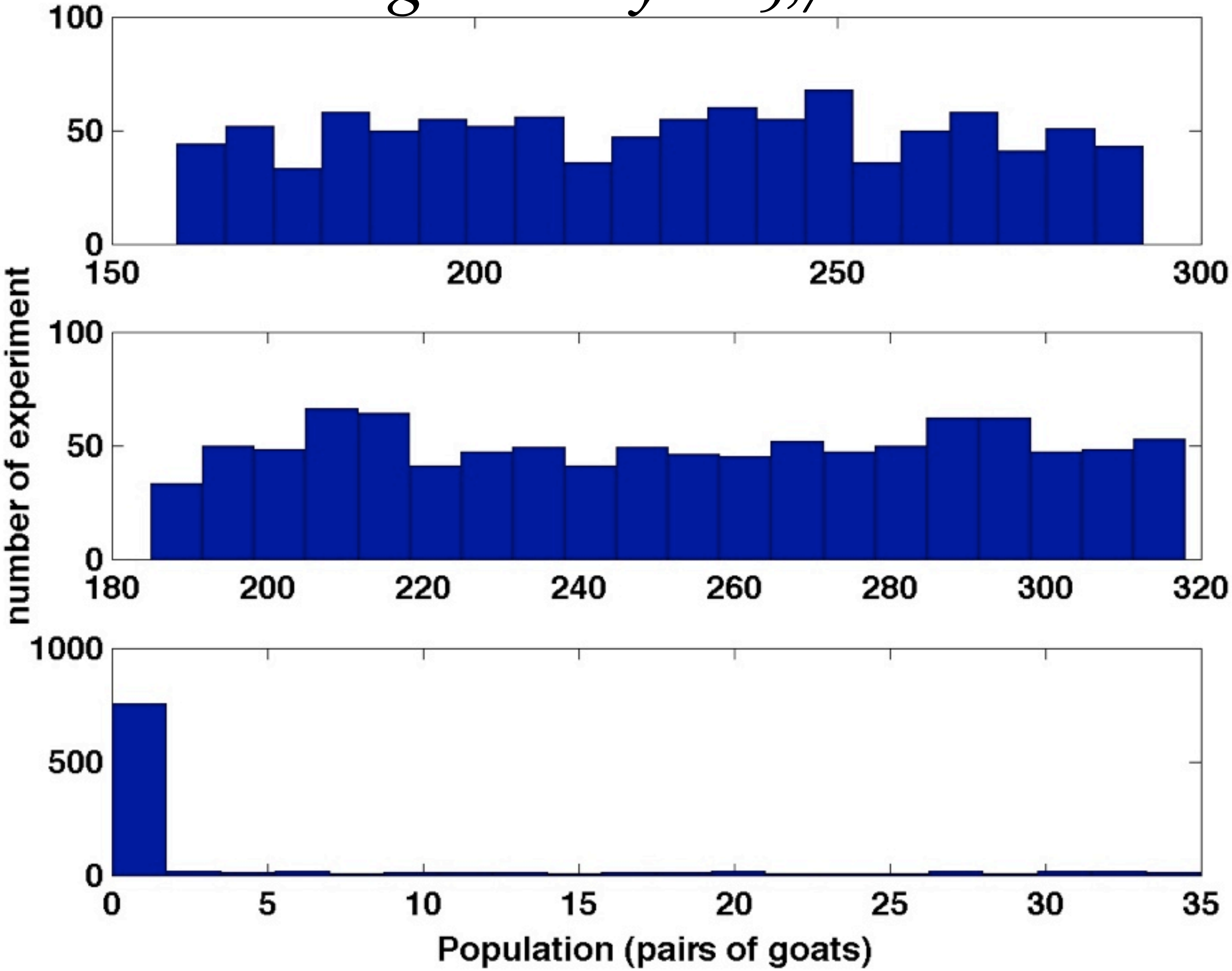
- *You can see how the "coin flip" changes the model result a little bit in some cases, and more than a little bit in others.*
- *Mathematicians like to be a bit more precise than that.*
- *So they look at the statistics of the result, which means the "game" has to be played many, many times on a computer.*
- *The next slide shows an example of this with an initial population of 200 and up to 130 hunters.*



Experiments in blue
Average in red
350
300
250
200
150
100
50
0
Population (pais of goats)
0
5
10
15
20
25
Year

- *The last picture is nice in that the average is clearly represented (in red) and you have some idea how the different experiments behave (in blue).*
- *Another way to look at all the experiments is to make bar graphs, or histograms.*

Histograms at year 5,7 and 11



- *To summarize, randomness (flipping a coin, rolling dice or generating random numbers on a computer) is used in models to represent the "fuzziness" of our model.*
- *Computers let us "play the game" many times over and we can then make graphs of the average population, the population in any given experiment, or to use more complex graph types, like histograms to learn about what happens.*

Population Models Part IV

Marek Stastna

University of Waterloo

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- *I want to finish the session with one more bit of math.*
- *I have mentioned that modeling and the search for new mathematics are not necessarily the same thing, however sometimes new math comes naturally out of modeling.*
- *We have spoken about the average and even shown a graph of it, and you have computed some averages in the exercises.*
- *The histograms at the end of part III show that there is more to data than just the average, but there is no doubt that a single number is more convenient than an entire histogram.*
- *Let's define some notation and a new mathematical idea.*

$$< f >= \frac{1}{N}\sum_{i=1}^{N} f_i \qquad var(f) = \frac{1}{N}\sum_{i=1}^{N}(f_i - < f >)^2$$

$$N = 4 \qquad < f >= \frac{1}{4}(f_1 + f_2 + f_3 + f_4)$$

$$var(f) = \left[(f_1 - < f >)^2 + (f_2 - < f >)^2 + (f_3 - < f >)^2 + (f_4 - < f >)^2\right] / 4$$

$$f_i = (5, 7, 3, 10) \qquad < f >= 6.25 \qquad var(f) = 8.9167$$

- *You can see right away that computing the variance is time consuming by hand, and indeed computers are much better at handling data.*

$$f_i = (5, 7, 3, 10) \qquad < f >= 6.25 \qquad var(f) = 8.9167$$

- *If you are paying attention you might note that the variance is very different from the mean.*
- *This is partially due to the numbers I chose for the example, but partially due to the definition in terms of squares (incidentally why did we square the difference from the mean?).*
- *Usually people report what is called the **standard deviation**, often labelled "std", which is just the square root of the variance.*
- *This is because its value is often comparable to the data itself, at least for data whose mean is near zero (not true for our example).*

- *To answer the question of why we squared in the definition of the variance, let's imagine a simple gambling game involving a coin flip (recall we mentioned it early in Part III as well).*

- *If the coin comes up heads Player B gets +1 coins and if the coin comes up tails Player B gives away a coin, or in math speak gets -1 coins.*

- *If we play this game an even number of times for a very long time the average pay out is zero.*

- *As a clever math "gotcha" moment, if we play it an odd number of times the average is never zero. Can you explain why?*

- *For now let's assume we always play an even number of times.*

- *If I say H is heads and T is tails. Here is an example of a game played ten times: H,H,T,H,T,T,T,H,H,H*
- *At the end of these ten rounds Player B is up two coins.*
- *If we think about the number of extra coins player B has we have: 1,2,1,2,1,0,-1,0,1,2*
- *Now imagine the exact same end result but with a different set of coin flips: H,H,H,H,H,H,T,T,T,T*
- *The extra coins player B has now read: 1,2,3,4,5,6,5,4,3,2*
- *So at the end of the game we are in the same spot, but how far player B got from zero is very different this time.*

- *The variance works exactly the same way but for outcomes where the range of possibilities is bigger than just +1 or -1.*
- *To see that let's look at one last example, a game with dice:*
- *Again let's have two players who pay each other in poker chips or coins. A dice is rolled and Player A pays player B for a roll of 1, 2, 3 and player B pays player A for a roll of 4, 5, 6.*
- *To make the game math friendly let's focus on player B again and what he or she gains. The payouts will work like this:*

Roll	1	2	3	4	5	6
Payout	-3	-2	-1	1	2	3

Roll	1	2	3	4	5	6
Payout	-3	-2	-1	1	2	3

- *So a sample ten round game might go like:*
- *payout={3,1,-2,-2,3,1,-3,-2,-1,2}; balance={3,4,2,0,3,4,1,-1,-2,0}*
- *So player B comes right back to a balance of zero.*
- *The mean and variance are given below and you can see that the mean is zero but the variance is large, though the standard deviation is a lot like the values of payout.*

$$\langle \text{payout} \rangle = (3+1-2-2+3+1-3-2-1+2)/10 = 0$$
$$var(\text{payout}) = 5.111 \text{ and } std(\text{payout}) = 2.26$$

Double Payout Game

Roll	1	2	3	4	5	6
Payout	-6	-4	-2	2	4	6

- *Finally let's replay the game, with double the payout.*
- *payout={6,2,-4,-4,6,2,-6,-4,-2,4} balance={6,8,4,0,6,8,2,-2,-4,0}*
- *The mean is unchanged but the variance quadruples:*
- *var(payout)=20.444*
- *But the standard deviation is std(payout)=4.5216 and again that's quite representative of the data.*