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Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

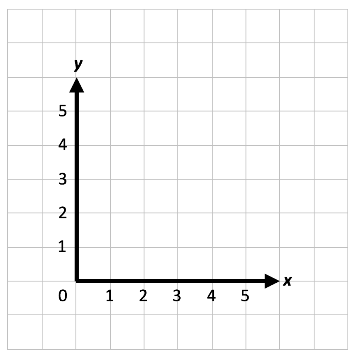
October 24/25, 2017

Shapeshifting

Plotting

Before we begin today, we are going to quickly go over how to plot points.

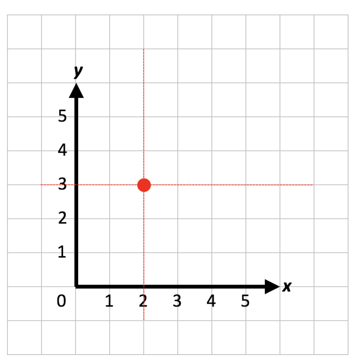
When you are plotting a point, you will be given coordinates in an (x, y) pair. What exactly is an (x, y) pair? First, lets look at some graph paper.



The two bolded lines seen above are called 'axes'. The 'y' axis counts how many spaces your point is upwards, and the 'x' axis counts how far your point is to the right.

Lets say for example you are given an (x, y) pair of (2, 3). The number on the left, will always represent the position of the point on the x axis. In this case it is 2. The number on the right will always represent the position of the point on the y axis. In this case it is 3.

In other words, starting from zero, the point will be 2 spaces to the right, and 3 spaces up.

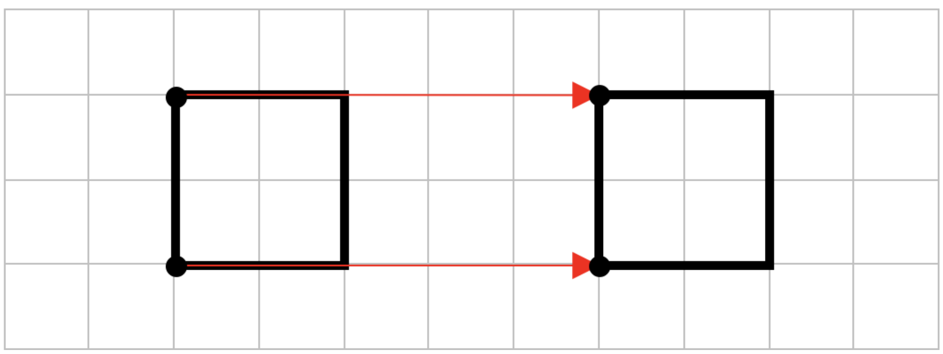


Introduction

Shapeshifting can refer to any kind of transformation that something goes through. You might be familiar with giant robots shapeshifting into vehicles, caterpillars turning into butterflies or even frogs transforming into princes, but these are not exactly the kinds of shapeshifting which we will be talking about today.

In mathematics, there are 4 main types of shapeshifting. The first one that we will discuss is known as a 'translation'.

A translation is the most basic form of shapeshifting. To perform a translation, all you have to do is move an object from one location to another location. For example, if you have a square which is made from 4 points, to perform a translation of 5 units to the right, simply move each of these 4 points, 5 spaces to the right.

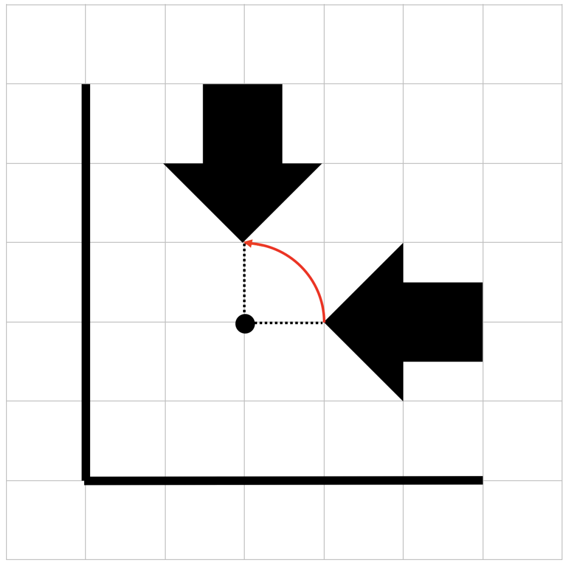


A translation of 5 units to the right.

The second type of shapeshifting we will look at is known as a rotation.

There are two key components to any rotation (in 2 dimensions); a vertex and an angle. A vertex is the point which an object is rotated around. For example, if you are riding on the edge of a merry-go-round, the point you are rotating around, or the vertex, is the center of the merry-go-round. The angle determines the amount that you will rotate by.

The next page shows an example of rotating a shape by 90° counter clockwise (CCW) around the point (2,2).

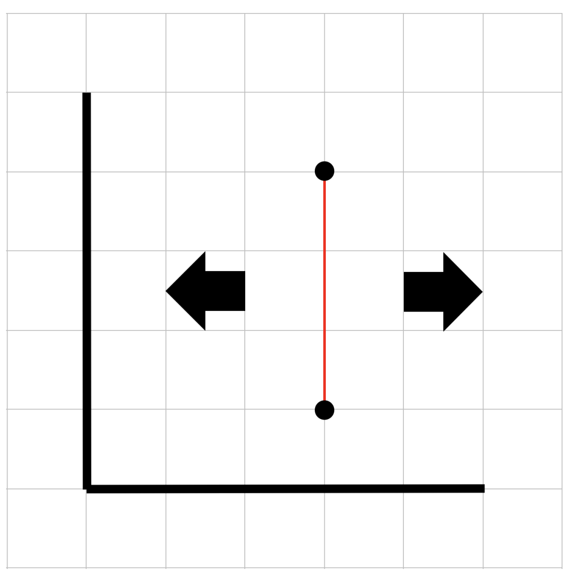


A 90° CCW rotation around the point (2,2).

Another important kind of shapeshifting is a reflection.

The first step of any reflection is to determine what your 'line of reflection' is. After you have done this, all you have to do is recreate the same object on the other side of this line, except it should be flipped in the other direction.

For example, reflecting an arrow across the line created by the points (3,1) and (3,4) would look like this:



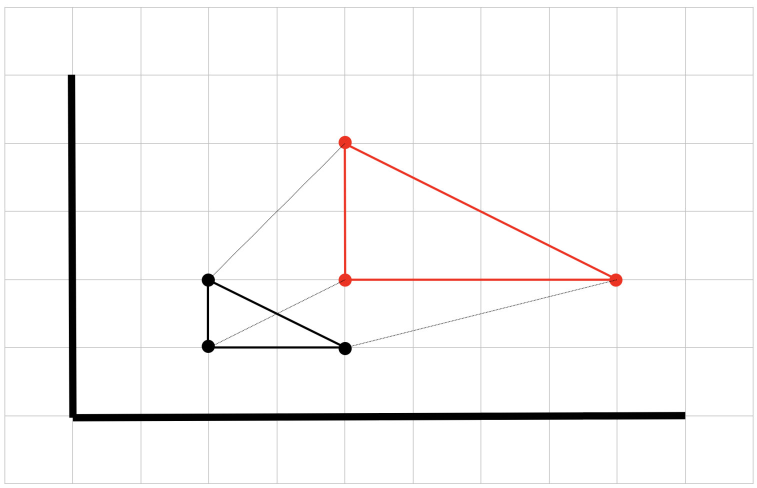
Reflecting an arrow.

The final kind of shapeshifting which we will discuss today is called 'scaling'.

Scaling an object means to change the size of the object, but keep all of its proportions the same. To do this, take every (x,y) point of the shape, and multiply the values by the 'scaling factor'.

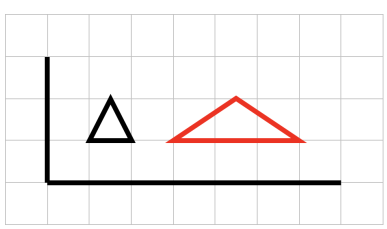
For example, scale the triangle made out of the points (2,1), (4,1) and (2,2) by a factor of 2.

First multiply each coordinate point by the scaling factor of 2. The first point (2,1) will become (4,2), the second will be (8,2) and the third will be (4,4). Now lets draw the original triangle along with the scaled version to see what they look like.

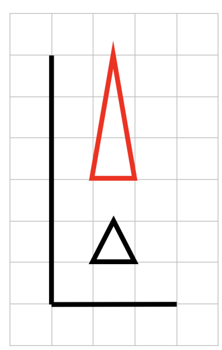


A triangle scaled by a factor of 2.

There are actually two other ways that you can scale an object as well. The first way is called a 'vertical stretch'. To do a vertical stretch, multiply your 'y' coordinate by the scale factor, and leave your 'x' values the same. The second kind of stretch is 'horizontal stretch', where you only multiply your 'x' values by the scale factor.



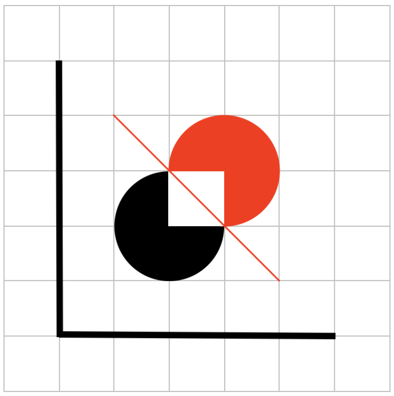
A horizontal stretch by a factor of 3.



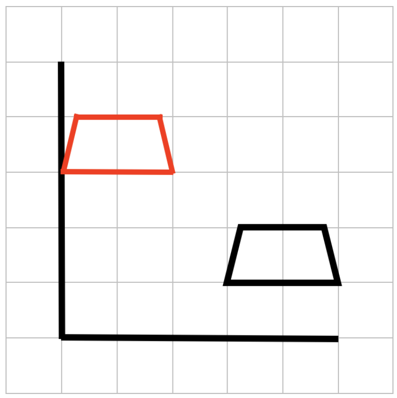
A vertical stretch by a factor of 3.

Problems

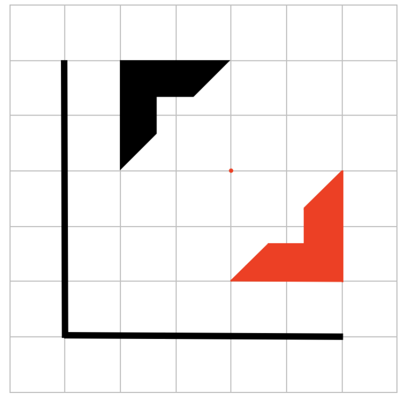
1. Perform the indicated shapeshifting techniques to each shape below.



Reflect across the line made by (1,4) and (4,1).



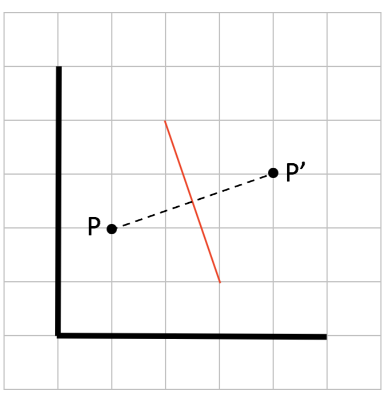
Translate 3 units left, and 2 units up.

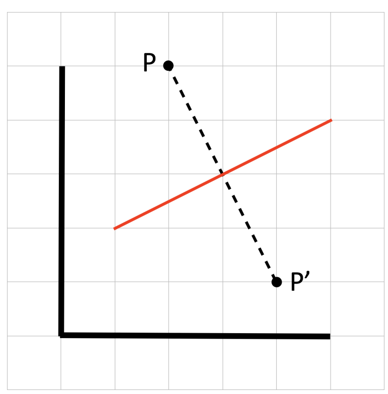


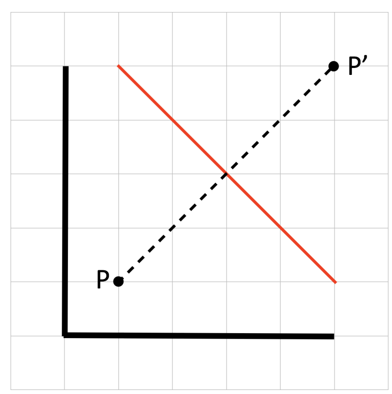
Rotate by 180° around the point (3,3).

2. To reflect point P to point P', what line of reflection must be used? Do you notice anything special about these lines?

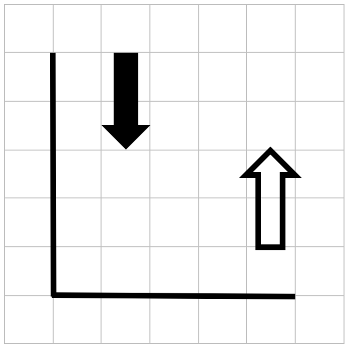
These lines are known as the 'perpendicular bisector'. They are exactly in the middle of the two points, and they are perpendicular to the line made from the two points!

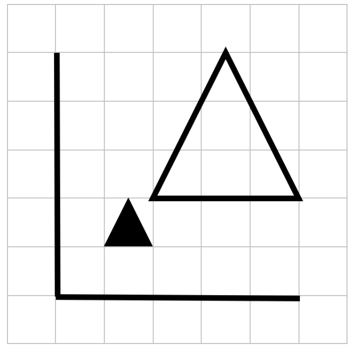


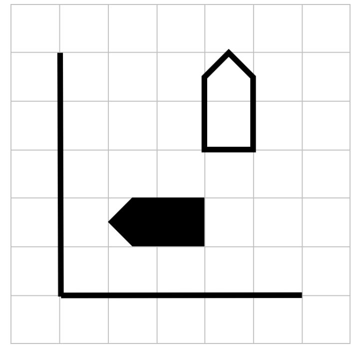




3. What transformations have been done on these shaded shapes to turn them into their respective outlined shapes? *There might be more than one answer!*

a) 

b) 

c) 

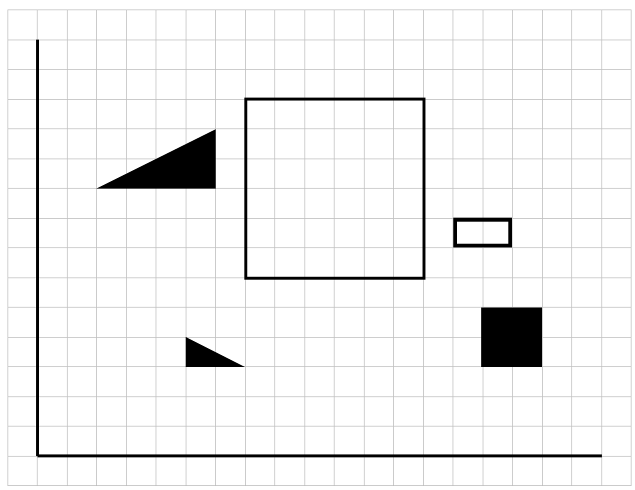
3a) Rotate 180° around the point (3,3)

3b) Scale by a factor of 3, then translate 1 unit down and 1 unit left.

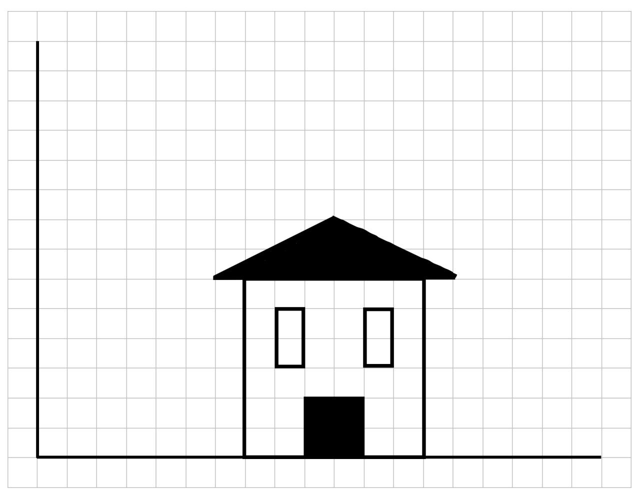
3c) Rotate 90° CW around the point (4,2)

4. Perform all of the shapeshifting methods below. What do all of these shapes make?

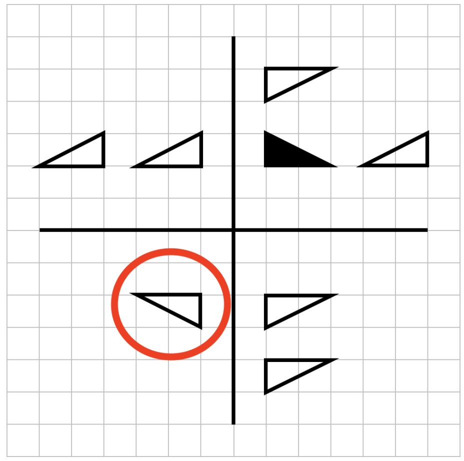
- Translate the triangle that has one of its vertices at (2,9), three units down and four units to the right.
- Reflect the biggest square across the line created by the points (6,6) and (14,6).
- Translate the solid black square down three units and six units to the left.
- Rotate the rectangle with one of its points at (14,7) by 90° CCW around the point (13,6), then translate it down by four units.
- Create a reflection of the rectangle from the last step by reflecting it across the line made from the points (10,0) and (10,6).
- Scale the small triangle by a factor of two.



Use the grid below to help clearly draw your final image.



5. Circle the triangle which can *not* be made from a *single* reflection of the shaded triangle.

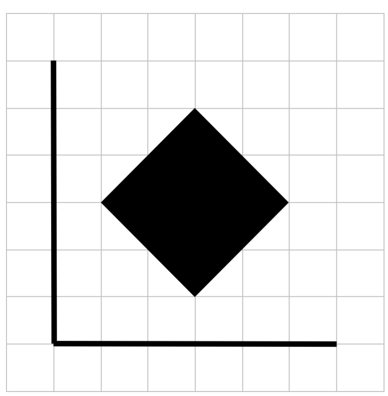


6. Why is shapeshifting important?

Shapeshifting can help us quickly manipulate sets of data, and it can help us find solutions to problems with different sets of initial conditions without having to redo all of the calculations.

7. Perform all of the transformations on the object below:

- Rotate around the point (3,3) by 90°.
- Rotate around the point (3,3) by 180°.
- Rotate around the point (3,3) by 270°.
- Reflect across the line made by the points (3,0) and (3,6).
- Reflect across the line made by the points (0,3) and (6,3).



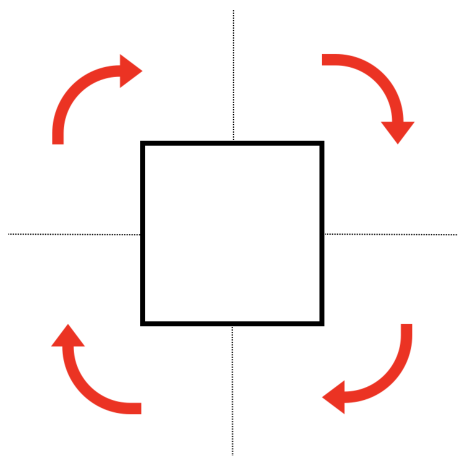
You should notice something special about each one of these transformations. This 'special' thing is happening because this shape is symmetrical!

Symmetry

There are some shapes which are not affected by certain transformations. These special shapes are known as being 'symmetrical' in some way. There are 2 main types of symmetry which we will discuss today; first there is rotational symmetry, and secondly there is bilateral symmetry.

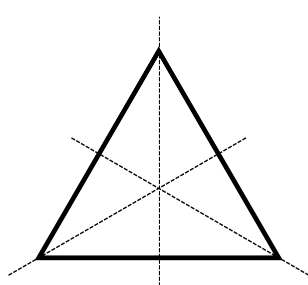
Rotational symmetry means that if you rotate an object around a certain point by a certain angle, you will get the exact same shape back that you began with. This 'certain point' will almost always be the center of the shape, and the 'certain angle' can be found by doing a quick calculation. For example, a square has 4 sides which are all identical. We know that a full rotation consists of 360°, so each one of these 4 sides of the square will have a rotational symmetry of 90°.

$$\frac{360°}{4} = 90° \tag{1}$$



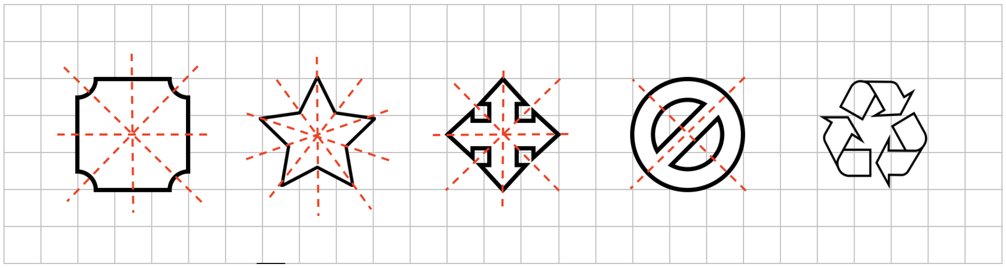
Bilateral symmetry occurs whenever you can *evenly* fold a shape in half. You can also think of this as having one side of a shape be the reflection of the other side.

You can see below that an equilateral triangle has 3 lines of bilateral symmetry.



Problems

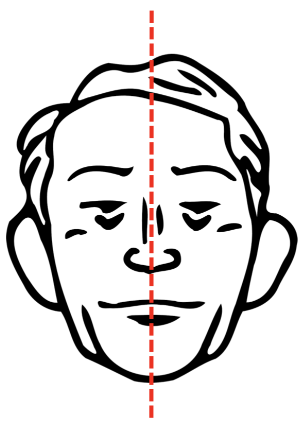
1. Find all of the rotational and bilateral symmetries in the following objects.

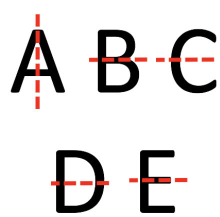


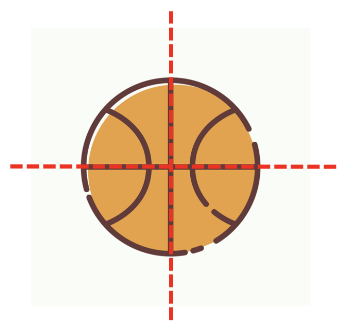
The first image has 4 bilateral symmetries, and a rotational symmetry of 90°.
The second image has 5 bilateral symmetries, and a rotational symmetry of 72°.
The third image has 4 bilateral symmetries, and a rotational symmetry of 90°.
The fourth image has 2 bilateral symmetries, and a rotational symmetry of 180°.
The fifth image has 0 bilateral symmetries, and a rotational symmetry of 120°.

2. Name 3 symmetric objects that you can find in the real world. In what ways are they symmetrical?

a) A human face b) Letters of the alphabet c) A basketball



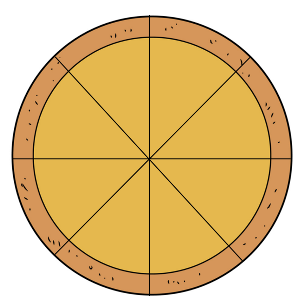




Neither the face, or the letters have rotational symmetry. The basketball has rotational symmetry of 180°.

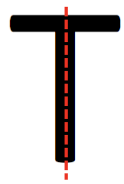
3. Draw a shape that has a rotational symmetry of 45°.

A medium pizza!

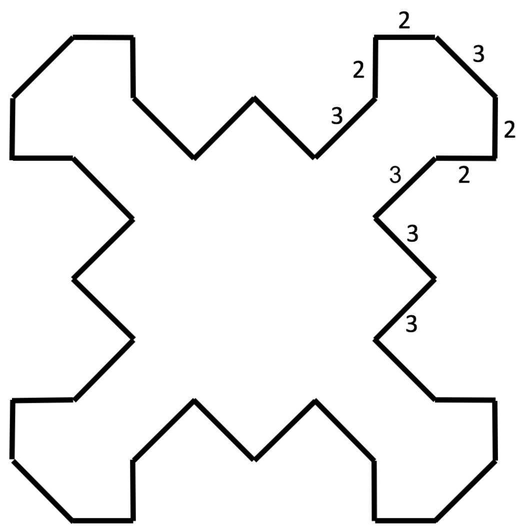


4. Draw a shape that has *only* 1 bilateral symmety.

The letter 'T'!



5. What is the perimeter of the following object? What method did you use to find the perimeter?



This shape has four symmetrical quadrants. We are given numbers for the side lengths of one quadrant. The total perimeter of the shape must be four times as much as the perimeter of one quadrant.

$$\begin{aligned} Perimeter &= 4 \times (3 + 2 + 2 + 3 + 2 + 2 + 3 + 3 + 3) \\ &= 4 \times (23) \\ &= 92\, units \end{aligned} \tag{2}$$

6. Why are symmetries useful?

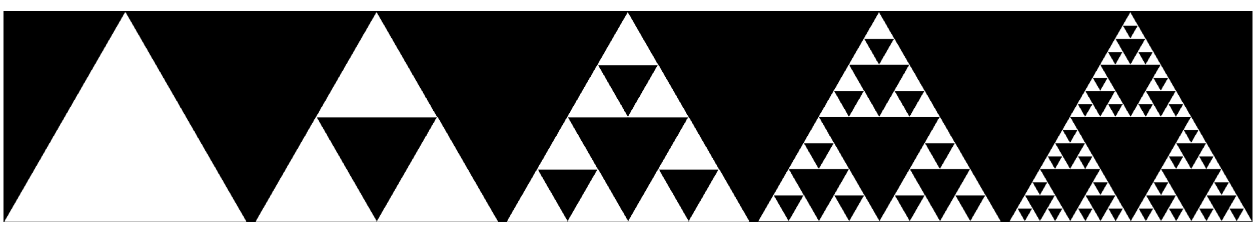
They can help reduce the number of calculations in a question (as seen in question 5).

Fractals

Fractals can be fun, beautiful, and highly mathematically constructions; but what exactly is a fractal? A fractal is the result of a repeated process which is done on an initial shape. A fractal can also be thought of as a never ending pattern.

One famous fractal is known as the 'Sierpinski Triangle'. To draw this fractal, follow the following steps:

- Draw an equilateral triangle.
- Divide this triangle into 4 identical smaller triangles and colour in the upside down triangle.
- Repeat the previous step with each of the smaller, un-shaded triangles as many times as you want!



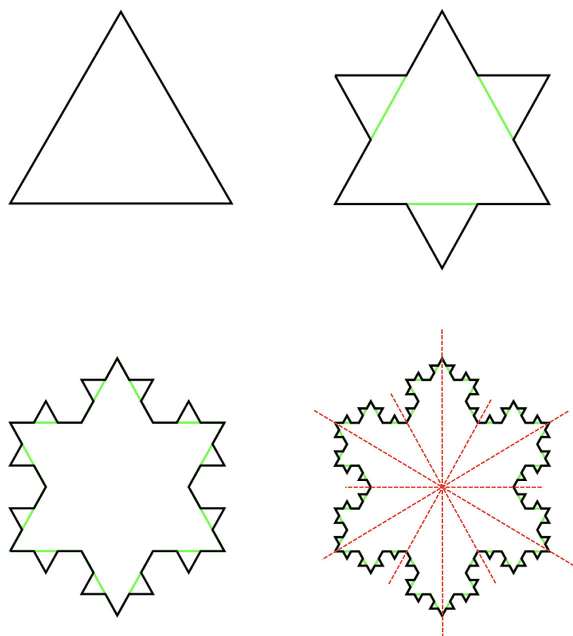
What is the fraction of the un-shaded area to the total area after each iteration? How about after n interations?

Step	Fraction of Area
1	1
2	3/4
3	$(3/4)^2 = (9/16)$
n	$(3/4)^{n-1}$

Problems

1. Draw the first 4 iterations of the 'Koch Snowflake'.

- Draw an equilateral triangle.
- Divide each side of your shape into 3 equal segments. Turn each side's middle segment into an outward pointing equilateral triangle.
- Remove the base of each of these smaller triangles.
- Repeat the previous two steps as many times as you want!



2. What symmetries does the Koch Snowflake have on the ninth iteration?

The Koch Snowflake will have the same six bilateral symmetries for every iteration past the first iteration. You can see these six symmetries in the diagram above.

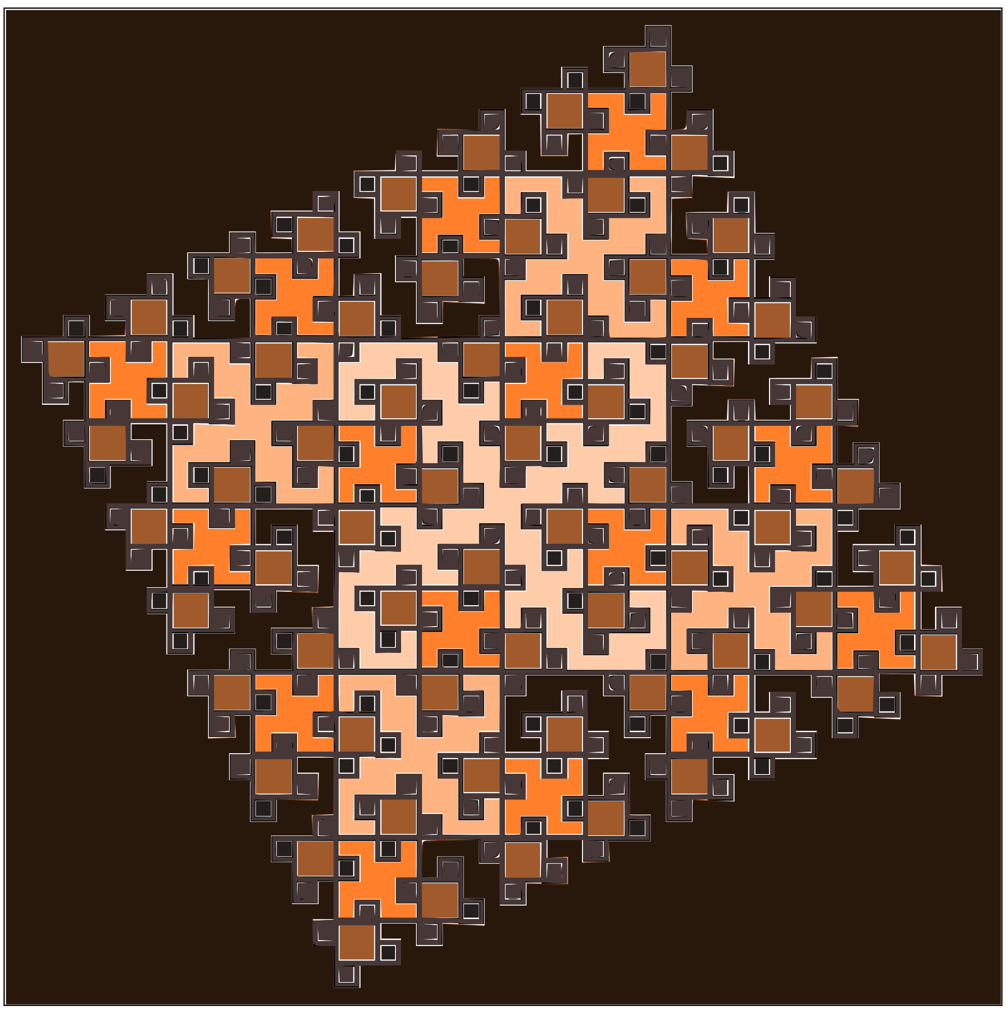
3. If the first iteration of the Koch Snowflake has a perimeter of 3 inches, what is the perimeter on the 2^{nd} iteration? How about after the n^{th} iteration?

For each iteration of the Koch Snowflake, each side of the shape is divided into 3 segments. The middle of these segments is turned into a triangle, which gives us an extra segment (4 in total). We began with 3 segments, and ended with 4, so we now have $\frac{4}{3}$ of the original perimeter.

2^{nd} step: Perimeter $= 3\,inches \times \frac{4}{3} = 4\,inches$ nth step: Perimeter $= 3 \times (\frac{4}{3})^{n-1}\,inches$

*4. Creating your own fractals can be fun also! Here is one that I came up with.

- Draw a big square in the center of the grid (I have done this for you to make sure the fractal will fit on the page!).
- Create a copy of the big square, scale it by a factor of $\frac{1}{2}$ and translate it so that it's bottom right corner touches the top right corner of the original square.
- Create three copies of this smaller square, and rotate them CW around the center of the big square by 90°, one by 180°, and one by 270°.
- Repeat the previous two steps using each of your new smaller squares as the big square.



Bonus: At home, once you have done multiple iterations, colour or shade the fractal to make a cool design!

5. Try making your own fractal here! It can be very simple, or you can try making a complex one with shapeshifting techniques. Come up with about 3 to 5 steps, then try drawing it!

Write your steps here!

Many possible answers!

6. Why are fractals useful?

There are a few uses for fractals in computer science, telecommunications, physics and a few other areas, although they are all fairly obscure. Fractals are also used in things like graphic design because the images which can be made are quite beautiful.

Fractals are mainly a pure mathematics topic.