Cardano's Formula

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Special thanks to Steve Furino!

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This Week... Girolamo Cardano

- Born 24 September 1501. Died 21 September 1576.
- Illegitimate child of Fazio Cardano (was close friends to Da Vinci)
- Prolific mathematician and scientist
- Early adopter of binomial coefficients, Binomial Theorem.

Reference: School of Mathematics and Statistics, University of St Andrews, WIkimedia Commons

Cardano the Thief

Girolamo Cardano (1501 - 1576) was a turbulent man of genius, very unscrupulous, very indiscreet, but of commanding mathematical ability ... He was interested one day to find that Tartaglia held a solution of the cubic equation. Cardano begged to be told the details, and eventually under a pledge of secrecy obtained what he wanted. Then he calmly proceeded to publish it as his own unaided work in the Ars Magna which appeared in 1545... He seems to have been equally ungenerous in the treatment of his pupil Ferrari, who was the first to solve a quartic equation. Yet Cardano combined piracy with a measure of honest toil, and he had enough mathematical genius in him to profit by these spoils...

H W Turnbull, “The Great Mathematicians”

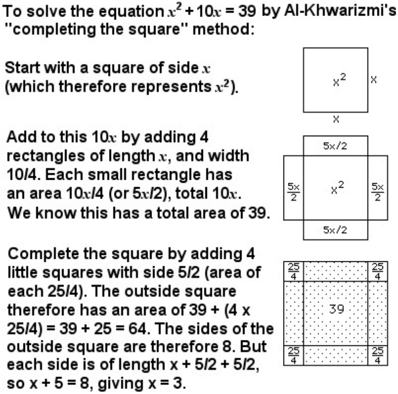
Revisit Muhammad ibn Musa al-Khwarizmi (c. 790 - 850)

- Father of modern algebra
- Wrote The Compendious Book on Calculation by Completion and Balancing, a book to solve quadratic equations.

Khwarizmi statute in Amir Kabir University, Tehran

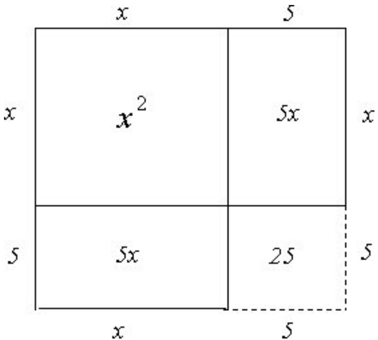
A Sample Problem

For instance, "one square and ten roots of the same amount to thirty-nine dirhems", that is to say, what must be the square which when increased by ten of its own roots amounts to thirty-nine? [Kat07, p. 543]



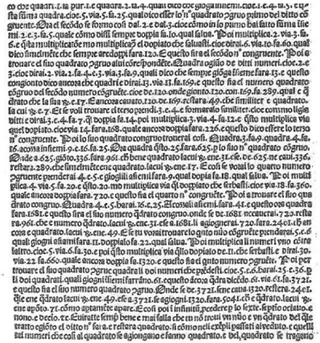
http://www.storyofmathematics.com/islamic_alkhwarizmi.html

Alternate Approach



Luca Pacioli (1445 - 1517)

- Close friend of Leonardo da Vinci
- In 1494 the first edition of his "Summa de Arithmetica, Geometrica, Proportioni et Proportionalita" appeared.
- Does not discuss cubic equations but does discuss quartics



jeremycripps.com/docs/Summa.pdf
https://archive.org/details/161Pacioli

Scipione dal Ferro (1465 - 1526)

- Chair of Arithmetic and Geometry at the University of Bologna
- Presumably met Pacioli when Pacioli was at Bologna in 1501 -1502
- Was able to solve cubics of the form $x^3 + px = q, \quad p, q > 0$
- Kept this knowledge secret. Why?
- Shortly before his death in 1526, he passed on the secret to one of his students, Antonio Maria Fiore

Antonio Maria del Fiore (dates c. 1500)

- Also known as Antonio Maria Fior
- Couldn't keep a secret
- Rumours began to spread that some form of the cubic equation had been solved in Bologna

Niccolò Fontana (Tartaglia)

- Niccolò (or Nicolo)Fontana (Tartaglia) (1500 - 1557), originally of Brescia, moved to Venice in 1534
- Family name was Fontana.
- From a very poor family.
- Father was a despatch rider (mailman). Killed in 1506 by robbers further impoverishing the Fontana family. [Str13, p. 188-193]
- This wasn't the end of the Fontana misfortune.
- February 18, 1512: The Sack of Brescia.

1512 - War of the League of Cambrai [Str13, p. 188-193]

- Part of War of the League of Cambrai a.k.a. War of the Holy League (1508-1516)
- Main parties: France, Papal States, Venitians
- A mess of alliances (First Papal States allied with King Louis XII which led to friction)
- Then in 1510, formed a Venetial-papal aliance against France
- They managed to defend northern Italy from France but then disagreements with the Venetians about the spoils caused another broken alliance.

Feb. 18th, 1512: The Sack of Brescia [Str13, p. 188-193]

- While the Italians managed to defend northern Italy from France, it wasn't without damage.
- The Sack of Brescia was a slaughtering of Italian citizens in Brescia
- Brescia revolted against French control - reinforcing itself with Venetian troops.
- Brescia was sacked and pillaged - more than 45000 citizens killed. (For comparison, in 1561, Brescia had 41,168).[1]

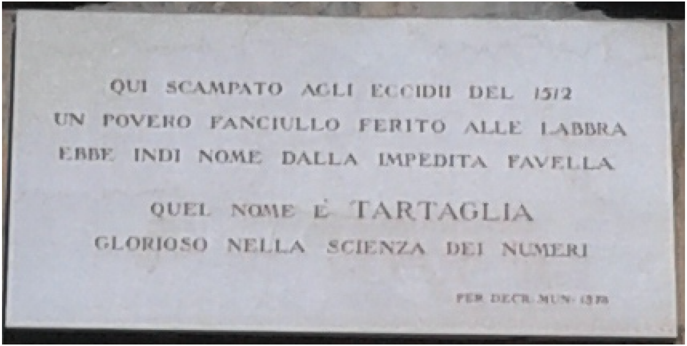
[1] Some estimate killed citizens 17,000-40,000 - see [HHH, p.10, 13]

Poor Fontana [Str13, p. 188-193]

- Niccolò, his mother and brothers took sanctuary in the local cathedral.
- Alas, a French soldier slashed Niccolò across the head leaving him for dead.
- Mother nursed him back to health but his speech was never the same.
- This gave him the nickname "Tartaglia" meaning "the stammerer"

https://upload.wikimedia.org/wikipedia/commons/0/0b/Niccol%C3%B2_Tartaglia.jpg

Tartaglia Plaque Currently at Cathedral in Brescia



Here escaped the massacre of 1512, a poor injured child unable to speak due to an injury to his lips. His name is Tartaglia who became glorious in the science of numbers.

Duomo at Brescia (from Google Maps)

Niccolò Tartaglia

- Tartaglia was very poor. Made his money as a teacher, tutor, and significantly, debater.
- Injury forced him to stay at home; became prolific at mathematics however he became unbearably proud and arrogant offsetting his lowly origins and gruesome figure [Str13]
- In 1534, he moved to Venice and was beating local mathematicians at contests that were becoming in vogue at the time.
- Translated Euclid's Elements into Italian (formerly only Greek and Latin).
- Tartaglia solved equations of the following form but kept the technique secret $x^3 + ax^2 = b, \quad a, b > 0$

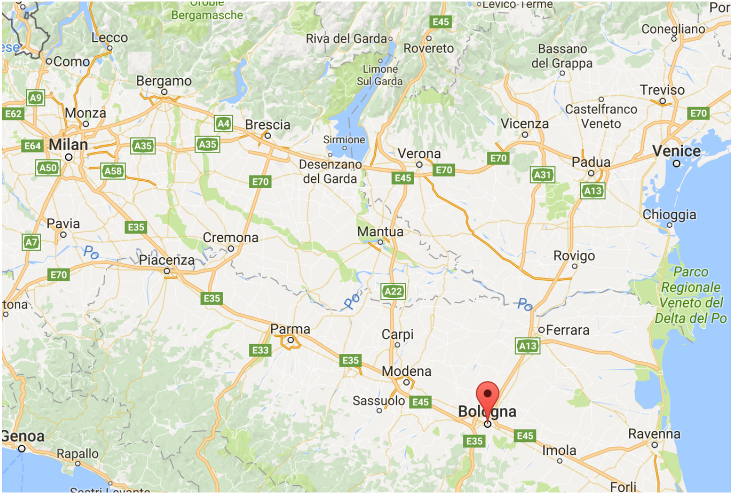
Tartaglia Statue in Brescia at Piazetta Santa Maria Calchera

(Near Universita Cattolica Del Sacro Cuore)

First ever Selfie?

Actual First Ever Selfie (Thanks J.P. Pretti)

Map of Italy


Bergamasche
Lecco
Como
Bergamo
Monza
Milan
Brescia
Riva del Garda
Rovereto
Limone
Sul Garda
Sirmione
Desenzano
del Garda
Verona
Bassano
del Grappa
Conegliano
Treviso
Castelfranco
Veneto
Vicenza
Padua
Venice
Chioggia
Pavia
Cremona
Piacenza
Mantua
Rovigo
Po
Parco
Regionale
Veneto del
Delta del Po
Parma
Carpi
Ferrara
Modena
Sassuolo
Bologna
Ravenna
Imola
Forlì
Genoa
Rapallo
A9
E62
E64
A35
A50
A58
A4
E70
E35
A7
E45
A22
A31
A13
E33

Fiore vs Tartaglia

- Fiore challenged Tartaglia to a public contest: each gave the other 30 problems with 40 or 50 days in which to solve them, the winner being the one to solve most but a small prize was also offered for each problem.
- In what follows, I have two dates for the contest. On [ORb], it quotes “Glory to God, 1534, and the 22nd day of February, in Venice.” however, most other references claim that the contest took place on February 20th, 1535
- I've learnt that “In the Republic of Venice until its end in 1797 the year did not begin on January 1 as today but officially on March 1. Therefore, February 12 and 13 of our year 1535 were still in the Venetian year 1534.” [Kat]

Contest

The challenge begins as follows:
Glory to God, 1534, and the 22nd day of February, in Venice.
These are the thirty problems proposed by me Antonio Maria Fiore to you Master Niccolò Tartaglia.

1. Find me a number that when its cube root is added to it, the result is six, that is 6.
2. Find me two numbers in double proportion such that when the square of the larger number is multiplied by the smaller, and this product is added to the two original numbers, the result is forty, that is 40.

Oops!

- However, only 8 days before the problems were to be collected, Tartaglia had found the general method for all types of cubics.
- Tartaglia solved all Fiore's problems in the space of 2 hours
- Fiore couldn't solve most of Tartaglia's non-cubic related problems [Kat93, p. 359]
- Fiore was humiliated. Tartaglia becomes a celebrity
- The winners prize: 30 banquets prepared by the loser for the winner and his friends (Tartaglia declined) [Kat93, p. 359]

Contest Simulation

- Divide down the middle two teams (so yes you may have to move a bit).
- Nominate two captains
- We will have Team Tartaglia and Team del Fiore
- What follows is 10 polynomial equations. Each side will solve the questions and hand their solution to a team captain who will be responsible for official submissions. The captain can also work on them.
- You must find all real roots to each problem.
- You get only 3 chances as a team - if you don't get the answer in 3 tries you can't get the point for that problem.

Cardano to Tartaglia, February 12, 1539 [ORb]

I wonder much, dear Master Niccolò, at the unhandsome reply you have made to one Zuan Antonio de Bassano, bookseller ... I would pluck you out of this conceit, as I plucked out lately Master Zuanne da Coi, that is to say, the conceit of being the first man in the world, wherefore he left Milan in despair; I would write to you lovingly and drag you out of the conceit of thinking you are so great - would cause you to understand from kindly admonition, out of your own words, that you are nearer to the valley than the mountain-top. In other things you may be more skilled and clever than you have shown yourself to be in your reply; and so I must in the first place state that I have held you in good esteem, and as soon as your book on Artillery appeared, I bought two copies, the only ones Zuan Antonio brought, of which I gave one to Signore the Marquis ...

Cardano to Tartaglia, February 12, 1539 [ORb]

The third point is, that you told the said bookseller that if one of my questions were solved all would be solved, which is most false, and it is a covert insult to say that while thinking to send you six problems, I had sent but one, which would argue in me a great confusion of understanding; and certainly, if I were cunning, I would wager a hundred scudi upon that matter; that is to say, that they could not be reduced either into one, or into two, or into three questions. And, indeed, if you will bet them, I will not refuse you, and will come at an appointed time to Venice, and will give bank security here if you will come here, or will give it to you there in Venice if I go thither. This is not mere profession, for you have to do with people who will keep their word ...

Cardano to Tartaglia, February 12, 1539 [ORb]

I send you two questions with their solutions, but the solutions shall be separate from the questions, and the messenger will take them with him; and if you cannot solve the questions he will place the solutions in your hand. You shall have them each to each, that you may not suppose I have sent rather to get than to give them; but return first your own, that you may not lead me to believe that you have solved the questions, when you have not. In addition to this, be pleased to send me the propositions offered by you to Master Antonio Maria Del Fiore, and if you will not send me the solutions, keep them by you, they are not so very precious. And if it should please you, in receiving the solutions of my said questions - should you yourself be unable to solve them, after you have satisfied yourself that my first six questions are different in kind - to send me the solution of any one of them, rather for friendship's sake, as for a test of your great skill, than for any other purpose, you will do me a very singular pleasure.

Cardano to Tartaglia, February 12, 1539 [ORb]

1. Make me of ten four quantities in continued proportion whose squares added shall make sixty.
2. Two persons were in company, and possessed I know not how many ducats. They gained the cube of the tenth part of their capital, and if they had gained three less than they did gain, they would have gained an amount equal to their capital. How many ducats had they?

Cardano Persuades Tartaglia To Tell His Secret

- Cardano invites Tartaglia to visit him in Milan
- Cardano was just about to publish Practica Arithmeticae (1539)
- The following conversation between the two men on 25 March 1539 in Cardano's home is reported by Tartaglia later

Continuing the Conversation [ORb]

Cardano: And I also wrote to you that if you were not content that I should publish them, I would keep them secret.
Tartaglia: Enough that on that head I was not willing to believe you.
Cardano: I swear to you by the sacred Gospel, and on the faith of a gentleman, not only never to publish your discoveries, if you will tell them to me, but also I promise and pledge my faith as a true Christian to put them down in cipher, so that after my death nobody shall be able to understand them. If you will believe me, do; if not, let us have done.

Continuing the Conversation [ORb]

Tartaglia: If I could not put faith in so many oaths I should certainly deserve to be regarded as a man with no faith in him; but since I have made up my mind now to ride to Vigevano to the lord marquis, because I have been here already three days, and am tired of awaiting him so long, when I am returned I promise to show you the whole.

Cardano: Since you have made up your mind at any rate to ride at once to Vigevano to the lord marquis, I will give you a letter to take to his excellency, in order that he may know who you are; but before you go I should wish you to show me the rule for those cases of your, as you have promised.

Tartaglia: I am willing ...

(Warning: Poem Incoming)

Poem [ORb]

When the cube and things together
Are equal to some discreet number,
Find two other numbers differing in this one.
Then you will keep this as a habit
That their product should always be equal
Exactly to the cube of a third of the things.
The remainder then as a general rule
Of their cube roots subtracted
Will be equal to your principal thing
In the second of these acts,
When the cube remains alone,
You will observe these other agreements:

Poem [ORb]

You will at once divide the number into two parts
So that the one times the other produces clearly
The cube of the third of the things exactly.
Then of these two parts, as a habitual rule,
You will take the cube roots added together,
And this sum will be your thought.
The third of these calculations of ours
Is solved with the second if you take good care,
As in their nature they are almost matched.
These things I found, and not with sluggish steps,
In the year one thousand five hundred, four and thirty.
With foundations strong and sturdy
In the city girdled by the sea.

Finishing the Conversation [ORb]

Tartaglia: This verse speaks so clearly that, without any other example, I believe that your Excellency will understand everything.
Cardano: How well I understand it, and I have almost understood it at the present. Go if you wish, and when you have returned, I will show you then if I have understood it.

Cardano Asks For Help [ORb]

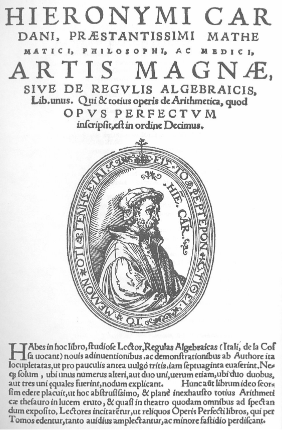
Tartaglia to Cardano (August 1539): Master Girolamo, I have received a letter of yours, in which you write that you understand the rule; but that when the cube of one-third of the coefficient of the unknown is greater in value than the square of one-half of the number you cannot resolve the equation by following the rule, and therefore you request me to give you the solution of this equation 'One cube equal to nine unknowns plus ten'. To which I reply, and say, that you have not used the good method for resolving such a case; also I say that such your proceeding is entirely false. And as to resolving you the equation you have sent, I must say that I am very sorry that I have given you already so much as I have done, for I have been informed, by person worthy of faith, that you are about to publish another algebraic work, and that you have gone boasting through Milan of having discovered some new rules in Algebra. But take notice, that if you break your faith with me, I certainly shall not break promise with you (for it is my custom); nay, even undertake to visit you with more than I have promised. ...

Note to Self [ORb]

Tartaglia (note to himself): I propose to see whether I can perhaps alter the data he possesses, that is, turn him away from the right track and make him take some other ...

Ars Magna

- The Great Art or Rules of Algebra by Girolamo Cardano
- Outstanding Mathematician, Philosopher and Physician In One Book, Being the Tenth in the Order of the Whole, Work on Arithmetic Which is called the Perfect Work.
- Published "Cardano's Formula".



HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

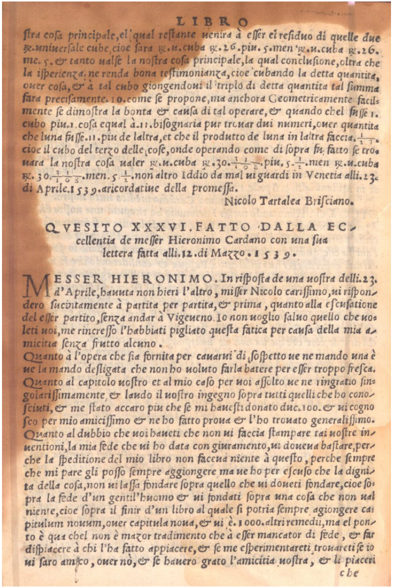
HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cossa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neque solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc aût librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarētur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.

Tartaglia Retaliates

Tartaglia publishes a book to defend his honour

Tartaglia Retaliates

Records the villainy of Cardano



LIBRO

ſtra coſa principale, el qual reſtante uenira à eſſer el reſiduo di quelle due R.uniuerſale cube, cioe ſara R.u.cuba R.26.piu.5.men R.u.cuba R.26.me.5.& tanto ualſe la noſtra coſa principale, la qual concluſione, oltra che la iſperienza ne renda bona teſtimonianza, cioe cubando la detta quantita, ouer coſa, & à tal cubo giongendoui il triplo di detta quantita tal ſumma ſara preciſamente.10.come ſe propone, ma anchora Geometricamente facilmente ſe dimoſtra la bonta & cauſa di tal operare, & quando chel fuſſe 1.cubo piu.1.coſa equal à.11.biſognaria pur trouar dui numeri, ouer quantita che luna fuſſe.11.piu de laltra, et che il produtto de luna in laltra faccia.$\frac{1}{27}$.cioe il cubo del terzo delle coſe, onde operando come di ſopra fu fatto ſe trouara la noſtra coſa ualer R.u.cuba R.30.$\frac{31}{108}$.piu.5.$\frac{1}{2}$.men R.u.cuba R.30.$\frac{31}{108}$.men.5$\frac{1}{2}$.non altro Iddio da mal ui guardi in Venetia alli.23.di Aprile.1539.aricordatiue della promeſſa.

Nicolo Tartalea Briſciano.

QVESITO XXXVI. FATTO DALLA ECcellentia de meſſer Hieronimo Cardano con una ſua lettera fatta alli.12.di Mazzo.1539.

MESSER HIERONIMO. In riſpoſta de una uoſtra delli.23.d'Aprile, hauuta non hieri l'altro, miſſer Nicolo cariſſimo, ui riſpondero ſucintamente à partita per partita, & prima, quanto alla eſcuſatione del eſſer partito, ſenza andar à Vigeueno. Io non uoglio ſaluo quello che uoleti uoi, me rincreſſo l'habbiati pigliato queſta fatica per cauſa della mia amicitia ſenza frutto alcuno.

Quanto à l'opera che ſia fornita per cauarui di ſoſpetto ue ne mando una è ue la mando deſligata che non ho uoluto farla batere per eſſer troppo freſca.

Quanto al capitolo uoſtro et al mio caſo per uoi aſſolto ue ne ringratio ſingolariſſimamente, & laudo il uoſtro ingegno ſopra tutti quelli che ho conoſciuti, & me ſtato accaro piu che ſe mi haueſti donato duc.100.& ui cognoſco per mio amiciſſimo & ne ho fatto proua & l'ho trouato generaliſſimo.

Quanto al dubbio che uoi haueti che non ui faccia ſtampare tai uoſtre inuentioni, la mia ſede che ui ho data con giuramento, ui doueua baſtare, perche la ſpeditione del mio libro non faceua niente à queſto, perche ſempre che mi pare gli poſſo ſempre aggiongere ma ue ho per eſcuſo che la dignita della coſa, non ui laſſa fondare ſopra quello che ui doueti fondare, cioe ſopra la fede d'un gentil'huomo & ui fondati ſopra una coſa che non ual niente, cioe ſopra il finir d'un libro al quale ſi potria ſempre agiongere capitulum nouum, ouer capitula noua, & ui è.1000.altri remedii, ma el ponto è qua chel non è mazor tradimento che à eſſer mancator di fede, & far diſpiacere à chi l'ha fatto appiacere, & ſe me eſperimentareti trouareti ſe io ui ſaro amico, ouer nò, & ſe hauero grato l'amicitia noſtra, & li piaceri

che

Ferrarri Enters The Dispute [ORb]

Ferrari to Tartaglia: You have the infamy to say that Cardano is ignorant in mathematics, and you call him uncultured and simple-minded, a man of low standing and coarse talk and other similar offending words too tedious to repeat. Since his excellency is prevented by the rank he holds, and because this matter concerns me personally since I am his creature, I have taken it upon myself to make known publicly your deceit and malice.

Ferrari to Tartaglia (again): As for the twenty-second problem in the disputation. You at first say that it is not a question for a mathematician. To which I reply, that, if by a mathematician you mean someone like you, that is, someone who spends the whole time on roots, fifth powers, cubes and other trifles, then you are quite right. I promise you that if it were up to me to reward you, taking example from the custom of Alexander, I would load you up so much with roots and radishes that you would never eat anything else in your life.

Solving Cubic Equations

- Tartaglia (and Cardano) argued geometrically
- Our presentation will be mainly algebraic.
- We first solve the simplified equation given by

$$x^3 + px = q$$

  where $p, q > 0$.

Brilliant Idea

Notice that for a and b numbers with $a > b > 0$,

$$\begin{aligned}
(a-b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3 \\
(a-b)^3 &= -3ab(a-b) + a^3 - b^3 \\
(a-b)^3 + 3ab(a-b) &= a^3 - b^3
\end{aligned}$$

Pattern matching this to $x^3 + px = q$, we see that $x = a - b$ is a solution to

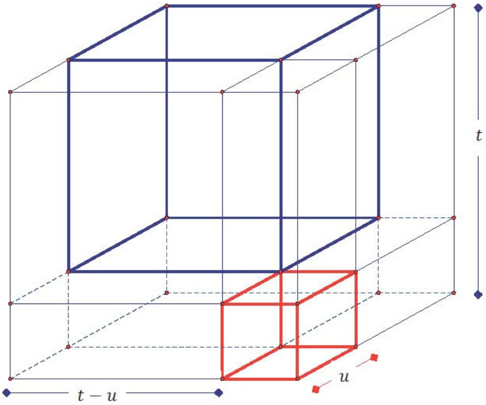
$$x^3 + px = q$$

where $p = 3ab$ and $q = a^3 - b^3$.

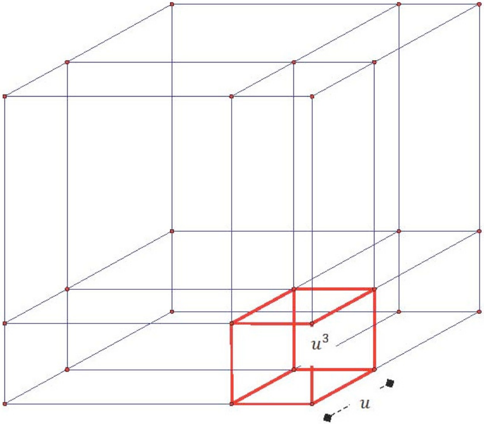
Dissecting the Cube

Tartaglia (and others) however thought of this dissection geometrically. The slides that follow are due to Marty Bonsangue at Cal State Fullerton.

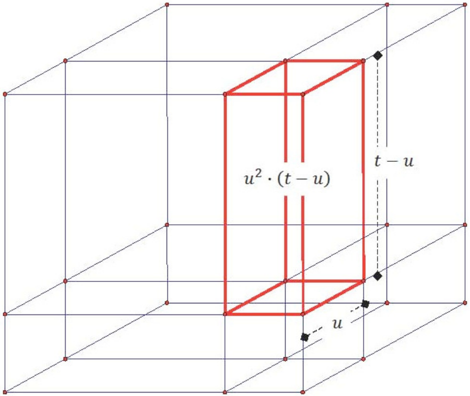
Tartaglia’s Cube



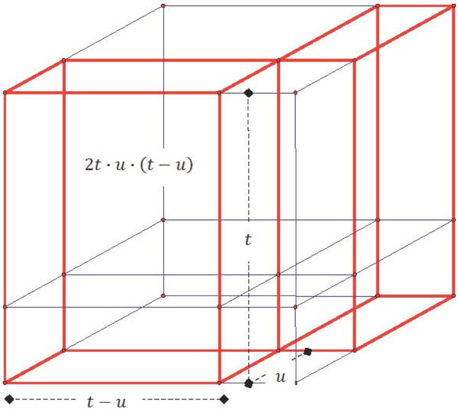
Small Cube



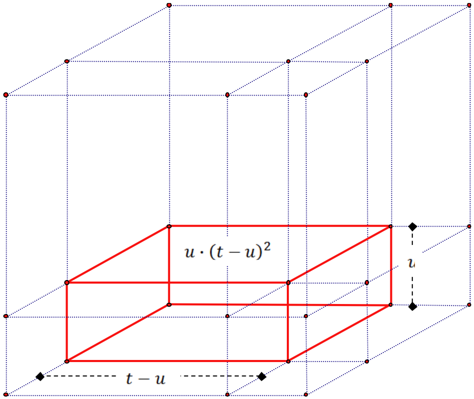
Tower



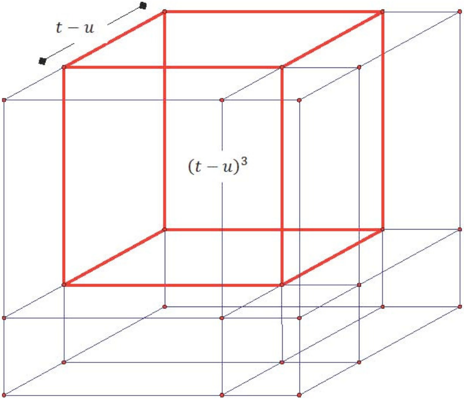
Two Slabs



Base



Big Cube



Summing

Big Cube	$(t-u)^3$
Small Cube	u^3
Tower	$u^2(t-u)$
Slabs	$2tu(t-u)$
Base	$u(t-u)^2$
Total	t^3

Simplifying the algebra

$$
\begin{aligned}
t^3 &= (t-u)^3 + u^3 + u^2(t-u) + 2tu(t-u) + u(t-u)^2 \\
t^3 - u^3 &= (t-u)^3 + 2tu(t-u) + u(t-u)(u + (t-u)) \\
t^3 - u^3 &= (t-u)^3 + 2tu(t-u) + tu(t-u) \\
t^3 - u^3 &= (t-u)^3 + 3tu(t-u)
\end{aligned}
$$

which is precisely the algebraic deconstruction we had from before.

Finding a and b

Hence, it suffices to find values for a and b satisfying

$$\begin{aligned} a^3 - b^3 &= q \\ ab &= \frac{p}{3} \end{aligned}$$

and then we can compute $x = a - b$ which will be a solution to our given equation.

Another idea

If we could find a value r satisfying $a^3 + b^3 = r$, then this with $a^3 - b^3 = q$, we can easily solve for a and b. Summing these two equations gives

$$2a^3 = r + q \quad \Rightarrow \quad a = \sqrt[3]{\tfrac{1}{2}(r+q)}$$

and subtracting these two equations gives

$$2b^3 = r - q \quad \Rightarrow \quad b = \sqrt[3]{\tfrac{1}{2}(r-q)}$$

Getting $a^3 + b^3$

Notice that

$$(a^3 + b^3)^2 = a^6 + 2(ab)^3 + b^6$$
$$(a^3 - b^3)^2 = a^6 - 2(ab)^3 + b^6$$

Taking the difference of these two equations gives

$$(a^3 + b^3)^2 - (a^3 - b^3)^2 = 4(ab)^3$$

However, we know that $a^3 - b^3 = q$ and $ab = p/3$ and thus,

$$(a^3 + b^3)^2 = q^2 + \frac{4p^3}{27}$$

Summary

Therefore, as

$$(a^3 + b^3)^2 = q^2 + \frac{4p^3}{27}$$

we see that

$$a^3 + b^3 = \sqrt{q^2 + \frac{4p^3}{27}}.$$

Notice above we can take the positive root since a and b are positive. It turns out even if these weren't positive, we could still take the positive root because the value of $a - b$ would eventually be the same (exercise).

Solving for a and b

Now, $a^3 - b^3 = q$ and $a^3 + b^3 = \sqrt{q^2 + \frac{4p^3}{27}}$ gives

$$a^3 = \frac{1}{2}\left(q + \sqrt{q^2 + \frac{4p^3}{27}}\right) = \frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

$$b^3 = \frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right) = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

Solving For x

This yields

$$a = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$
$$b = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

and thus,

$$x = a - b = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Cardano's Example

Cardano used the example $x^3 + 6x = 20$. In this case, $p = 6$ and $q = 20$ and hence

$$\frac{p^3}{27} = 8 \qquad \frac{q^2}{4} = 100$$

and thus, the formula

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

yields

$$x = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}$$

Wait a Minute...

If you try to solve this on Wolfram Alpha, you'll actually get the roots

$$x = 2, -1 \pm 3i.$$

Two questions:

- What does $-1 \pm 3i$ mean?
- How on earth is

$$x = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}$$

  equal to one of those roots?

Let's answer the second question first.

Magic!!!

Well if we start by believing that there must be a nice solution, it probably means that

$$10+\sqrt{108}=10+6\sqrt{3}=(a+b\sqrt{3})^3$$

for some values of a and b. Expanding:

$$10+6\sqrt{3}=(a+b\sqrt{3})^3=a^3+9ab^2+(3a^2b+3b^3)\sqrt{3}$$

and comparing coefficients of the constant term and the $\sqrt{3}$ term gives us that

$$a^3+9ab^2=10 \qquad 3a^2b+3b^3=6$$

Finishing the magic

Simplifying gives

$$a(a^2 + 9b^2) = 10 \qquad b(a^2 + b^2) = 2$$

We know the answer is positive so if this is solvable for a and b integers, a solution has to exist when $b \leq 2$ and sure enough, $b = 1$ and $a = 1$ gives a solution, namely that $10 + 6\sqrt{3} = (1 + \sqrt{3})^3$. Similarly, $-10 + \sqrt{108} = (-1 + \sqrt{3})^3$ and so

$$\begin{aligned} x &= \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}} \\ &= \sqrt[3]{(1 + \sqrt{3})^3} - \sqrt[3]{(-1 + \sqrt{3})^3} \\ &= 1 + \sqrt{3} - (-1 + \sqrt{3}) \\ &= 2 \end{aligned}$$

Ars Magna - On the Rule for Postulating a Negative Ch. XXXVII

The second species of negative assumption involves the square root of a negative. I will give an example: If it should be said, Divide 10 into two parts the product of which is 30 or 40, it is clear that this case is impossible. Nevertheless, we will work thus: We divide 10 into two equal parts, making each 5. These we square, making 25. Subtract 40, if you will, from the 25 thus produced, as I showed you in the chapter on operations in the sixth book, leaving a remainder of -15, the square root of which added to or subtracted from 5 gives parts the product of which is 40. These will be $5+\sqrt{-15}$ and $5-\sqrt{-15}$.

Mental Tortures

Putting aside the mental tortures involved, multiply $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$, making $25 - (-15)$. Hence this product is 40. Yet the nature of [the square] AD is not the same as that of 40 or of [the line segment] AB, since a surface is far from the nature of a number and that of a line, though somewhat closer to the latter. This truly is sophisticated ...

... So progresses arithmetic subtlety the end of which, as is said, is as refined as it is useless.

[From the 38th problem of Ars Magna Arithmeticae] Note that $\sqrt{9}$ is either 3 or -3, for a plus [times a plus] or a minus times a minus yields a plus. Therefore $\sqrt{-9}$ is neither 3 nor -3 but is some recondite third sort of thing.

A Brief Digression on Complex Numbers

- Define i to be the 'thing' such that $i^2 = -1$. (I will call it a number)
- We call $a + bi$ a complex number where a and b are real.
- We define the sum or difference of two complex numbers to be

$$(a \pm bi) + (c \pm di) = (a \pm c) + (b \pm d)i$$

- We define the product of two complex numbers to be

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$$

Practice

Take some time to practice complex numbers computations on the handout.

Another type of cubic

Cardano (and previously Scipione and Tartaglia) solved

$$x^3 + px = q \qquad p, q > 0$$

Cardano also solved

$$x^3 = px + q \qquad p, q > 0$$

in much the same way as $x^3 + px = q$. His value for x was

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}$$

Things are Getting Complex...

Contrasting the two solutions, for $x^3 + px = q$ with $p, q > 0$, Cardano's solution

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

gives only real solutions. However, Cardano's solution to $x^3 = px + q$ with $p, q > 0$:

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}$$

will give imaginary solutions whenever $\dfrac{q^2}{4} < \dfrac{p^3}{27}$.

Bombelli's Approach [Kat93, p.365-366]

- Keep in mind that imaginary numbers are not known to have any real meaning yet.
- *Ars Magna* was a difficult text to absorb and read
- 15 years later, Bombelli decided to write down his version of the text, *Algebra*
- He created an algebraic way to handle numbers that were neither positive (*più*) or negative (*meno*)
- He called bi and $-bi$ (in modern notation) *più di meno* and *meno di meno*
- He also gave our modern day version of the operations with complex numbers.

Brief Biography of Bombelli [ORa]

- Baptized Jan 20, 1526 died 1572 in Bologna
- Son of Antonio Mazzoli, a powerful family name in Bologna forcing him to eventually change his name to Bombelli to avoid the association
- Received no formal college education; was taught by engineer-architect Pier Francesco Clementi

Bombelli's Algebra

L'ALGEBRA

OPERA

Di RAFAEL BOMBELLI da Bologna

Diuisa in tre Libri.

Con la quale ciascuno da se potrà venire in perfetta cognitione della teorica dell'Arimetica.

Con vna Tauola copiosa delle materie, che in essa si contengono.

Posta hora in luce à beneficio delli Studiosi di detta professione.

IN BOLOGNA,

Per Giouanni Rossi. MDLXXIX.

Con licenza de' Superiori

Bombelli's Approach

Consider $x^3 = 15x + 4$. Cardano's formula gives

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

and the three roots are

$$4, -2 \pm \sqrt{3}$$

Bombelli's Idea

Notice that

$$\sqrt[3]{2+\sqrt{-121}} \quad \text{and} \quad \sqrt[3]{2-\sqrt{-121}}$$

differ only by a sign. What if the imaginary numbers they represented also only differed by a sign? That is, do there exist positive real numbers a and b such that

$$\sqrt[3]{2+\sqrt{-121}} = a + bi \quad \text{and} \quad \sqrt[3]{2-\sqrt{-121}} = a - bi$$

Bombelli's Thoughts [Bur91, p. 325]

It was a wild thought in the judgement of many; and I too for a long time was of the same opinion. The whole matter seemed to rest on sophistry rather than on truth. Yet I sought so long, until I actually proved this to be the case.

Attempting to solve for a and b

Setting $\sqrt[3]{2+\sqrt{-121}} = a + bi$, we see that

$$\begin{aligned} 2+\sqrt{-121} &= (a+bi)^3 \\ &= a^3 + 3a^2bi - 3ab^2 - b^3 i \\ &= (a^3 - 3ab^2) + (3a^2 b - b^3)i \end{aligned}$$

and so $2 = a(a^2 - 3b^2)$ and $b(3a^2 - b^2) = 11$

So...

If integer solutions are going to work, a is one of 1 or 2 and $a^2 - 3b^2 > 0$ so $a = 2$ either works or there is no integer solution. Checking yields $a = 2$ and $b = 1$. Hence,

$$\sqrt[3]{2 + \sqrt{-121}} = 2 + i \quad \text{and} \quad \sqrt[3]{2 - \sqrt{-121}} = 2 - i$$

Therefore,

$$\begin{aligned} x &= \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \\ &= (2 + i) + (2 - i) \\ &= 4 \end{aligned}$$

The Final Step

- In much the same way, one can handle $x^3 + px + q = 0$ for and p and q with knowledge of complex numbers.
- What about handling equations of the form $x^3 + ax^2 + bx + c = 0$?
- Idea: Eliminate the ax^2 term.

Eliminating ax^2

Set $x = y - a/3$. This gives

$$
\begin{aligned}
0 &= \left(y - \frac{a}{3}\right)^3 + a\left(y - \frac{a}{3}\right)^2 + b\left(y - \frac{a}{3}\right) + c \\
&= \left(y^3 - 3y^2\frac{a}{3} + 3y\frac{a^2}{9} - \frac{a^3}{27}\right) + a\left(y^2 - 2y\frac{a}{3} - \frac{a^2}{9}\right) + by - \frac{ab}{3} + c \\
&= y^3 + \left(b - \frac{a^2}{3}\right)y + \left(\frac{2a^3}{27} - \frac{ab}{3} + c\right)
\end{aligned}
$$

The rest of the story

- As I alluded to earlier, the quartic was solved by Cardano's student Ferrari using similar types of tricks (eliminating the cubic term, doing some weird completing tricks etc.)
- For fifth degree and higher, despite efforts, no solutions were found.
- It wasn't until 1824 when Abel from work of Ruffini managed to show that if a polynomial had degree at least 5, then one could not in general find an algebraic solution to every equation.
- This coincided with the founding of Galois Theory which helped explain the proof in a different way.

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