Intermediate Math Circles February 13: Solutions to Contest Preparation Problems (Geometry)

February 15, 2019

Apart from questions 8,10,11,12,14, and 15, the solutions can be found on the CEMC website.

1. #6, 2002 Pascal Contest

2. #17, 2003 Cayley Contest

3. #15, 2007 Cayley Contest

4. #18, 1998 Pascal Contest

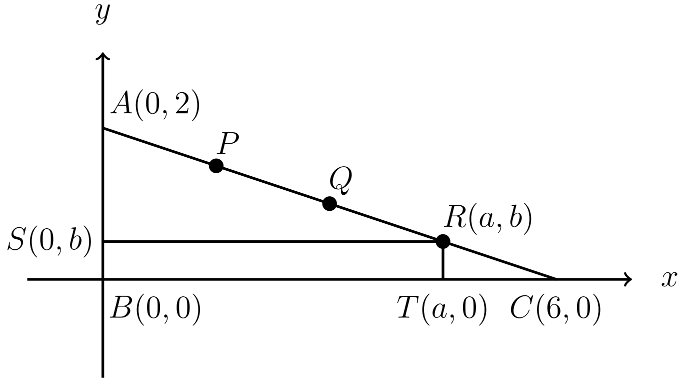
5. #15, 2001 Pascal Contest

6. #21, 1998 Pascal Contest

7. #19, 2001 Pascal Contest

8. #20, 1995 Cayley Contest

   Label the point $R = (a, b)$, draw points $S = (0, b)$ and $T = (a, 0)$, and connect S and T to R:



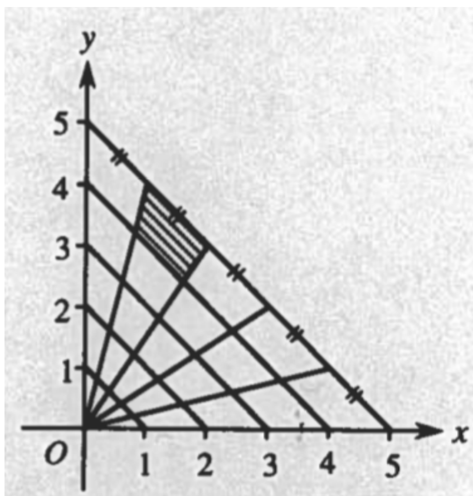
   Since $SR \parallel BC$ and $RT \parallel AB$, we have that $\angle ASR = \angle RTC = \angle ABC = 90°$. Also, $\angle TRC = \angle BAC$. We also have that $\angle BAC = \angle SAR$, so $\triangle ABC$, $\triangle ASR$, and $\triangle RTC$ are all similar because they each have two (and hence, three) common angles. We know that

RC is one quarter of the length of AC, or $\frac{RC}{AC} = \frac{1}{4}$. Since $\triangle ABC \sim \triangle RTC$, this means $\frac{TC}{BC} = \frac{RC}{AC} = \frac{1}{4}$. Rearranging, this gives $TC = \frac{1}{4}BC$, but $BC = 6$, so $TC = \frac{6}{4} = \frac{3}{2}$. This means the point T has coordinates $\left(6 - \frac{3}{2}, 0\right) = \left(\frac{9}{2}, 0\right)$, so $a = \frac{9}{2}$. Similarly, since $\triangle ABC \sim \triangle ASR$, we have $\frac{AR}{AC} = \frac{AS}{AB}$, but $\frac{AR}{AC} = \frac{3}{4}$, so $AS = \frac{3}{4}AB = \frac{3}{4}(2) = \frac{3}{2}$. This means the coordinates of S are $\left(0, 2 - \frac{3}{2}\right) = \left(0, \frac{1}{2}\right)$, so $b = \frac{1}{2}$. We now have that R is the point $\left(\frac{9}{2}, \frac{1}{2}\right)$, so the slope of the line BR is

$$\frac{\frac{9}{2} - 0}{\frac{1}{2} - 0} = \frac{\frac{9}{2}}{\frac{1}{2}} = \frac{1}{9}$$

9. #24, 2002 Cayley Contest

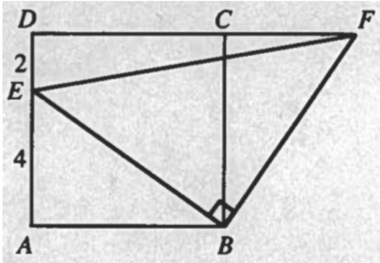
10. #22, 1990 Cayley Contest



By the Pythagorean Theorem, the length of the line segment connecting $(0,5)$ to $(5,0)$ is $\sqrt{5^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$. Thus, the longer base of the shaded trapezoid is one fifth of this length, or $\frac{5\sqrt{2}}{5} = \sqrt{2}$. An argument involving similar triangles will show that the length of the shorter base of the trapzoid is one fifth of the distance between $(0,4)$ and $(4,0)$. The distance between these two points is $\sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$, so the length of the shorter base is $\frac{4\sqrt{2}}{5} = \frac{4}{5}\sqrt{2}$. The height of the trapezoid is the length of a perpendicular from $(0,4)$ to the line connecting $(0,5)$ to $(5,0)$. The small triangle created by doing this is a 45°-45°-90° triangle, with a hypotenuse of length 1, which means the height will be $\frac{1}{\sqrt{2}}$ [This follows from a fact in the slides.] Recall that the area of a trapezoid with base lengths b_1 and b_2 and height h is $\frac{1}{2}h(b_1 + b_2)$. Therefore, the area of the trapezoid is

$$\frac{1}{2}\frac{1}{\sqrt{2}}\left(\sqrt{2} + \frac{4}{5}\sqrt{2}\right) = \frac{1}{2}\left(1 + \frac{4}{5}\right) = \frac{1}{2} \times \frac{9}{5} = \frac{9}{10}$$

11. #24, 1990 Cayley Contest.



Note that

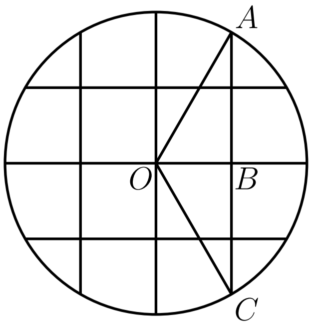
$$\angle ABE + \angle EBF = \angle ABF = \angle ABC + \angle CBF$$

We also have that $ABCD$ is a square and are given that $\angle EBF = 90°$, which means $\angle EBF = \angle ABC$, so the above equation implies $\angle ABE = \angle CBF$. Also, $\angle DAB = 90°$, and DCF is a straight line, so $\angle BCF = 180° - \angle DCB = 180° - 90° = 90°$. Finally, we have $AB = BC$ since they are sides of the same square, so we can conclude that $\triangle AEB \cong \triangle BCF$ by angle-side-angle congruence. Since $AB = AD = AE + ED = 4 + 2 = 6$, we have, by the Pythagorean Theorem that $EB^2 = AE^2 + AB^2$, or $EB = \sqrt{4^2 + 6^2} = \sqrt{16 + 36} = \sqrt{52}$. Since $\triangle AEB \cong \triangle BCF$, this means $BF = \sqrt{52}$ as well. Triangle EBF is right, so its area is

$$\frac{1}{2} \times BF \times EB = \frac{1}{2}\sqrt{52}\sqrt{52} = \frac{1}{2}(52) = 26$$

12. #19, 1995 Cayley Contest

The length of each diameter is $2 \times 2 = 4$, and the other chords are all equal to each other by symmetry, so the total will be $4 + 4 + 4x$ where x is the length of any of the shorter chords. We partially label the diagram as follows:



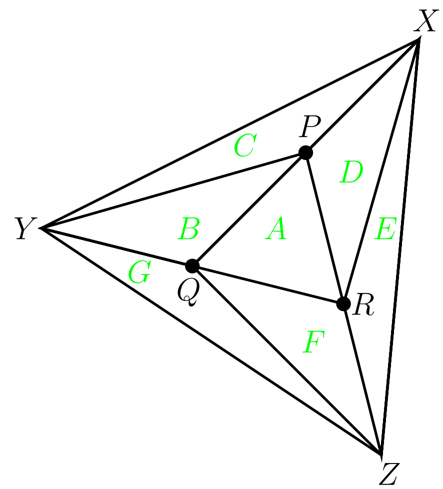
The diameters are perpendicular, and AC is parallel to the vertical diameter, which means $AC \perp OB$. We also have OB equal to itself and $OA = OC$ because they are radii of the same circle, which means $\triangle OBA \cong \triangle OBC$ because they are right triangles with an equal leg and hypotenuse. Therefore, $AB = AC$, so the value of x is $2AB$. We are given that $OA = 2$ and $OB = 1$, so by the Pythagorean Theorem, $OA^2 = OB^2 + AB^2$ or $AB = \sqrt{2^2 - 1^2} = \sqrt{3}$. Thus, the total length of the all of the chords is

$$4 + 4 + 4 \times 2\sqrt{3} = 8(1 + \sqrt{3})$$

13. #25, 2000 Pascal Contest

14. #21, 1995 Cayley Contest

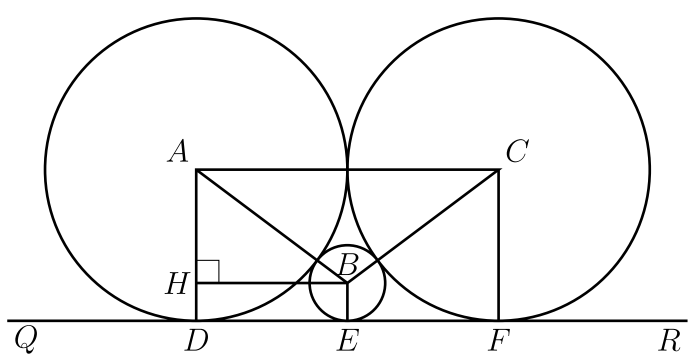
Connect P to Y, Q to Z, and R to X and let A, B, C, D, E, F and G represent the areas of various triangles as shown:



$\triangle PQR$ and $\triangle PXR$ each have height which is the distance from the point R to the line QX. Also, since $QP = PX$, these triangles have equal bases. Thus, $\triangle PQR$ and $\triangle PXR$ have the same area, so $A = D$. Similarly, triangles $\triangle PRX$ and $\triangle ZRX$ have the same height, and since $PR = RZ$, we also have that $D = E$. So far, we have $A = D = E$. Similar reasoning shows that $A = B = C = D = E = F = G$. The area of $\triangle XYZ$ is equal to $A + B + C + D + E + F + G$, which, by the previous fact, equals $7A$. Therefore, we have $7A = 420$, so $A = 60$ which means the area of $\triangle PQR$ is 60.

15. #23, 1990 Cayley Contest

Label the centres of the circles, from left to right by A, B, and C. Let D, E, and F be on QR so that AD, BE, and CF are each perpendicular to QR, as shown. As well let H be on AD so that $BH \parallel QR$



Let r_1 denote the radius of the large circles, and r_2 denote the radius of the small circle. Line segment AD is parallel to BE, so HD is parallel to BE. HB was constructed so that $HB \parallel DE$, so $HDEB$ is a parallelogram, which means $HD = BE$. By properties of circles, we have $AD = r_1$ and $BE = r_2$. Putting this together with $HD = BE$, we have $AH = r_1 - r_2$. By another property of circles, $AB = r_1 + r_2$. By the Pythagorean Theorem, we then have $AB^2 = AH^2 + HB^2$, so

$$(r_1 + r_2)^2 = (r_1 - r_2)^2 + HB^2$$

or

$$r_2^2 + 2r_1r_2 + r_2^2 = r_1^2 - 2r_1r_2 + r_2^2 + HB^2$$

Simplifying gives $4r_1r_2 = HB^2$. Again using that $HDEB$ is a parallelogram, we have that $HB = DE$. By the symmetry in the diagram, $DE = EF$, so $DF = 2DE = 2HB$. Also, $\angle ADF = \angle DFC = 90°$ and $AD = CF$, so $ADFC$ is a rectangle which means $AC = DF = 2HB$. By properties of circles, $AC = 2r_1$. Combining all of this, we have that

$$4r_1r_2 = HB^2 = \left(\frac{AC}{2}\right)^2 = \left(\frac{2r_1}{2}\right)^2 = r_1^2$$

Dividing both sides by r_1r_2 gives $4 = \frac{r_1}{r_2}$. Therefore, the ratio of the area of the smaller circle to one of the larger circles is $1 : 4$.