Math Circles: Intermediate Contest Preparation (Counting)

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A very common type of contest problem asks you to count the number of ways something can happen, or asks for a probability that a randomly chosen object has some property. These two questions are closely related.

Here are 4 questions I will discuss tonight.

1. How many arrangements (words) may be made using the letters a,b,b,c,d?
2. (Pascal 2016, 16) An integer from 10 to 99 inclusive is randomly chosen so that each such integer is equally likely to be chosen. What is the probability that at least one digit of the chosen integer is a 6?
3. (Cayley 1997, 21) Three balls numbered 1, 2, and 3 are placed in a bag. A ball is drawn from the bag and the number is recorded. The ball is then returned to the bag. After this has been done three times, what is the probability that the sum of the three recorded numbers is less than 8?
4. (Simpler Version of Cayley 2018, 24) In how many ways may 4 red balls and 9 green balls be placed in a line so that, between any two red balls there are at least 2 green balls?

How many arrangements (words) may be made using the letters a,b,b,c,d?

- For a positive integer n, we define $n!$, read as "n factorial", to be the number $n \times (n-1) \times \cdots \times 3 \times 2 \times 1$.
- For example, $4! = 4 \times 3 \times 2 \times 1 = 24$. (you don't have to write the " times 1" at the end).
- We also make the convention that $0! = 1$. This is a convention, sort of like the convention that $5^0 = 1$.

$n!$ is the number of ways that n distinct objects may be arranged from left to right. Why?

- Suppose we have n distinct objects that we want to arrange from left to right.
- There are n choices for the leftmost item.
- There are $n - 1$ choices for the second-leftmost item: every item except the first one chose.
- This continues, all the way down to having 2 choices for the second-to-last item (the 2 objects not yet chosen), and then 1 choice for the final item (every other item has already been chosen

Each choice made was independent of the other choices, so the total number of arrangements is $n \times (n-1) \times \cdots \times 3 \times 2 \times 1$, which is exactly $n!$. Note that since all the objects are different, two different sets of choices cannot give the same arrangements.

Time for the first question: How many arrangements (words) may be made using the letters a,b,b,c,d? **Solution**:

- Note the answer isn't just 5! since the objects are not all different.
- Let's replace one of the b's with B and instead count the arragements of a, b, B, c, d. There are $5! = 120$ arrangement of these 5 distinct objects.
- Each arrangement of a,b,b,c,d gives rise to two arrangements of a,b,B,c,d: one from replacing the first b with B and one from replacing the second b with B. For example, bcabd gives rise to both Bcabd and bcaBd.
- Two distinct arrangements of a,b,b,c,d cannot give the same arrangement of a,b,B,c,d.
- Therefore the number of arrangements of a,b,b,c,d is equal to $\frac{5!}{2} = 60$.

Second Question: (Pascal 2016, 16) An integer from 10 to 99 inclusive is randomly chosen so that each such integer is equally likely to be chosen. What is the probability that at least one digit of the chosen integer is a 6?

- Connection between counting and probability: Consider instead we were asked the question "How many integers from 10 to 99 inclusive have the property that at least one digit is 6?". This is pretty much just as hard as the given problem.
- Let A be the answer to this new question.

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- Let A be the answer to this new question.
- Let B be the total number of two-digit numbers.
- Then the answer to the probability question is $\frac{A}{B}$ (or $(100 \times \frac{A}{B})\%$)
- The probability question requires knowing B while the counting problem doesn't, but B isn't very hard to determine. Once you know A, you are done with the new question, and almost done with the original question.

What about "at least"? Say you want to answer "How many ways does at least one of X or Y occur?" there are two good approaches:

- (# ways that at least one of X or Y occurs) = (# ways that X occurs) + (# ways that Y occurs) - (# ways that X and Y both occur)
- (# ways that at least one of X or Y occurs) = (total number of outcomes) - (# ways that X and Y both do not occur)

We will answer the question in two different ways, one for each approach.

Solution 1:

- To have a 6 appear in a two-digit integer, it must occur in at least one of the tens digit spot and the ones digit spot.
- There are 10 integers with 6 in the leftmost spot: 60,61,62, ..., 68,69.
- There are 9 integers with 6 in the rightmost spot: 16,26, ..., 86,96.
- There is 1 integer with 6 in both the leftmost and rightmost spot: 66.
- Therefore the number of integers with a 6 in at least one spot is $10+9-1=18$.
- There are 90 integers between 10 and 99, inclusive.
- Therefore the probability that a randomly chosen two-digit integer has at least one 6 as a digit is equal to $\frac{18}{90}$ (which simplifies to $\frac{1}{5}$ if you like).

Solution 2: Here we instead count the number of two-digit integers where a 6 doesn't occur in either spot.

- There are 8 choices for the first digit: 1,2,3,4,5,7,8,9 (can't use 0 since we want two-digit integer)
- There are 9 choices for the second digit: 0,1,2,3,4,5,7,8,9
- Therefore the number of two-digit integers without a 6 as a digit is $8 \times 9 = 72$.
- There are 90 integers between 10 and 99, inclusive. Therefore the number of two-digit integers with at least one 6 as a digit is equal to $90\text{-}72 = 18$.
- Therefore the probability that a randomly chosen two-digit integer has at least one 6 as a digit is equal to $\frac{18}{90}$ (which simplifies to $\frac{1}{5}$ if you like).

Third Problem: (Cayley 1997, 21) Three balls numbered 1, 2, and 3 are placed in a bag. A ball is drawn from the bag and the number is recorded. The ball is then returned to the bag. After this has been done three times, what is the probability that the sum of the three recorded numbers is less than 8?

Solution:

- Let the desired probability be p. We will instead calculate the probability that the sum of the three recorded numbers is not less than 8; call this probability q. Note that $p+q=1$ since the sum is always less than 8 or not less than 8, and both events cannot occur. The tootle number of outcomes is $3^3 = 27$.
- Let S be the sum of 3 numbers from the bag. We have that $3 \leq S \leq 9$.
- Therefore q equals the sums of the probability that $S=8$ and the probability that $S=9$ (no overlap here).
- if $S=8$ then the numbers must be 3,3,2 in some order. There are 3 ways this can happen.
- if $S=9$ then the numbers must be 3,3,3. There is only 1 way this can happen.
- There there for 3+1=4 ways for which $S >= 8$. Therefore $q = \frac{4}{27}$ and so $p = 1 - \frac{4}{27} = \frac{23}{27}$

(Simpler Version of Cayley 2018, 24) In how many ways may 4 red balls and 9 green balls be placed in a line so that, between any two red balls there are at least 2 green balls?

- We first say a word about binomial coefficients. Given a positive integer n and another integer b for which $0 \leq b \leq n$, we define $\binom{n}{b}$ to be the number of ways an unordered group of b objects may be chosen from a group of n (distinct) objects. People often say "n choose b" to refer to $\binom{n}{b}$
- Cool fact: $\binom{n}{b} = \frac{n!}{b!(n-b)!}$.

Why does $\binom{n}{b} = \frac{n!}{b!(n-b)!}$?

- lets explain why when $n = 7$ and $b = 3$. Consider the question of lining up our 7 objects from left to right where we created a barrier between the 3rd and 4th spot.
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- There are 7! ways to arrange the items from left to right. Let's say the 3 items on the left is our group of 3.
- We now take into account the fact that we don't care about order on either side of the barrier. There are 3! ways to arrange the members on the left and 4! ways to arrange the members on the right. These arrangements are independent. Therefore the total number of groups of 3 is $\frac{7!}{3!4!}$
- Notice that $\binom{n}{b} = \binom{n}{n-b}$, $\binom{n}{0} = 1$, and $\binom{n}{1} = n$.

Solution:

- Note that 6 of the G's are already spoken for: 2 between each pair of R's: RGGRGGRGGR
- We need to place the 3 remaining G's into 5 possible spots.
- We divide into cases based on how many of these 3 G's stay together.
- All 3 G's are together: 5 ways (5 empty spots for the group of 3 G's)
- 2 G's together, 1 G alone: $5(4)=20$ ways (5 empty spots for the group of 2 G's, then 4 empty spots for the single G, and the order matters)
- All 3 G's alone: $\binom{5}{3} = 10$ ways (We are choosing 3 of 5 empty spots and we don't care about the order of the 3 spots chosen.)

In total, there are 35 ways to do this.

Note that in the previous two problems, we divided a counting problem into several more manageable cases - this was especially useful in the previous question for counting where the remaining three G's went. Figuring out a good way to break up a counting problem into manageable cases is often the hardest part and requires lots of practice!

For any questions related to mathematics please feel free to contact me at rgarbary@uwaterloo.ca