Math Circles
18 October 2017
Logic and Problem Solving, Part 2
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EXERCISES

1. Each letter represents a different digit, and pairs of letters (e.g., AC) represent two-digit numbers.

   We know:

$$\begin{array}{ccc} A \times B = B & B \times C = AC & C \times D = BC \\ D \times E = CH & E \times F = DK & F \times H = CJ \\ H \times J = KJ & J \times K = E & K \times L = L \\ & A \times L = L & \end{array}$$

   Find the values of all the letters with explanation.

   Solution:
   Since $A \times B = B$, we must have that $A = 1$.

   Since $K \times L = L$ and $A \times L = L$, we must have that $L = 0$.

   Since $B \times C = 1C$, then $(B, C) = (6, 2)$ (to multiply to 12) or $(B, C) = (3, 5)$ (to multiply to 15): all other numbers with leading 1 are not possible.

   Since $C \times D = BC$, if $(B, C) = (6, 2)$, then no two single digit numbers multiply to 62 (which factors as 2×31). Thus $(B, C) = (3, 5)$ and $D = 7$.

   Since $D \times E = CH$, we know $7 \times E = 5H$. The only two digit number starting with 5 which is a multiple of 7 is 56, thus $E = 8$ and $H = 7$.

   Since $E \times F = DK$, we know $8 \times F = 7K$, and so $F = 9$ and $K = 2$ (since no other multiple of 8 begins with 7).

   Since $J \times K = E$, we know $J \times 2 = 8$, and so $J = 4$.

   In summary we have:

A	B	C	D	E	F	H	J	K	L
1	3	5	7	8	9	6	4	2	0

2. Harold and Tabby are playing a coin game.

   There are 12 coins in one pile, and each player can take either two or three coins from the pile. The player who takes the last coin loses.

   Harold goes first, and then they alternate turns.

   Each player makes a move that allows them to win, if possible. If there is no way to win, then the player will make a move that allows a tie, if possible.

   Show that there is a winner (i.e., that there is no tie), demonstrate who the winner is and what the winning strategy is. Give some reasoning and analysis why the strategy works.

   Solution:
   Tabby can always win. The key idea is to make the intermediate values 7 and 2 after Tabby's move. If the pile contains 2, then Harold must pick 2 and will lose.

In other words, Tabby plays the "opposite" move of Harold to get 7 and then 2. If Harold plays 3, Tabby plays 2; if Harold plays 2, then Tabby plays 3. In both cases, the pile will be reduced by a total of 5 after Tabby plays.

The sequence looks like:

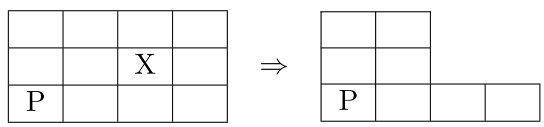
$$12 \xrightarrow{H} \{10, 9\} \xrightarrow{T} 7 \xrightarrow{H} \{5, 4\} \xrightarrow{T} 2 \xrightarrow{H} 0$$

3. Chomp is a simple two-player game played on an $n \times m$ grid, which you can imagine is a chocolate bar.

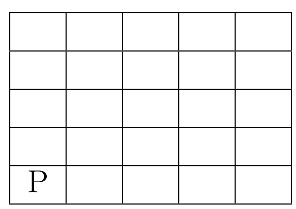
   Players take turns picking a square, and "eating it", removing that square and all squares to the right and above this square.

   In the bottom-left corner is a poisoned square: the player that eats this square loses.

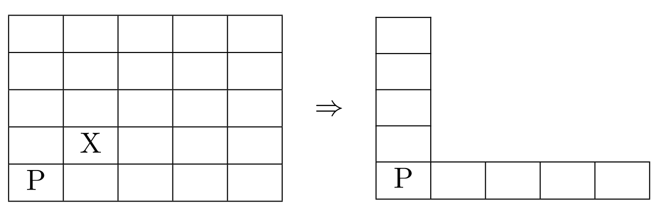
   For example, on the 3×4 board, the first player selects the cell labelled X, which results in the board shown on the right:



   (a) Who wins in the 5×5 game, shown below (where P marks the poison block), and what is the winning strategy?



   Solution:
   If player 1 picks the square marked with X, then we end up with L-shaped picture on the right.



   Notice there are 9 squares, which is an odd number. The winning strategy is for player 1 to do whatever player 2 does on the "opposite" leg of the L, which will cause an even number of squares to be removed. That way, after player 1's move, there will still be an odd number of squares. After player 1's last turn, since there will be one square left, and since one is odd, and player 2 will be forced to eat the square marked with P.

   (b) Who wins in the 2×8 game, shown below (where P marks the poison block), and what is the winning strategy?

P							

Solution:
Player 1 picks the cell marked X and we end up with the picture on the right.

							X
P							

$\Rightarrow$

P							

Then, whatever player 2 does (labelled B_1 or B_2), player 1 does the "diagonal opposite" (labelled A_1 or A_2) to ensure that there are an odd number of squares left.

		A_2			B_1		
P			B_2			A_1	

Since A can force B to start with an odd number of squares, and since 1 is odd, then B will eventually start with 1 square, which is the poisoned square, and will lose

(b) Who wins in the 2×8 game, shown below (where P marks the poison block), and what is the winning strategy?

P							

Solution:
Player 1 picks the cell marked X and we end up with the picture on the right.

							X
P							

$\Rightarrow$

P							

Then, whatever player 2 does (labelled B_1 or B_2), player 1 does the "diagonal opposite" (labelled A_1 or A_2) to ensure that there are an odd number of squares left.

		A_2			B_1		
P			B_2			A_1	

Since A can force B to start with an odd number of squares, and since 1 is odd, then B will eventually start with 1 square, which is the poisoned square, and will lose

(c) Suppose the starting board is larger than a 1×1 rectangle (i.e., it has more than one square). Prove that there is a winning strategy for one of the players. Hint: you don't need to show or construct the winning strategy, but just prove that there exists one. Hint 2: use contradiction.

Solution:
The proof relies on "strategy-stealing".

That is, suppose, by way of contradiction, that player 2 has a winning strategy regardless of what player 1 does.

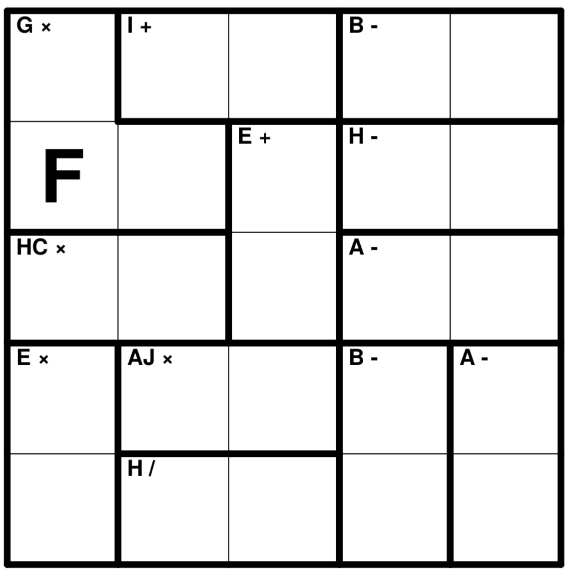
That means that player 1 could make a move removing the top-right single square, and player 2 would make a move, say M, that would ensure winning. Notice that whatever M does, it would have removed this top-right square if player 1 had not removed it.

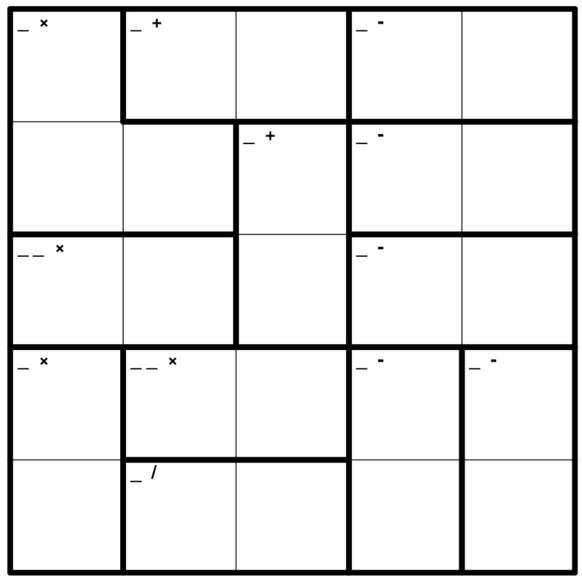
However, player 1 could have made the move M first, and thus player 1 would have the winning strategy.

Thus, player 2 cannot have a winning strategy.

4. Below is an encrypted KenKen puzzle. Each digit has been replaced with a single letter. In the solved puzzle, each row and each column will contain all of the digits from 1 through 5. As in "normal" Kenken, the heavy lines indicate "cages" which contain numbers which can be combined (in any order) to produce the result shown in the cage. For example, $AB/$ means that the numbers in the cage can be arranged, in some order, to divide to give the number AB. Note that numbers within a cage may repeat, so long as the repetition is not in the same row or the same column.

   Fill in the 5×5 grid, as well as the 10 digits in the table below. Additionally, explain how H, A and C were determined. (Hint: start with H, A and C.)





A	B	C	D	E	F	G	H	I	J

Solution:
To derive H, notice that we have $HC\times$, meaning that we get a two digit multiplication by multiplying two integers between 1 and 5. Thus, the only possible values for H are 1 or 2. However, if $H = 1$, then $H/$ would imply there are two identical values in the same cage and row, since the only way to get 1 by division is to divide a number by itself. Thus, $H = 2$. Since the only way to get a two-digit number by multiplication with a leading 2 is to take $5 \times 4 = 20$, and thus $C = 0$.

Notice also we have $AJ\times$, which means that $A = 1$, since it cannot be 2 (since H is).

1	3	4	5	2
4	2	5	3	1
5	4	1	2	3
2	5	3	1	4
3	1	2	4	5

A	B	C	D	E	F	G	H	I	J
1	3	0	9	6	4	8	2	7	5