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Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

October 31/November 1, 2017

Spatial and Visual Thinking

Introduction

One very important skillset used in mathematics which is commonly overlooked, is spatial and visual thinking. Almost every math question which you will *ever* encounter will have some form of data, or some kind of diagram which can be drawn from the question. Spatial and visual reasoning is the skill which we use to interpret these diagrams, which eventually helps us solve the problem.

Some examples of when you would use spatial and visual thinking are:

- Playing Tetris
- Solving a Rubiks Cube
- Viewing 3 dimensional (3D) diagrams drawn on 2 dimensional (2D) surfaces
- Using any mathematical shapeshifting techniques

Practicing your spatial and visual thinking skills is a great way to improve at mathematics; there have been many studies done which link strong mathematical and spatial/visual skills together. Today we will try some exercises to push our visualization skills to the maximum!

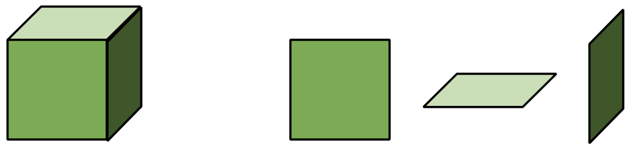
Three Dimensional Visualization

First of all, what do I mean by 'Three Dimensional'? A 3D object is anything that has a volume, or in other words, anything that has a length, a width *and* a height! A few examples of these would be a cube, a sphere, a prism, a human, this building and just about everything else we see in the entire world.

2D objects on the other hand have an area, or in other words, they only a length and a width. Some examples of 2D objects would be a square, a circle, an image on a computer screen, or the writing on this paper.

One way to tell if an object is 2D or 3D is to imagine running your hand across it. If you ever feel any bumps, you know it has to be 3D, because bumps only ever occur if the object has height!

We live our lives in 3D, but we unfortunately have to do the majority of our mathematics on 2D surfaces (sheets of paper and computer screens). Fortunately, with the aid of our spatial/visual skills, we can represent 3D objects on these 2D surfaces! For example, look at the cube below.

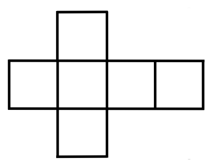


You can see that the object on the appears to be a 3-dimensional object; but how can this be? If you look to the right, you can see that it is made from only 2-dimensional shapes!

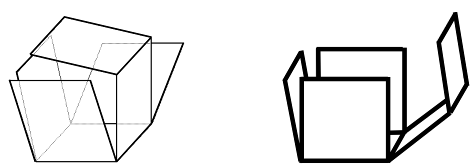
The key here is that the object drawn on the left is really only 2-dimensional. Your brain however, using its spatial/visual powers, perceives the 2-dimensional image as a 3-dimensional cube!

Nets

Another way to represent a 3D object on a 2D surface is to draw the 'net' of the object. Below is an image of the net of a cube.

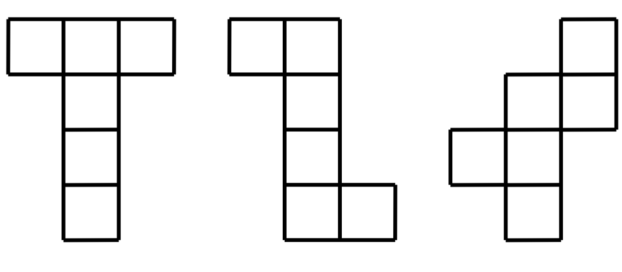


How does this represent a 3D object? Imagine unfolding each side of a cube and flattening them all down on the ground. When you do this, you will get one possible net of the shape.



Two possible perspectives of unfolding a cube.

There can also be multiple different nets for some 3D shapes. The cube for example has 11 different nets. Here are a few of them below. Can you find any more of them?

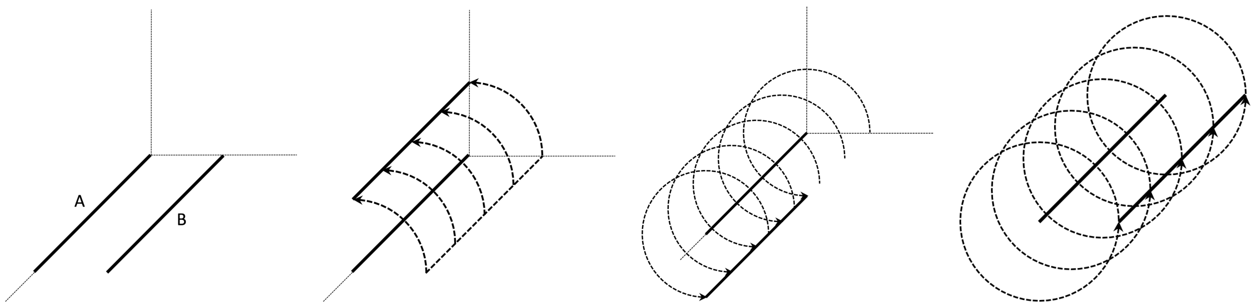


Rotations

Rotations in 3 dimensions work differently than rotations in 2 dimensions. Rather than rotating around a point, we now have to rotate around a line!

Example

Rotate line B around line A by a full 360°. What shape is traced out by line B?



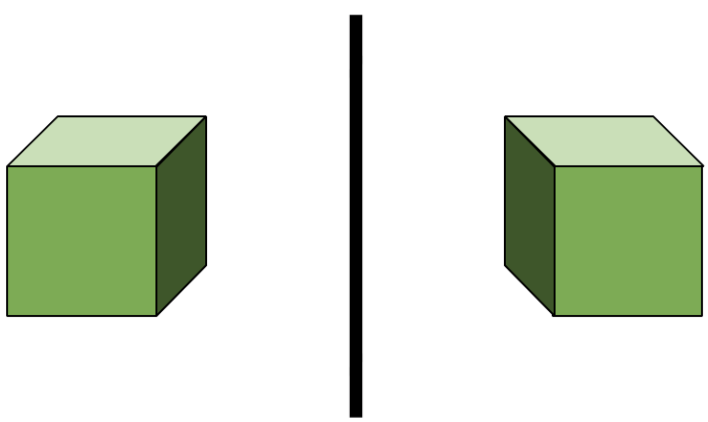
From the diagrams above, we can see that line B traced out the shape of a cylinder.

Reflections

Reflections work differently in 3 dimensions as well. Instead of reflecting an object across a line, you now have to reflect it across a plane. A plane is a flat 2D surface, like a piece of paper for example.

Example

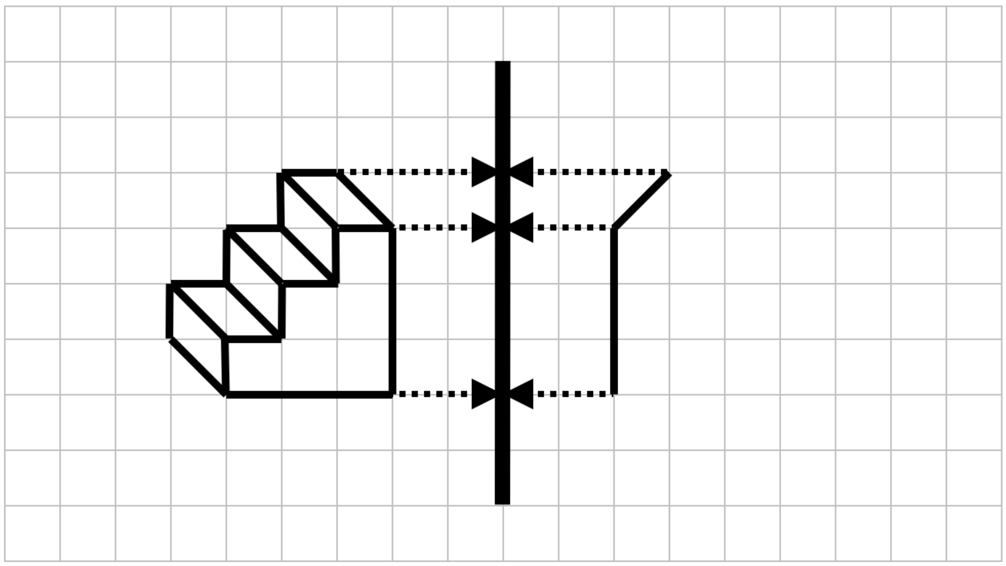
Here we are given a cube (on the left). This cube has been reflected across the plane in the middle, giving us the reflected version of the original cube on the right.



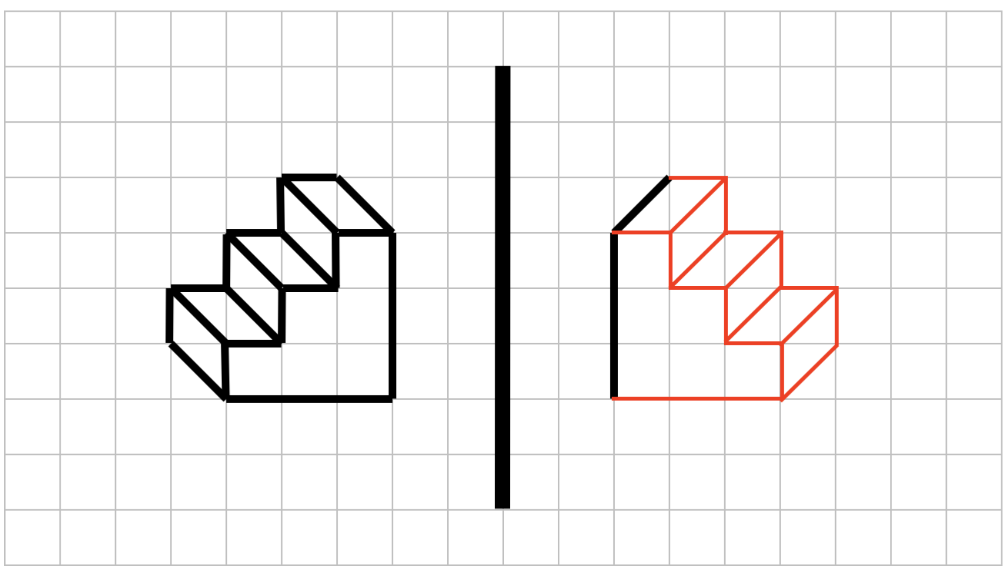
When reflecting a 3D object it is important to try to visualize what the object looks like first; this is what exercises your brain! After you have done that, you can draw the reflection, which I will show you how to do on the next page!

Now you may have already been doing this last week, but I feel like I should mention it here just because 3D reflections can be much harder to visualize than 2D ones.

To perform *any* reflection in 2D *or* 3D, all you have to do is count the number of spaces that each point on your shape is away from either the line or plane of reflection.

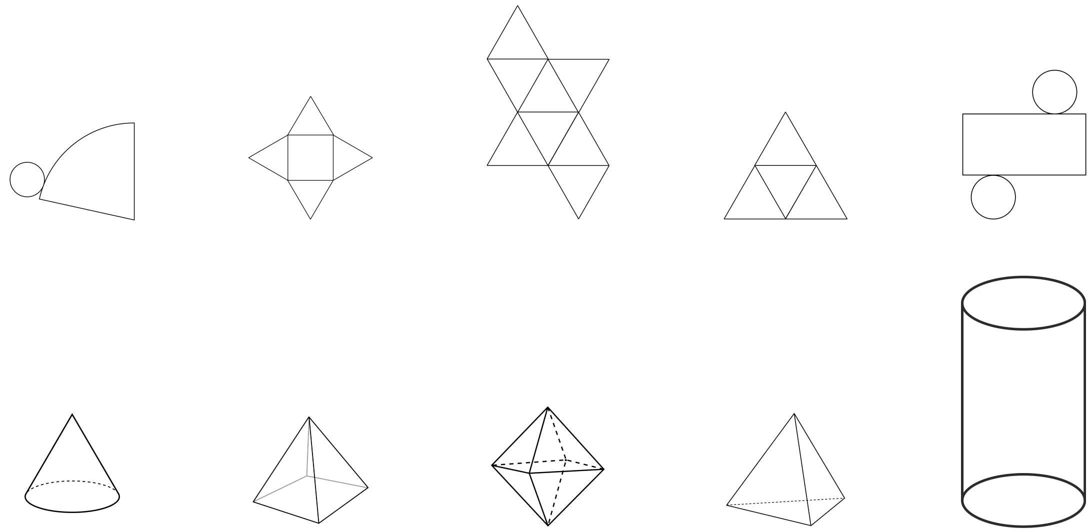


I have started this one off for you. Try to finish reflecting the remaining points.



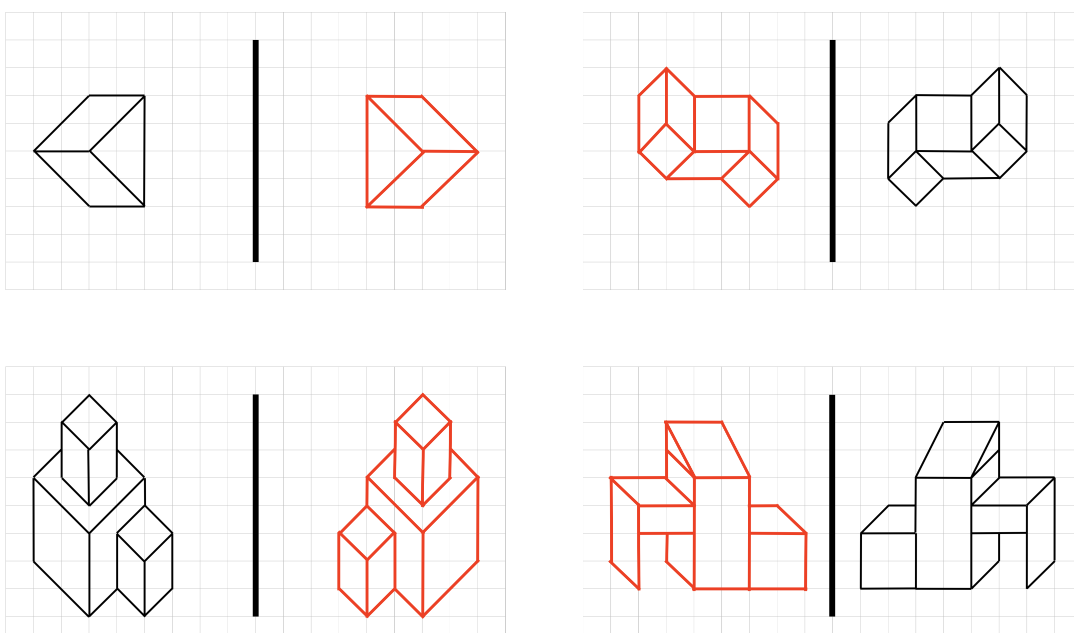
Problems

1. What shape does each net represent? Either name the shape, or draw a picture of it.

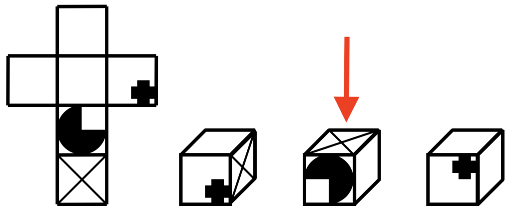


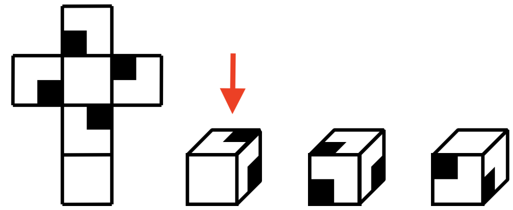
Cone Pyramid Octahedron Tetrahedron Cylinder

2. First try to visualize, and then try to draw the reflections of these 3D shapes.

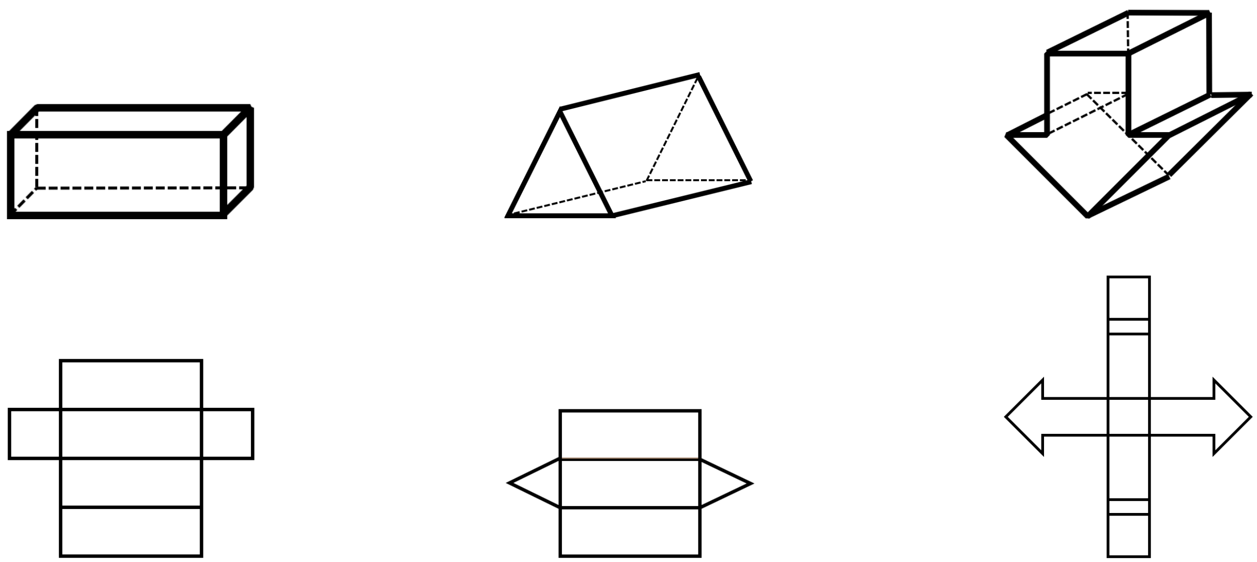


3. Which of the three cubes is a possible layout for the given net?

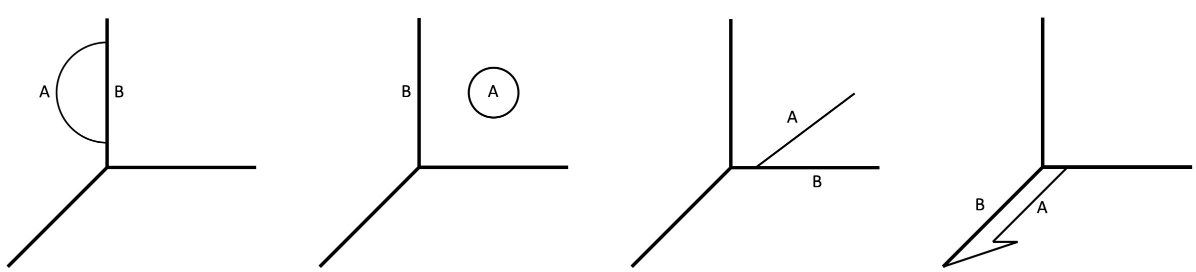




4. What will the nets of these shapes look like?

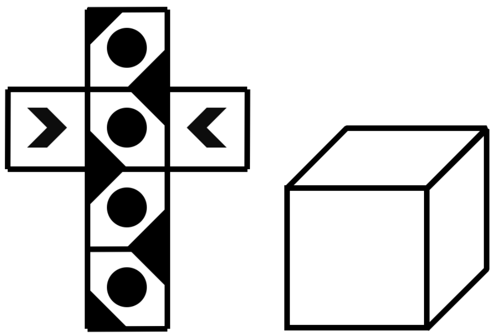


5. If line/shape A is rotated by 360° around line B in each of the 4 diagrams below, what shapes will be traced out by line A?



Sphere Torus (Donut) Cone 3D Arrow

*6. Draw one possible orientation of this net on the blank cube. How many different *unique* orientations does this cube (i.e. how many ways can you position the cube so it looks different)?



There are many possible answers for the first part of this question.

For the second part of the question, first consider how many possible orientations there are overall.

(1) Pick any of the six faces of the cube, and imagine pointing that face in one direct. There are four possible ways you can have that same face pointing in that direction because you can rotate your cube by either 90°, 180°, 270° or 360°.

You can do this same process for each face of your cube, so each face has four different orientations associated with it. Overall that means:

$$\begin{aligned} Total\ Orientations &= 4\ per\ face \times 6\ faces \\ &= 24\ Orientations \end{aligned}$$

(2) Now we have to find out how many of these orientations are *not* unique (i.e. how many of these orientations look the same).

If two orientations do look the same, that would mean that the cube must have some kind of symmetry. Looking at the net, we can find a total of 2 symmetries. The very top square and the second square from the bottom are identical, and the middle square along with the very bottom square are identical.

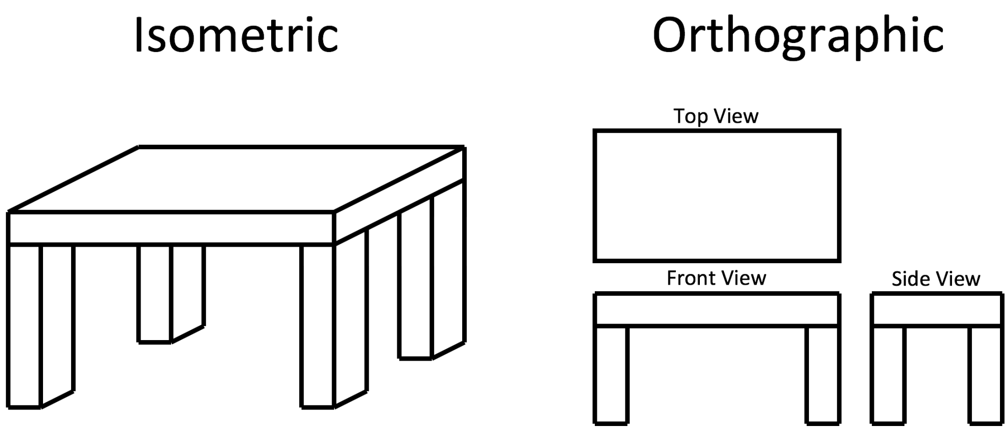
Because out of these four faces, only two of them are unique, we only have to count the orientations from the two unique faces (i.e. overall, we only have four unique faces).

$$\begin{aligned} Total\ Orientations &= 4\ per\ face \times 4\ faces \\ &= 16\ Orientations \end{aligned}$$

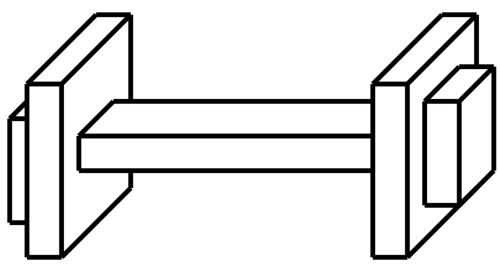
7. A basic technical drawing usually comes in two forms; 'isometric' and 'orthographic'. An isometric drawing shows you how the object would look in three dimensions with the front, top, and side views all visible.

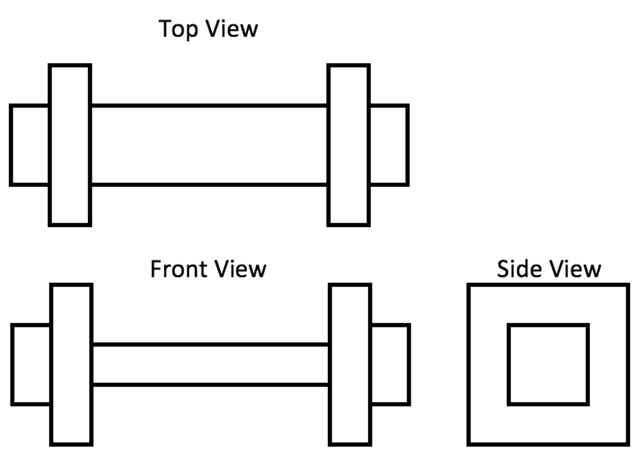
An orthographic drawing shows you exactly how the object would look from the front view, top view, and side view individually.

Here is an example of both an isometric and a orthographic drawing of a desk.



Draw the orthographic images of this dumbbell.



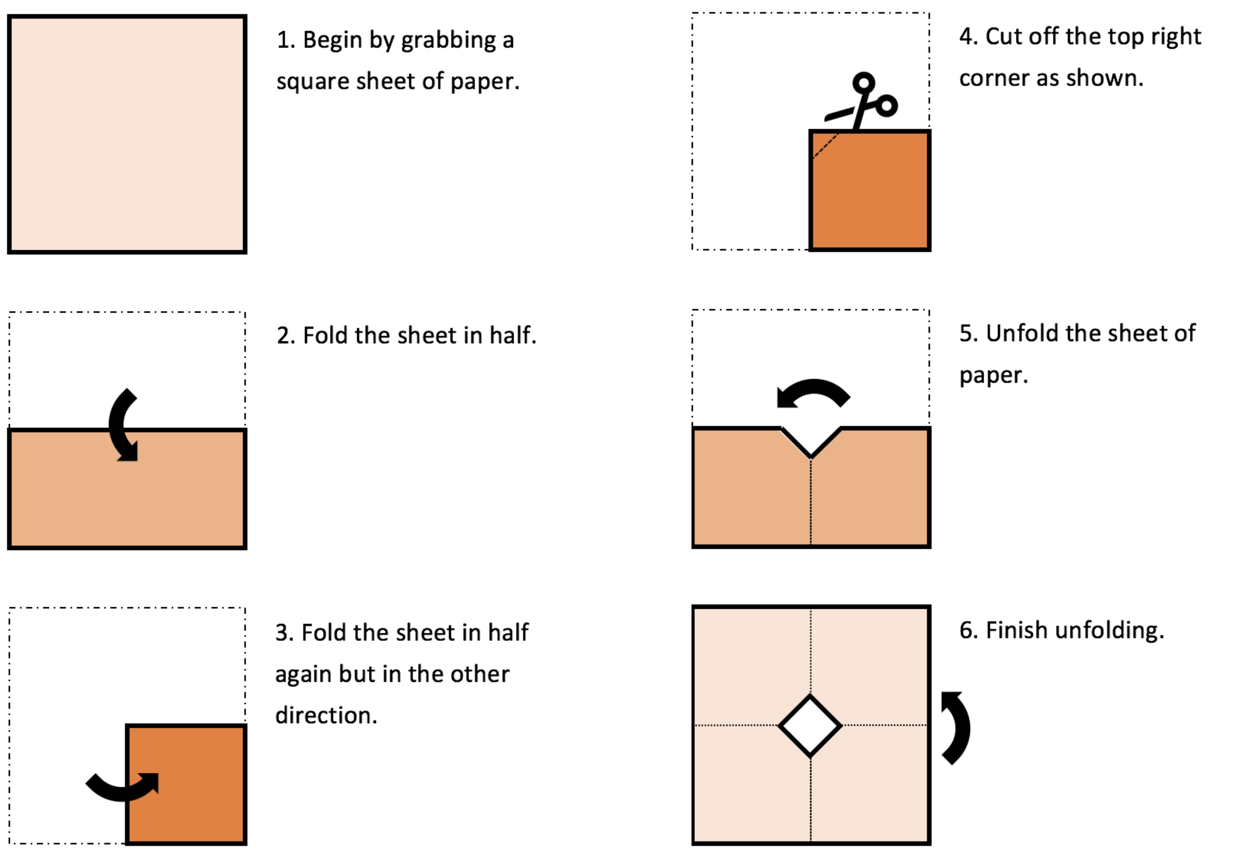


Paper Folding

Another challenging visualization exercise has to do with paper folding. If a sheet of paper is folded multiple times, where will all of the crease marks be? What if we cut the paper in some way? How will it look in its unfolded state?

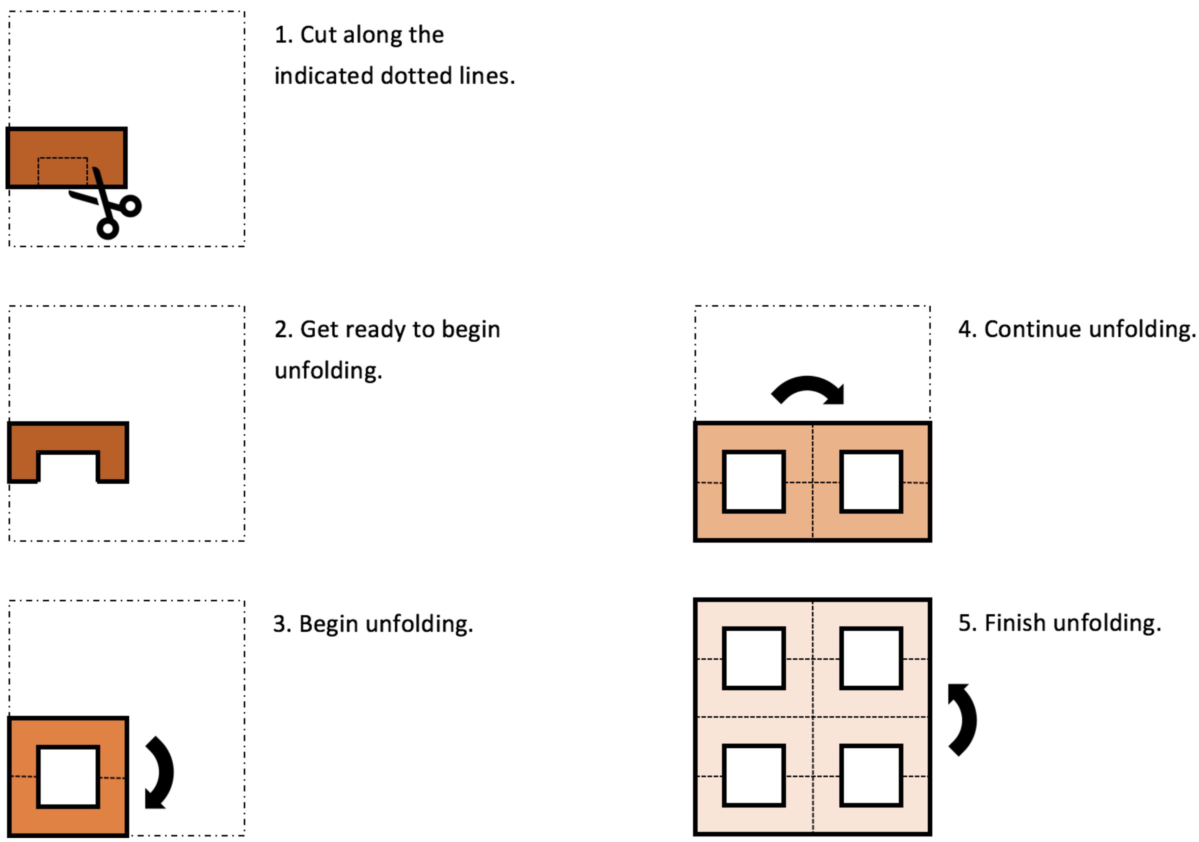
Example

Using only a *single* straight cut with a pair of scissors, how can a rhombus be cut out of a sheet of paper?



Example

You can also do these kinds of questions in reverse. For example, if a sheet of paper is folded in half 3 times, and a rectangle is cut out of the bottom as shown, what will the sheet of paper look like when it is completely unfolded?



These questions can be very tricky to visualize all in one step. One method to make these questions easier is to draw what the paper will look like after each unfold. If you do this, you are effectively doing a 2D reflection where the line of reflection is the crease mark made by the fold!

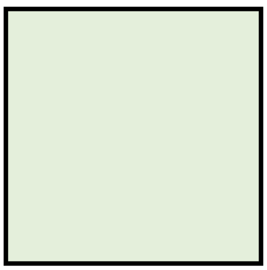
I will hand out some sheets of paper so you can try these two examples, and you will need some for the upcoming examples and problems. Please do not waste any paper, there is a limited supply. If you make a mistake ask me and I can give you more.

Folding Geometry

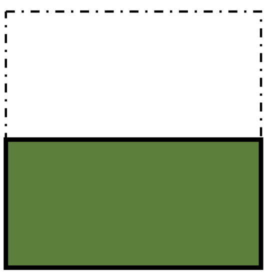
The last challenge that we will face today is creating different shapes with paper folding. You can think of this as math-ified origami!

Example

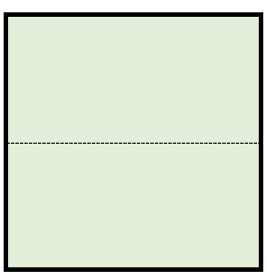
How can you fold a perfect equilateral triangle out of a square sheet of paper?



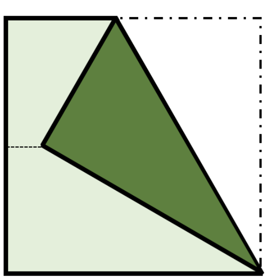
1. Begin by grabbing a square sheet of paper.



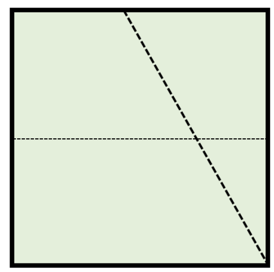
2. Fold the sheet of paper in half.



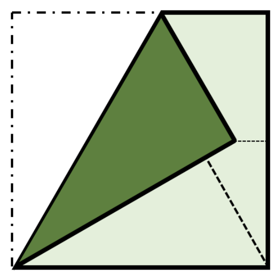
3. Crease the paper and unfold.



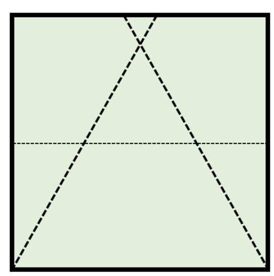
4. Fold the top right corner down and make it touch the crease. Move the top right corner along the crease until the bottom right corner forms a perfect point.



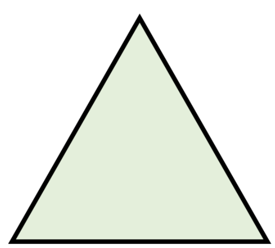
5. Unfold.



6. Repeat step 4 except on the opposite corners.



7. Unfold. The triangle you see formed from the crease marks should be an equilateral triangle!

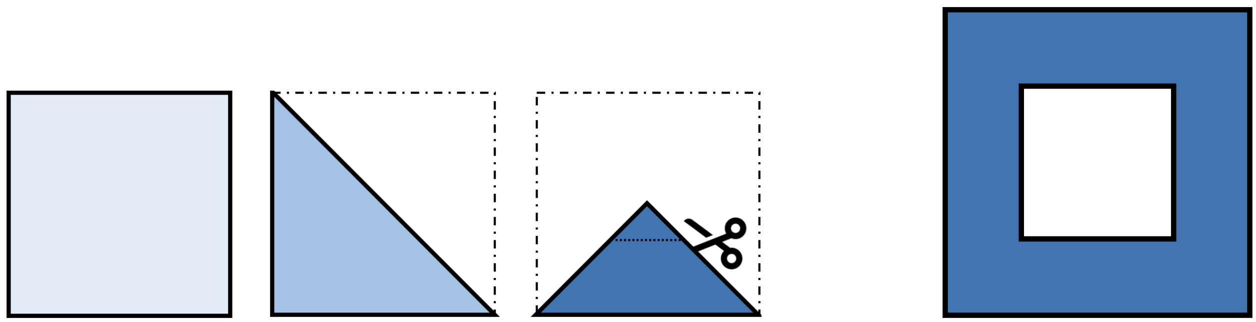


Problems

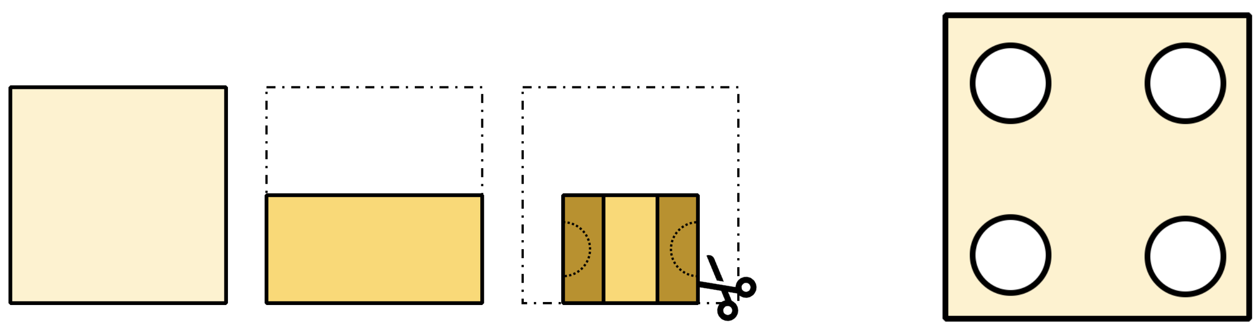
1. Prove using only paper folding, that the triangle in the previous example is *actually* equilateral (i.e. show that all of the angles all 60°).

If you fold the triangle in half along each of its three bilateral symmetries, you will see that all of the sides of your triangle line up perfectly, which means all three sides of the triangle are the same length, meaning it must be an equilateral triangle.

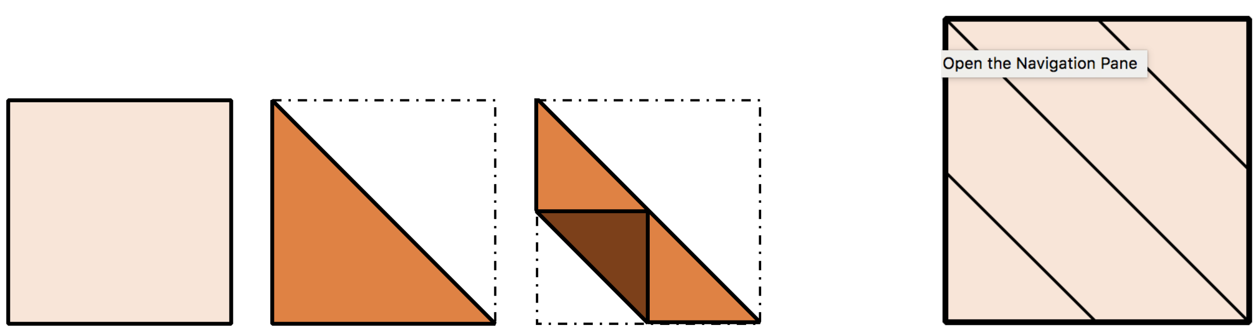
2. Using only a single cut, how can you make this shape out of a sheet of paper?



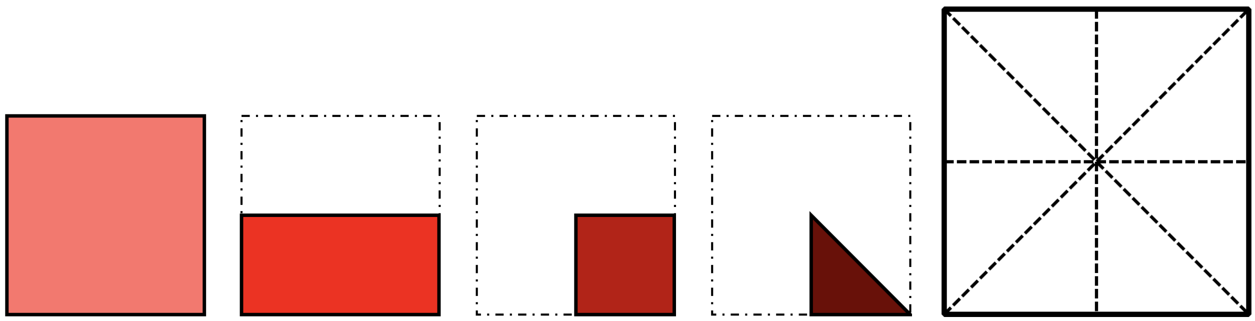
3. If the sheet of paper below is folded and cut along the dotted lines as shown, what will the unfolded sheet of paper look like? Draw your answer in the square to the right.



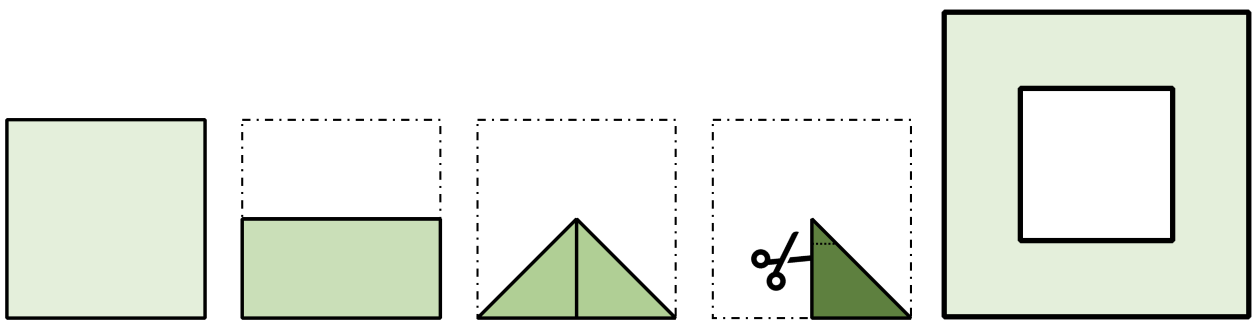
4. What crease marks will appear when the sheet of paper below is unfolded? Draw your answer in the square to the right.

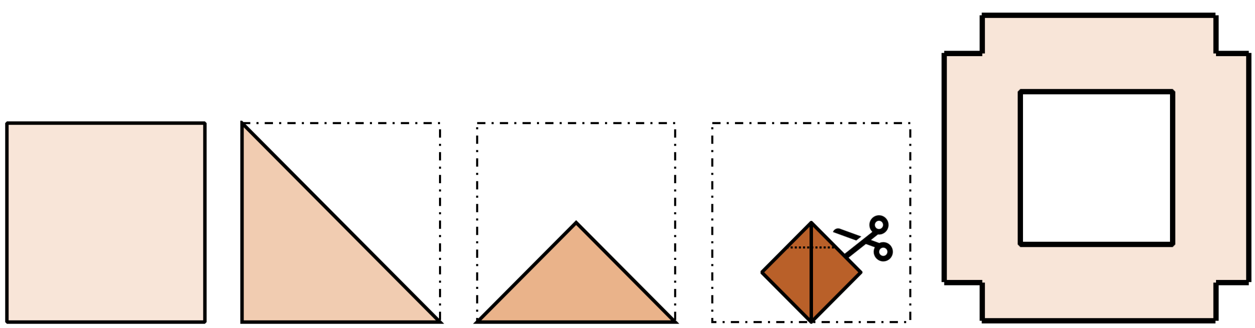


5. What folds have to be done to make these crease marks? Draw each step of the folding process below.



6. If the sheets of paper below are folded and cut along the dotted lines as shown, what will the unfolded sheets of paper look like? Draw your answers in the squares to the right.





*7. A standard sheet of paper is about 0.1mm thick.

a) How thick will a standard sheet of paper be if it is folded in half 5 times?

The paper will double in thickness after each fold. If the paper is folded 5 times, the thickness will double five times as well!

$$Thickness = 0.1\,mm \times 2 \times 2 \times 2 \times 2 \times 2 = 3.2\,mm$$

b) How many times do you need to fold a standard sheet of paper in half for it to be more than 1km thick?

The first thing to notice is that the paper thickness is in millimeters, and we are given a thickness in kilometers in our question. Convert 1 kilometer into terms of millimeters.

$$1\, kilometer = 1000\, meters = 100,000\, centimeters = 1,000,000\, millimeters$$

So how many times do we have to multiply 0.1mm by 2 until we get a number larger than 1,000,000mm (i.e. how many times do we have to fold the paper in half)?

Using a calculator, you can see that after 23 folds you would have a thickness of 838,860.8 mm, so to get more than 1 kilometer, or 1,000,000 millimeters of thickness, 24 folds must be done.

Another faster, more mathematical way of solving this question would be to use logarithms. This is something you will likely see in high school!

c) How thick will a standard sheet of paper be after n folds?

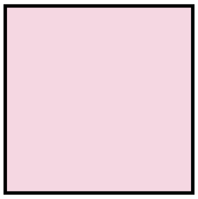
For every fold of the paper, we must multiply by an additional factor of 2. For example, after one fold, we multiply by 2 once. After two folds, we multiply by 2 twice etc.

After many folds, we don't want to have to write out '×2' for every single fold, so instead we can use something known as exponents!

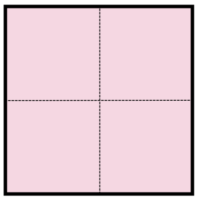
1 Fold	$Thickness = 0.1\, mm \times 2$
2 Folds	$Thickness = 0.1\, mm \times 2 \times 2$
3 Folds	$Thickness = 0.1\, mm \times 2 \times 2 \times 2$
n Folds	$Thickness = 0.1\, mm \times 2 \times 2 \times 2 \times 2 \times 2$ ...... $\times$ 2 with n 2s overall

1 Fold	$Thickness = 0.1\, mm \times 2^1$
2 Folds	$Thickness = 0.1\, mm \times 2^2$
3 Folds	$Thickness = 0.1\, mm \times 2^3$
99 Folds	$Thickness = 0.1\, mm \times 2^{99}$
n Folds	$Thickness = 0.1\, mm \times 2^n$

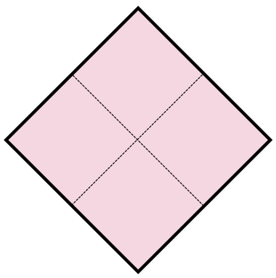
Bonus: Create the following origami box.



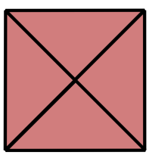
1. Begin by grabbing a square sheet of paper.



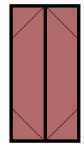
2. Fold the sheet of paper in such a way that you get crease marks down the middle both vertically and horizontally.



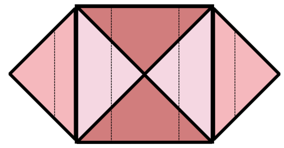
3. Rotate the sheet of paper by 45 degrees.



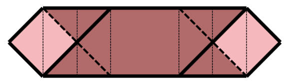
4. Fold in each corner of the sheet of paper to the intersection of the two creases.



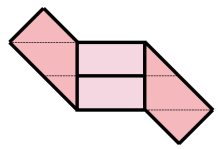
5. Fold the left and right sides of the paper to meet in the center.



6. Unfold the previous step and unfold the left and right triangles.

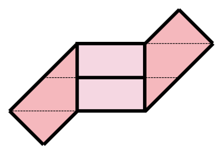


7. Fold the top and bottom of the sheet of paper to meet at the center.

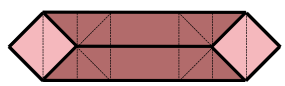


8. Fold the left side up and the right side down at 90 degree angles.

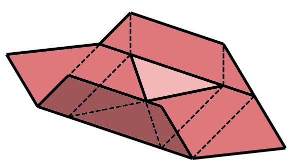
(Fold along the bolded, dashed line from the previous step)



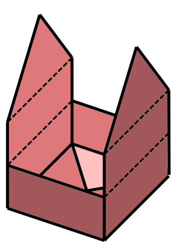
9. Unfold the previous step, then repeat the previous step, except fold in the other direction.



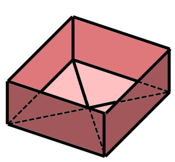
10. Unfold the previous step.



11. Fold the two trapezoidal shaped flaps of paper up at a 90 degree angle. There should be two triangular flaps attached to them, keep them flat on the ground.



12. While keeping the two trapezoidal shapes vertical, push inwards on the top left and right corners of each trapezoid. The left and right sides of the paper should fold upwards to make this shape.



13. Fold down and inwards along the dotted lines in the previous diagram. The final result should have four triangles with their points touching at the center of the box.