SET THEORY - LECTURE 1

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1. Basics of sets

Sets. A set is any collection of objects. For example, $\{1, 2, 3\}$, or $\{5, 11, 13\}$. A set may contain non-'mathy' objects. For instance, {cat, dog, mouse}, or {banana, 17, π, mouse}. Sets may also contain other sets! For instance, {1,2, {6,7,8}, 11, 12}.

We denote $|A|$ the number of elements of a set A. So $|\{5, 11, 13\}| = 3$. Note that {6,7,8} is a single element of the set $\{1, 2, \{6, 7, 8\}, 11, 12\}$. Also, we don't double count. So {1,2,2} only has 2 elements. On the other hand, {1,2, {2}} has 3 elements: 2 and {2} are not the same element!

Special sets.

(1) The empty set, the set that doesn't contain any elements. { }, also denoted $\emptyset$.
(2) $\mathbb{N} = \{0, 1, 2, \ldots\}$ - the natural numbers.
(3) $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$ - the integers.
(4) $\mathbb{Q} = \{\frac{5}{9}, 2, 6, \frac{4}{13}, \ldots\}$ -the rational numbers.
(5) $\mathbb{R} = \{3, 8, 9.43267, \pi, \sqrt{2}, \ldots\}$ - the real numbers.

Useful symbols.

- $\in$ - Indicates that an element is a member of a set. $3 \in \{2, 3, 4\}, \frac{6}{7} \in \mathbb{R}$. Also, $\{1, 2\} \in \{5, 6, 7, \{1, 2\}\}$. But, $0.34 \notin \mathbb{N}, \{2\} \notin \{1, 2, 3\}$.
- $\subseteq$ - Indicates that a set is contained in another. $\mathbb{N} \subseteq \mathbb{Z}$, $\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$. But $\{1, 2\} \not\subseteq \{4, 5, 6\}, \{3, 4, 5\} \not\subseteq \{4, 5, 6, 7, 8\}, \mathbb{Q} \not\subseteq \mathbb{Z}$. Also, $\{2\} \subseteq \{1, 2, 3\}$.

2. Operations on sets

Union. The *union* of A and B, denoted $A \cup B$, is the set consisting of the elements of A and the elements of B. For instance, $\{1, 2, 3\} \cup \{4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}$. Again, we don't double count, so $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$.

Intersection. The intersection of A and B consists of the elements that are in both A and in B. For instance, $\{1, 2, 3\} \cap \{4, 5, 6\} = \emptyset, \{1, 2\} \cap \{2, 3\} = \{2\}$,

$$\{n \in \mathbb{Z} | n \geq 8\} \cap \{n \in \mathbb{N} | n \leq 9\} = \{8, 9\}.$$

Distributivity. For any sets A, B, C, we have

(1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. We only show the first part. The second part is similar and is left as exercise. We need to show that any member of the RHS (righthand side) is also a member of the LHS (lefthand side), and vice versa. Let x be a member of $A \cup (B \cap C)$. Then by definition, x is either a member of A, or a member of $B \cap C$. If x is a member of A, then it is a member of $A \cup B$ and of $A \cup C$. Therefore, it belongs to the RHS. If x belongs to $B \cap C$, then it belongs to B and to C. Therefore, it belongs to $A \cup B$ and to $A \cup C$, and in particular to the RHS.

Now let x be a member of the RHS $(A \cup B) \cap (A \cup C)$. Then it belongs to both $(A \cup B)$ and $(A \cup C)$. If it belongs to A, then it also belongs to $A \cup (B \cap C)$, which is the LHS. Otherwise, it belongs to both B and C, so it belongs to $B \cap C$. In particular, it is a member of$A \cup (B \cap C)$. □

Complements of sets. If A and B are sets, then the complement of A in B, denoted $B \setminus A$ consists of all the elements of B that do not belong to A. For instance, $\{1, 2, 3, 4, 5, 6\} \setminus \{1, 2, 3\} = \{4, 5, 6\}$. Note that the complement flips unions and intersections: for any sets A, B, C, we have $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ and $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Power set. The power set of A consists of all the subsets of A. For instance,

- $\mathcal{P}(\{1, 2\} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.
- $\mathcal{P}(\mathbb{Z}) = \{\{1, 2, 3\}, \{-7, 12, 19\}, \mathbb{N}, \ldots\}$.

Theorem 1. *If A is a finite set, then $\mathcal{P}(A)$ contains $2^{|A|}$ elements.*

Why? To construct an element B of $\mathcal{P}(A)$, we need to decide for each element of A whether or not it will be in B. There are $|A|$ choices to be made, and each time there are 2 options. Altogether $2^{|A|}$ options. □

Ordered pairs, n-tuples, and Cartesian products. An ordered pair (a, b) is just like a set, except that the order does matter. An n-tuple is the same but with n elements instead of 2. In this case we also allow double counting. So each of the following pairs is distinct: $(1, 2), (2, 1), (1, 2, 2)$.

Given two sets A and B, their Cartesian product consists of all the pairs (a, b) such that $a \in A, b \in B$.

Example 2. If $A = \{3, 4\}$, and $B = \{8, 9\}$, then $A \times B = \{(3, 8), (3, 9), (4, 8), (4, 9)\}$.

The set $\mathbb{R} \times \mathbb{R}$ consists of all the pairs (r, s) of real numbers. We can think of it as the standard 2-dimensional plane.

3. Functions

A function is an assignment from one set to another. The input set, the one where elements are coming from is called the *domain*. The output set, the one where elements are assigned to is called the *range* or *codomain*. We denote a function f by

$$f : A \to B,$$

where A is the domain, and B is the range. Note that not every element in B needs to be achieved by f. If $f(a) = b$, we say that b is the *image* of a.

Example 3.

$$f : \mathbb{Z} \to \mathbb{Z}, f(a) = 5k.$$
$$g : \{4, 3, 2\} \to \{5, 6, 7\}, g(4) = 5, g(3) = 7, g(2) = 6.$$
$$h : \mathbb{Q} \to \mathbb{R}, h(x) = 4x + 7.$$

Note that the image by h of every rational number is rational, so we could have defined the range of h to be $\mathbb{Q}$.

Useful terminology.

- We say that a function is *surjective* or *onto* if every element in the range is being hit. For instance, f above is not surjective, g is, and h is not. However, if we change the range of h to $\mathbb{Q}$ then it becomes surjective.
- We say that a function is *injective* or *one-to-one* if every element in the range is only being hit once. Namely, if no element in the range is the image of two different elements. All the functions f, g, h are injective. On the other hand, the functions

  $$k : \{4, 5, 6\} \to \{1, 2\}, k(4) = 1, k(5) = 2, k(6) = 2,$$

  $$s : \mathbb{R} \to \mathbb{R}, s(x) = x^2$$

  are not.
- A function is called a *bijection* or an *isomorphism* if it is both surjective and injective.