Egyptian Mathematics

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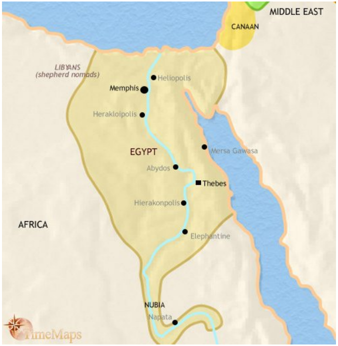
Three Part Series

- **Egyptian Mathematics**
- Diophantus and Alexandria
- Tartaglia, Cardano and the Renaissance Age

Egypt and Babylon

Egypt circa 1500 BCE

- Egyptian civilization established on the shores of the Nile about 5000 years ago
- The yearly flood of the Nile brings sediments and nutriments that enriched the surrounding soils in the valley
- The Nile becomes a method of transportation for both material and people and with the flood allowing for crops to grow, Egypt becomes a self-sufficient country (unlike other civilizations)



Papyrus

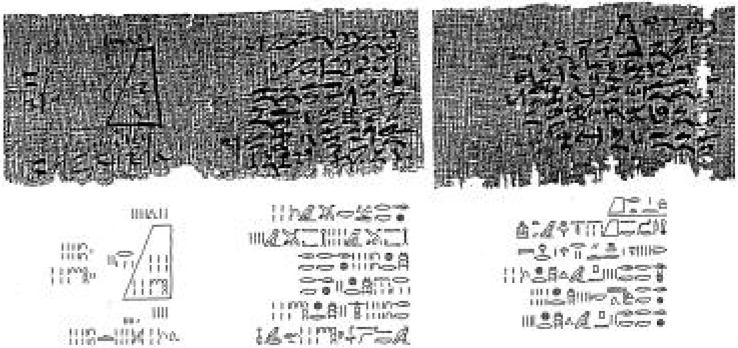
- Egyptians were the first known source of papyrus, a writing surface made from the pith (tissue in the stems) of a papyrus plant.
- A lot of what we know from Egyptian mathematics originates from papyri.
- We will be discussing content from two main papyri: the Rhind Mathematical Papyrus and the Moscow Mathematical Papyrus.

Rhind Papyrus

Written by the scribe Ahmes (also known as Ahmose or A'hmosè).

(Source: Wikimedia Commons)

Moscow Papyrus



(Source: Wikimedia Commons)

Mathematics Arising Out of Necessity

For Egyptians, mathematics was used as a tool to solve many of the problems they encountered.

- Areas
- Taxes
- Calendars
- Food
- Volumes

Areas

Common units of measurement:

- Cubit: length from the tip of the middle finger to the bottom of the elbow. Approximately 52cm.
- Khet: 100 cubits
- Setat/Arura/Aroura: A square khet. (About 2.7 square km or 2/3 of an acre).
- Digit, palm, hand, fist: Other smaller units of measurement.

This allowed them to give parcels of land to people and to keep track of these lands for taxes.

Area of a Circle

- Egyptians used an approximation to compute the area of a circle.
- In Problem 50 of the Rhind Papyrus [Bur91, p. 57]

  > *Example of a round field of diameter 9 khet. What is its area? Take away $\frac{1}{9}$ of the diameter, namely 1; the remainder is 8. Multiply 8 times 8; it makes 64. Therefore, it contains 64 setat of land.*

- This translates to $A = \left(\frac{8d}{9}\right)^2$ where d is the diameter of a circle

Compared to Modern Knowledge

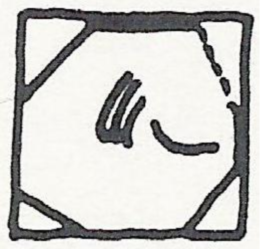
- It's not clear that the Egyptians knew that the circumference of a circle and it's diameter are in proportion to each other (we call this value π now).
- This being said, we can use the Egyptian approximation to see how they approximated a value for π.
- Solving

$$\pi r^2 = A = \left(\frac{8d}{9}\right)^2 = \left(\frac{8(2r)}{9}\right)^2 = \frac{256r^2}{81}$$

  we see that the Egyptians would have computed π to be $256/81 = 3.1605$. This is reasonably close; the error in using this value would be 0.0189.

Hypothesis On This Approximation

- To this day, scholars are uncertain of exactly how the Egyptians came to this approximation.
- We have some evidence from the Rhind Papyrus with problem 48 on computing the area of an octagon that is inscribed in a square of side length 9.



https://numberwarrior.files.wordpress.com/2008/03/problem48.jpg

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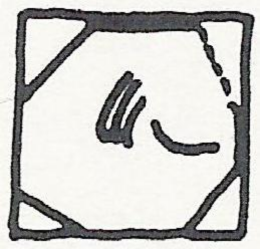

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- We see we get the answer $A = 9^2 - 4(9/2) = 63$. This is almost the value obtained by taking $d = 9$ in $A = (8d/9)^2$.

Taxes

- With an ability to compute areas, Egyptians were able to compute how much certain properties should be taxed.
- Taxes were collected by or on behalf of the Pharaoh who owned everything.
- Grain was one of the most importantly taxed items as it could be used for both food or trade.
- Egyptians were taxed about 1.5 tons of grain per setat.

Calendars

- Given how important the harvest was, Egyptians needed an accurate way to determine things like seasons, the tide of the Nile and so on.
- Egyptians used the sun to help determine days, seasons and years.
- They also used the moon to help determine months to produce many calendars.
- A preliminary calendar consisted of alternating 29 and 30 month long days based on a lunar cycle for a total of 354 days in a year.
- Other calendars had 12 months of 30 days based on the sun and Egyptians added 5 days at the end of the year for 365 days [Kat93, p. 26].

Food

Divide 9 pitas amongst 10 people evenly.

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- Idea: First, split 5 pitas in half and give everyone a half.
- Then, take the 5 remaining pitas and give everyone one third of a pita leaving 2/3 of a pita.
- Take these two thirds and split them up into fifths and give everyone one of these fifths.
- Then each person has received $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{9}{10}$ of a pita.

Fibonacci's Greedy Algorithm

- This procedure can generalize very nicely.
- The idea is we want to break up a fraction into a bunch of unit fractions (a fraction where the numerator is 1).
- Take any such fraction and then find the largest unit fraction less than the given fraction (This value is $\frac{1}{\lceil n/m \rceil}$).
- Subtract the unit fraction from the given fraction, reduce and repeat until you are left with a unit fraction, that is, compute $\frac{m}{n} - \frac{1}{\lceil n/m \rceil}$.
- This always converges (see `https://math.stackexchange.com/questions/458238/fractions-in-ancient-egypt` for an induction proof)

Another Example

Try to use the algorithm with $\frac{3}{7}$.

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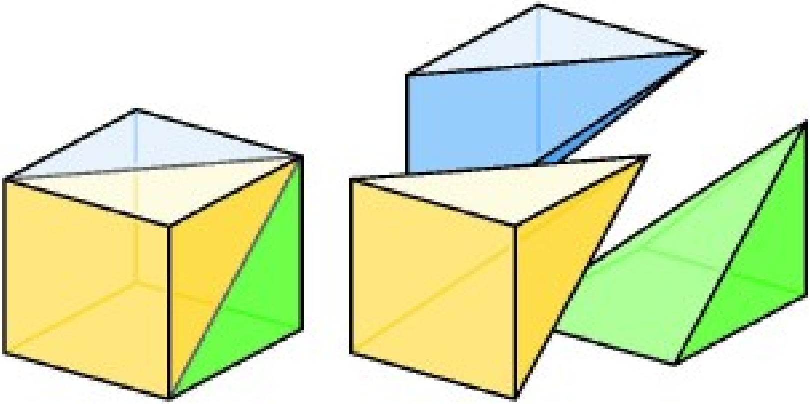
$$\frac{3}{7} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231}$$

Pyramids

We cannot in good conscious talk about Egyptian mathematics without mentioning pyramids. (Source: Wikipedia)

Volume of Pyramids

$$V = \frac{\ell w h}{3}$$

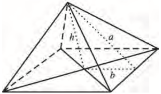


http://www.math.brown.edu/~banchoff/Beyond3d/chapter2/section02.html

Dimensions of the Pyramids

- Great Pyramid at Gizeh, erected in 2600 BCE by Khufu (Greek: Cheops).
- Herodotus (a Greek historian) wrote in a passage that the dimensions of the Great Pyramid should be chosen such that the area of a square with side length equal to the height of the pyramid would be equal to the area to the face of a triangle of the pyramid [Bur91, p. 62].
- Calling h the height, a the slant length and $2b$ the size of the square length, we have (picture form [Bur91, p. 62]

$$h^2 = \frac{1}{2}(2ba) = ab$$



Dimensions of the Pyramids

- With $h = ab$, we have the Pythagorean Theorem telling us that $b^2 + h^2 = a^2$.
- Combining gives $b^2 + ab = a^2$ and diving by a^2 gives $\left(\frac{b}{a}\right)^2 + \frac{b}{a} = 1$.
- This means that $x = b/a$ is a positive root of $x^2 + x - 1 = 0$.
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$$\frac{b}{a} = x = \frac{\sqrt{5} - 1}{2} = 0.6180339...$$

which is the reciprocal of the golden ratio, $\varphi = 1.6180339...$

Golden Ratio

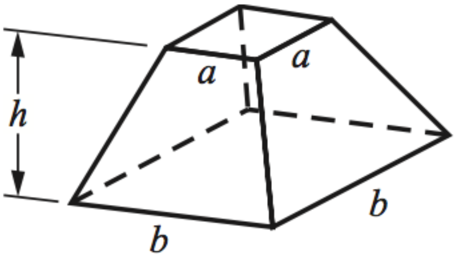
The Golden Ratio is ubiquitous in nature

- Snail Shells
- Art
- Pantheon
- Sunflowers
- Tool's song Lateralus
- Human proportions
- ... and so much more!

Frustum Volume

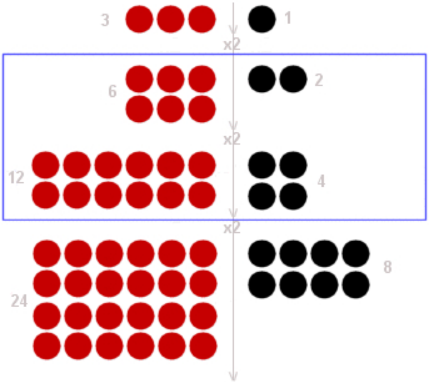
- In problem 14 of 25 from the Moscow Papyrus, we find an interesting formula for the volume of a truncated square based pyramid (frustum).
- Using the diagram below [Bur91, p. 56], we have

$$V = \frac{h}{3}(a^2 + ab + b^2)$$



Multiplication

Example: To multiply 3 by 6

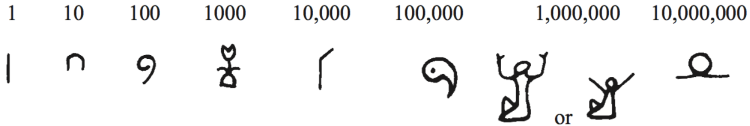


Take the combination of blocks on the 1's side which add up to 6 (2 + 4), and count up the corresponding blocks on the 3's side (6 + 12), to give a total of 18

http://www.storyofmathematics.com/egyptian.html

Numeral System

Each number for Egyptians were their own symbol [Bur91, p.13-14].

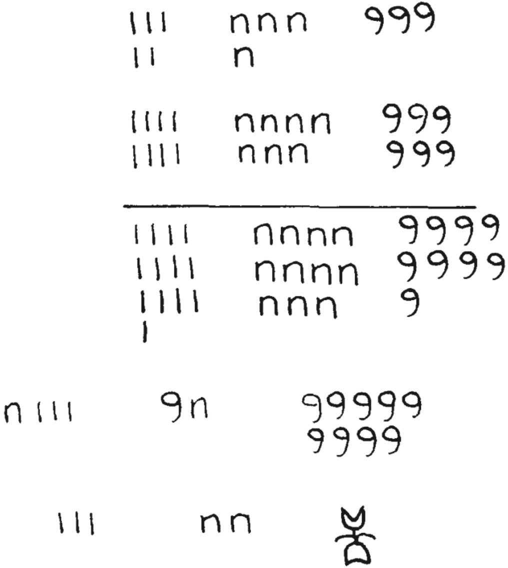


To represent fractions, Egyptians would put an eye  over the number with the following two exceptions:

- One half was denoted by 
- Two thirds was denoted by 

Sample Computation

345
678
$\overline{}$
1023

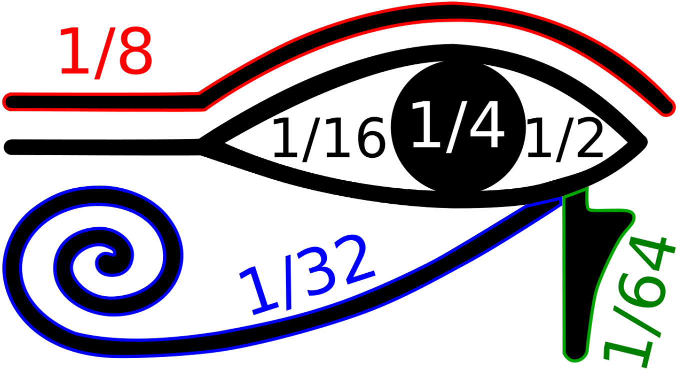


Horus

- Horus was the son of Osiris, king of the gods
- Osiris was killed by his brother Set.
- Horus tried to avenge his father, but in doing so he lost an eye that was restored to him by Thoth, the god of wisdom
- Horus instead decided to give his eye to his father who, following his assassination, became the god of hell so that his father could "keep" an eye on the world of the living.
- The lost eye of Horus represents the moon and the restored eye of Horus represents the sun.

Horus Eye and Fractions

Note unit fractions, geometric progression. binary numbers, etc.



For More Information

Check out "The Story of Maths" by Marcus du Sautoy as well as some of the references on the next slide.

Thank you!

References I

David M. Burton, *The history of mathematics*, second ed., W. C. Brown Publishers, Dubuque, IA, 1991, An introduction. MR 1223776

Victor J. Katz, *A history of mathematics*, HarperCollins College Publishers, New York, 1993, An introduction. MR 1200894