Intermediate Math Circles
Wednesday April 3, 2019
Introduction to Vectors III

Review

A **vector** must have a magnitude and a direction.

If $\vec{u} = [a, b]$ and $\vec{v} = [c, d]$, then we have the following properties:

1. Vector Addition: $\vec{u} + \vec{v} = [a + b, c + d]$

2. Scalar Multiplication: $t\vec{v} = [ta, tb]$

3. Length of a Vector: $|\vec{v}| = \sqrt{(c)^2 + (d)^2}$

Unit Vector: $\hat{v} = \dfrac{1}{|\vec{v}|}\vec{v}$

Dot Product

If $\vec{u} = [a, b]$ and $\vec{v} = [c, d]$, then the **dot product** is a scalar value defined as:

Example: If $\vec{u} = [3, 4]$ and $\vec{v} = [-2, 1]$, find the dot product of $\vec{u}$ and $\vec{v}$.

Example: Consider $\vec{u} = [3, 4]$.

1. What is the length $\vec{u}$?

2. What is $\vec{u} \cdot \vec{u}$?

3. How does this relate to the length of $\vec{u}$?

DEFINITION: The length of $\vec{u} = [a, b]$ is

FACT: We often square this result to get

TRY THIS: Find the following dot products. What is special about the two vectors in each case?

1. $\vec{a} = [1, 1] \quad \vec{b} = [-1, 1]$
2. $\vec{c} = [1, 0] \quad \vec{d} = [0, 1]$
3. $\vec{e} = [6, 8] \quad \vec{f} = [-2, \frac{3}{2}]$
4. $\vec{g} = [2, 5] \quad \vec{h} = [0, 0]$

1.

2.

3.

4.

In each pair, the vectors are perpendicular to one another.

DEFINITION: Two vectors, $\vec{u}$ and $\vec{v}$, are perpendicular (orthogonal)
if

FACT: The zero vector ($\vec{0}$) is perpendicular to every other vector.

A MORE FORMAL DEFINITION: If $\vec{u}$, $\vec{v}$ and θ is the angle between the two vectors, then

But this has trig reference so we will not use it.

DEFINITION: If $\vec{u} = [a, b]$ and $\vec{v} = [c, d]$, then

____

is the ______________ from $\vec{u}$ to $\vec{v}$.

EXAMPLE: Find the distance between $\vec{u} = [3, 1]$ and $\vec{v} = [4, 1]$.

Prove that $|t\vec{u}| = |t|\ |\vec{u}|$ for any vector $\vec{u} = [a, b]$ and $t \in \mathbb{R}$.

DEFINITION: The **cross product** of $\vec{u} = [a, b]$ and $\vec{v} = [c, d]$ is

NOTICE the order!!!!!!!!!!

Note that in 3D, the cross product of two vectors returns a vector rather than a scalar value.

Given $\vec{p} = [3, 2]$ and $\vec{p} = [2, 5]$. Find:

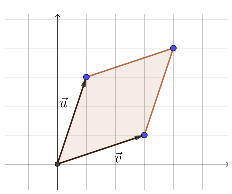
a) $\vec{p} \times \vec{q}$

b) $\vec{q} \times \vec{p}$

What do you notice?

CROSS PRODUCTS AND GEOMETRY

As defined, the cross product can be used to find the area of the parallelogram created by letting two sides of the parallelogram be $\vec{u}$ and $\vec{v}$ being tail to tail.



The area of the parallelogram $= |\vec{u} \times \vec{v}|$

EXAMPLE: Let $\vec{u} = [0, 6]$ and $\vec{v} = [8, 0]$. Find the area of the triangle created by the $\vec{u}$, and $\vec{v}$ being tail to tail.

Let $\vec{u} = [-1, -1]$ and $\vec{v} = [3, 4]$. Help discover the following properties of the cross product for yourself.

1. Find $\vec{u} \times \vec{v}$. Find $\vec{v} \times \vec{u}$. What do you notice?

2. Find $\vec{u} \times \vec{u}$. Find $\vec{v} \times \vec{v}$. What do you notice?

3. Find $(2\vec{u}) \times \vec{u}$. Find $(-5\vec{v}) \times \vec{v}$. What do you notice?

4. Find $(2\vec{u}) \times \vec{v}$. Find $2(\vec{u} \times \vec{v})$. What do you notice?

Example Let $\vec{u} = [-1, -1]$ and $\vec{v} = [3, 4]$, find $|\vec{u} \times \vec{v}|$ using the formula $|\vec{u} \times \vec{v}| = |\vec{u}|\ |\vec{v}| \sqrt{1 - (\hat{u} \cdot \hat{v})^2}$

Extension into 3D

Let $\vec{u} = [a, b, c]$ and $\vec{v} = [d, e, f]$.
ex. $\vec{p} = [2, 3, 4]$ and $\vec{q} = [5, 7, 9]$.

$\vec{u} + \vec{v} = [a + d, b + e, c + f]$
ex. $\vec{u} + \vec{v} = [2 + 5, 3 + 7, 4 + 9] =$

$t\vec{u} = [ta, tb, tc]$
ex. $3\vec{p} =$

$|\vec{u}| = \sqrt{a^2 + b^2 + c^2}$

ex. $|\vec{p}| =$

$\vec{u} \cdot \vec{v} = ad + be + cf$
ex. $\vec{p} \cdot \vec{q} =$

$\vec{u} \times \vec{v} = [bf - ce, cd - af, ae - bd]$
ex. $\vec{p} \times \vec{q} =$

(Notice that the cross product gives a vector but the magnitude of this vector is still the area of a parallelogram.)

Given $\vec{p} = [1, 4, 3]$ and $\vec{q} = [3, -2, 4]$ Find the following
a) $\vec{p} \cdot \vec{q}$ b) $|\vec{p} \times \vec{q}|$