Note: Most of these problems are directly taken from contests; for these problems a hyperlink to the official contest solutions is provided.

Extra Problems

1. The two digit number 36 has the property that when the digits are switched, the resulting number (in this case, 63) is larger than the original number. How many two-digit numbers have this property? (Ans: 36)

   Solution 1: We break into cases, fixing the first digit.

   When the first digit is 1, we have the numbers 12, 13, ..., 19 ; there are 8 such numbers.

   When the first digit is 2, we have 23, 24, ..., 29; there are 7 such numbers.

   This pattern continues, all the way down to 1 number for first digit 8 (89).

   The total number of numbers is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$.

   Solution 2: Here is an alternate solution, which doesn't actually explicitly list the desired numbers. The key observation here: a randomly selected two-digit number has "almost" a 50% chance to get larger when the digits are swapped. Why "almost"? Because the numbers with repeated digits and the numbers that end in a 0 break the symmetry.

   There are 90 two-digit numbers. Of these 90, 9 of them end in a 0 and 9 of them have repeated digits. (There is no overlap between these two groups of nine numbers.) Therefore the number is two-digit numbers where both digits are different and that don't end in a 0 is 90-9-9=72. Of these 72 numbers, exactly half of them get bigger when you swap the digits. Therefore our final answer is $\frac{72}{2} = 36$.

2. (Gauss G8 2006, 19) Bethany, Chun, Dominic, and Emily go to the movies. They choose a row with four consecutive empty seats. If Chun and Emily must sit next to each other, in how many different ways can the four friends sit? (Ans: 12)

   Solution:

   https://www.cemc.uwaterloo.ca/contests/past_contests/2006/2006GaussSolution.pdf (Note that this link includes solutions for both the G7 and G8 papers. You want to look at the G8 solutions, which come after the G7 solutions.)

3. (Pascal 2015, 20) Andre has an unlimited supply of $1 coins, $2 coins, and $5 bills. Using only these coins and bills, and not necessarily using some of each kind, in how many different ways can he form exactly $10? (Ans: 10)

   Solution:

   https://www.cemc.uwaterloo.ca/contests/past_contests/2015/2015PascalSolution.pdf

4. How many different ways may the letters b,b,c,c,c,c be ordered from left to right? (Ans: 15)

   Solution 1: There are 6! ways to arrange the 6 letters, pretending they are all distinct. How do we take into account the fact that there are two b's and four c's? There are 2! ways to move around the two b's and 4! ways to move around the four c's. Therefore the total number of arrangements is $\frac{6!}{2!4!} = 15$.

   Solution 2: Consider 6 blank spaces _ _ _ _ _ _. We are asking in how many ways can b's be put in two of the blanks, and c's in the remaining 4 blanks. Since it doesn't matter what order we put the b's into two blanks, there are $\binom{6}{2} = 15$ to put in the two b's. Once we put in the two b's, there are no choices to make about where the c's go. So the total number of choices is 15.

5. How many different ways may the letters a,a,b,b,c,c be ordered from left to right? (Ans: 90)

   Solution 1: Similar to the previous solution #1, the answer is $\frac{6!}{2!2!2!} = 90$.

   Solution 2: Similar to the previous solution #2, there are $\binom{6}{2}$ places to put down two a's. Once two a's are put down, there are $\binom{4}{2}$ places to put down two b's. Once the b's are down, there are no choices for where the c's go.

   So the total number of choices is $\binom{6}{2} \times \binom{4}{2} = 15 \times 6 = 90$.

6. (Cayley 2010, 19) How many 3-digit positive integers have exactly one even digit? (Ans: 350)

   Solution:

   https://www.cemc.uwaterloo.ca/contests/past_contests/2010/2010CayleySolution.pdf

7. (Cayley 2009, 19) How many integers n are there with the property that the product of the digits of n is 0, where $5000 \leq n \leq 6000$? (Ans: 272)

   Solution:

   https://www.cemc.uwaterloo.ca/contests/past_contests/2009/2009CayleySolution.pdf

8. (Euclid 2018, 7b) Eight people, including triplets Barry, Carrie and Mary, are going for a trip in four canoes. Each canoe seats two people. The eight people are to be randomly assigned to the four canoes in pairs. What is the probably that no two of Barry, Carrie and Mary will be in the same canoe? (Ans: $\frac{4}{7}$)

   Solution:

   https://www.cemc.uwaterloo.ca/contests/past_contests/2018/2018EuclidSolution.pdf

Extra Problems (Challenging)

1. (CIMC 2017, A6) In the diagram below, six squares form a 2×3 grid. The middle square in the top row is marked with an R. Each of the five remaining squares is to be marked with an R, S or T. In how many ways can the grid be completed so that it includes at least one pair of squares side-by-side in the same row or same column that contain the same letter? (Ans: 225)

	R	

   Solution:

   https://www.cemc.uwaterloo.ca/contests/past_contests/2017/2017CIMCSolution.pdf

2. (Cayley 2003, 24) In how many ways can a, b, c, and d be chosen from the set $\{0, 1, 2, ..., 8, 9\}$ so that $a < b < c < d$ and $a + b + c + d$ is a multiple of 3? (Ans: 72)

   Solution:

   https://www.cemc.uwaterloo.ca/contests/past_contests/2003/2003CayleySolution.pdf

3. (Pascal 2016, 25) A 0 or 1 is to be placed in each of the nine 1×1 squares in a 3×3 grid so that each row contains at least one 0 and at least one 1, and each column contains at least one 0 and at least one 1. How many ways can this be done? (Ans: 102)

   Solution:

   https://www.cemc.uwaterloo.ca/contests/past_contests/2016/2016PascalSolution.pdf

4. (CSMC 2012, A6) Lynne is tiling her long and narrow rectangular front hall. The hall is exactly 2 tiles wide and 13 tiles long. She is going to use exactly 11 black tiles and exactly 15 white tiles. Determine the number of distinct ways of tiling the hall so that no tow black tiles are adjacent (that is, share an edge). (Ans: 486)

   Solution:

   https://www.cemc.uwaterloo.ca/contests/past_contests/2012/2012CSMCSolution.pdf

5. (Euclid 2016, 9a) $BBABBAABBA$ is a 10 letter string(word) of A's and B's that contains $ABBA$ as a substring (subword), while $AAABBBBBAA$ is a 10 letter string of A's and B's that does not contain the substring $ABBA$. How many 10 letter strings of A's and B's are there which do not contain "$ABBA$" as a substring? (Ans: 631)

   Solution:

   https://www.cemc.uwaterloo.ca/contests/past_contests/2016/2016EuclidSolution.pdf