Exercises for Math Circles 2018

December 5

1. (a) How many k-color cellular automata are there?
   (b) How many k-color totalistic automata are there?
   (c) How many 2D cellular automata are there? (Assume that there are only two colors, black and white.)

   Solution. (a) There are k^3 possible color arrangements for a 3-cell neighborhood. For each of these rules, there are k possible output colors. So there are k^{k^3} possible rules.

   (b) A totalistic rule is defined by the *sum* of the values of the neighbors. Let's say the possible values are $0, 1, 2, \ldots, k$ (each value represents a color, so there are actually $k+1$ colors here if we count 0 as white). Then the possible sums of a 3-cell neighborhood are $0, 1, 2, \ldots, 3k$, which counts as $3k+1$ possibilities. And each of these possible states can give $k+1$ possible outputs. So the number of $(k+1)$-color totalistic rules is $(k+1)^{3k+1}$.

   (c) If there are are only two states, then there are 2^9 different possible 9-cell neighborhoods (a cell plus its eight neighbors). Each cell can output two different states. So in total there are 2^{2^9} possible 2D automata. These are called the **elementary** 2D automata because there are only two states. ∎

2. Let $f : \mathbf{N} \to \mathbf{N}$ be the Collatz function:

$$f(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

   (a) Determine the Collatz sequence starting at 9 — in particular, determine how many iterations are required before the sequence reaches 1.
   (b) For a positive integer n, let $m(n)$ be the least positive integer such that $f^m(n) = 1$. (It is an open question whether or not such $m(n)$ exists!) Find this $m(n)$ for each of $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.

   Solution. (a) The sequence is

$$9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, \ldots$$

   You have to iterate the function 19 times to reach 1.

   (b) Here is the table:

n	1	2	3	4	5	6	7	8	9	10
$m(n)$	0	1	7	2	4	8	16	3	19	6

∎

3. Let $g : \mathbf{R}^2 \to \mathbf{R}^2$ be the function

$$g(a, b) = (a - b, a).$$

   (a) Describe the orbit of the point $p = (1, 1)$ under this function.
   (b) Describe the set $\{n : g^n(p) = (1, 0)\}$.

Solution. (a) You should get the following orbit:

$$(1,1), \quad (0,1), \quad (-1,0), \quad (-1,-1), \quad (0,-1), \quad (1,0), \quad (1,1), \quad \ldots$$

After six iterations, we got back to the point $(1,1)$ — this just means $g^6(p) = p$. So the orbit is periodic.

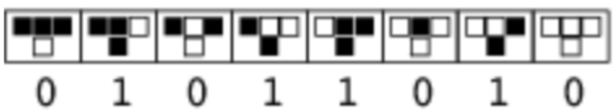
(b) From (a), we can see that $g^5(p) = (1,0)$. Since the orbit is periodic, the point $(1,0)$ will occur along this orbit every 6 iterations. So

$$\{n : g^n(p) = (1,0)\} = \{5, 11, 17, 23, 29, 35, 41, \ldots\} = \{5 + 6k : k \in \mathbf{N}\}.$$ ∎

4. **Rule 90**: In this question we will write Rule 90 as a formal cellular automaton.

   (a) Write the transition rules for Rule 90.
   (b) Find the memory rule μ for Rule 90.
   (c) Let $\tau : \{0,1\}^{\mathbf{Z}} \to \{0,1\}^{\mathbf{Z}}$ be Rule 90. Define τ in terms of μ. (Specifically, this means find an expression for $\tau(x)_n$ in terms of μ and x, where x is some state.)

Solution. (a) We've done this before actually: 90 in binary is 01011010, so its rule set is



(b) Rule 90 is the XOR automaton, so its memory rule is

$$\mu(a,b,c) := a \texttt{ xor } c.$$

Another way to write XOR is $(x \texttt{ xor } y) = x + y \pmod 2$.

(c) This is actually the same for any automaton:

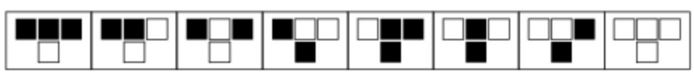
$$\tau(x)_n = \mu(x_{n-1}, x_n, x_{n+1}).$$

∎

5. **Rule 30**: In this question we will write Rule 30 as a formal cellular automaton.

   (a) Write the transition rules for Rule 30.
   (b) Find the memory rule μ for Rule 30.
   (c) Let $\tau : \{0,1\}^{\mathbf{Z}} \to \{0,1\}^{\mathbf{Z}}$ be Rule 30. Define τ in terms of μ. (Specifically, this means find an expression for $\tau(x)_n$ in terms of μ and x, where x is some state.)

Solution. (a) We've done this before actually: 30 in binary is 00011110, so its rule set is



(b) It's tricky to find it, but you might arrive at the following answer:

$$\mu(a,b,c) = a \texttt{ xor } (b \texttt{ or } c).$$

(c) This is the same as Problem 4! This is how every 1D cellular automaton is defined.

$$\tau(x)_n = \mu(x_{n-1}, x_n, x_{n+1}).$$ ∎

6. Let $\tau : \{0,1\}^{\mathbf{Z}} \to \{0,1\}^{\mathbf{Z}}$ be Rule 126, and let $x : \mathbf{Z} \to \{0,1\}$ be the state given as follows:

$$x_n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\tau(x)_{-2}$.

(b) Find $\tau(x)_{-1}$.

(c) Find $\tau(x)_0$.

(d) Find $\tau(x)_1$.

(e) Find $\tau(x)_2$.

Solution. The memory rule for Rule 126 is

$$\mu(a,b,c) = \begin{cases} 0 & \text{if } a+b+c \in \{0,3\} \\ 1 & \text{otherwise.} \end{cases}$$

Now we can use the equation $\tau(x)_n = \mu(x_{n-1}, x_n, x_{n+1})$.

(a) $\tau(x)_{-2} = \mu(x_{-3}, x_{-2}, x_{-1}) = \mu(0,0,0) = 0$.

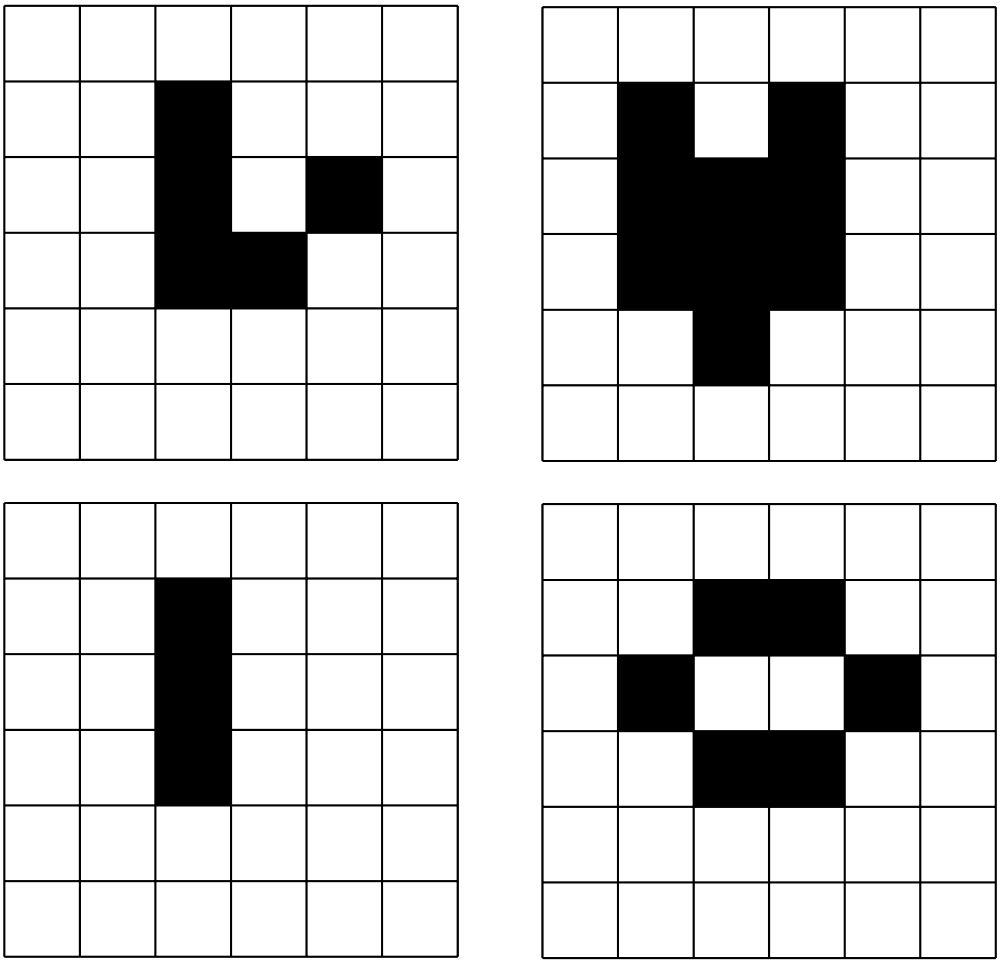
(b) $\tau(x)_{-1} = \mu(x_{-2}, x_{-1}, x_0) = \mu(0,0,1) = 1$.

(c) $\tau(x)_0 = \mu(x_{-1}, x_0, X_1) = \mu(0,1,0) = 1$.

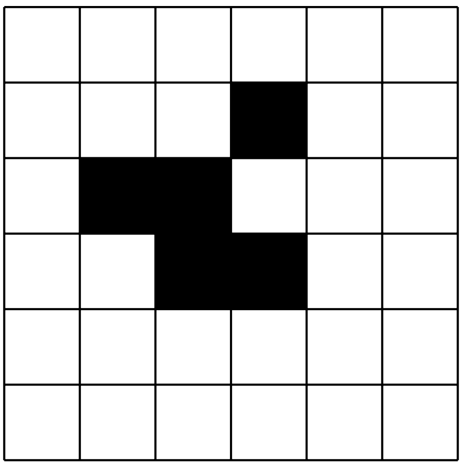
(d) $\tau(x)_1 = \mu(x_0, x_1, x_2) = \mu(1,0,0) = 1$.

(e) $\tau(x)_2 = \mu(x_1, x_2, x_3) = \mu(0,0,0) = 0$. ∎

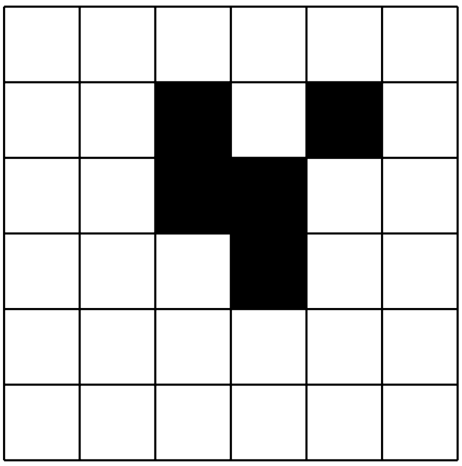
7. **Conway's Game Of Life.** For each of the following states, use the rules of Conway's Game Of Life to find (I) the next state, and (II) a predecessor state.



Solution. For the top-left state: here is the next state (left) and a predecessor (right).

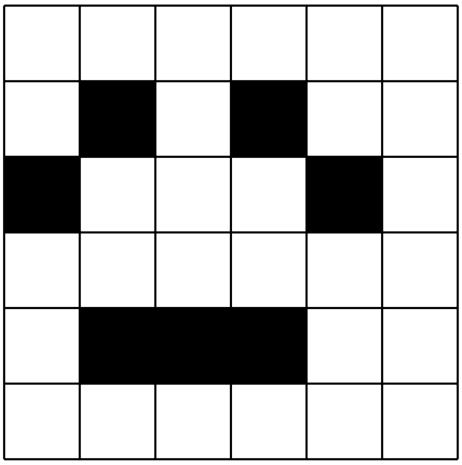


The next state



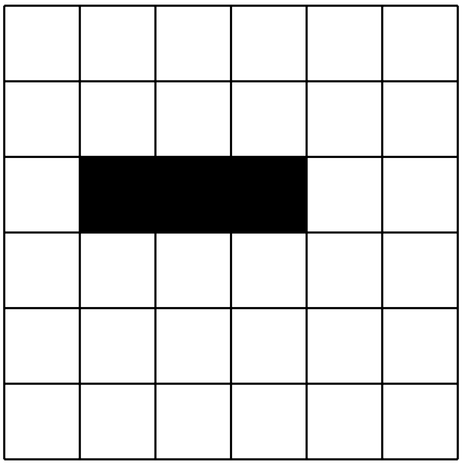
A predecessor

For the top-right state, here is the next state. I actually couldn't find a predecessor — if you find one, let me know!

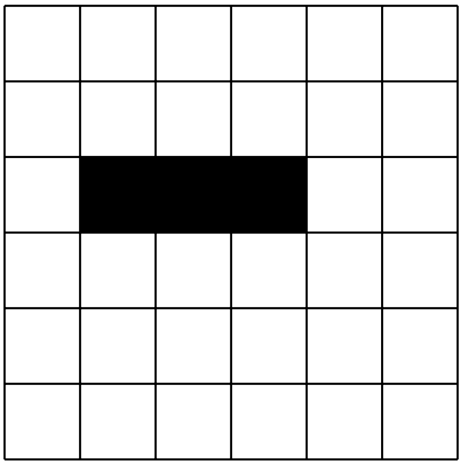


The next state

For the bottom-left state, here is the next state (left) and a predecessor (right). Note that they are the same!

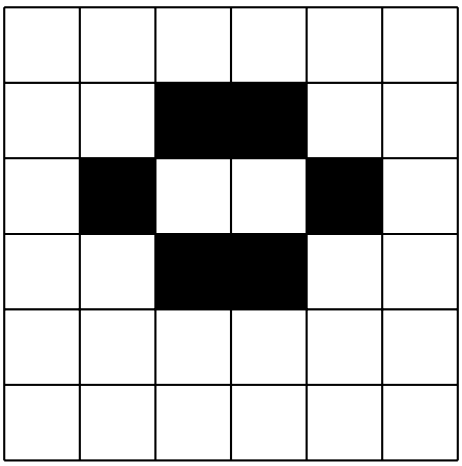


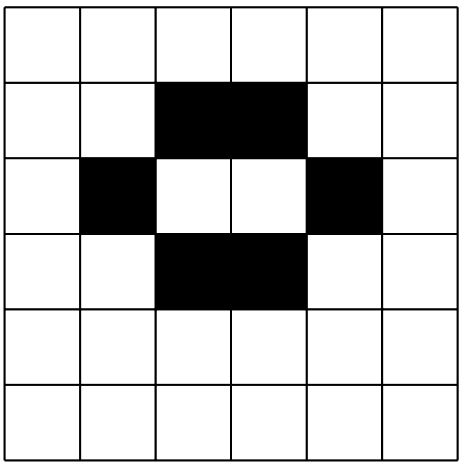
The next state



A predecessor

Finally, for the bottom-right state, here is the next state (left) and a predecessor (right). Note that they are the same as the state itself — it's a state that never changes!





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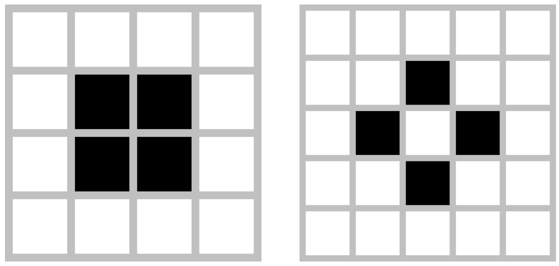
8. **Conway's Game Of Life.** Find a state — other than any of the ones in Problem 7 — that never changes.

Solution. Here are two:



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