UNIVERSITY OF
WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

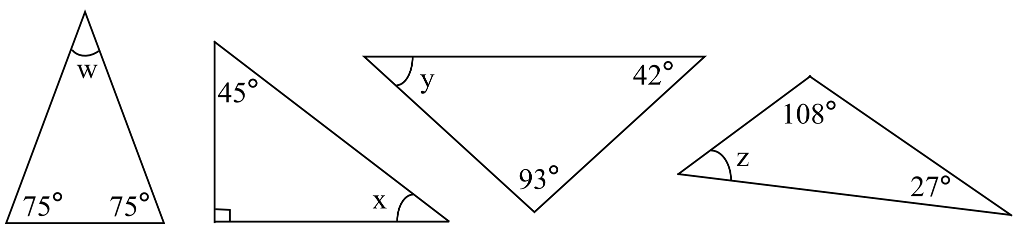
February 19th/20th

Tessellations

Warm-Up

What is the sum of all the angles inside of a triangle?

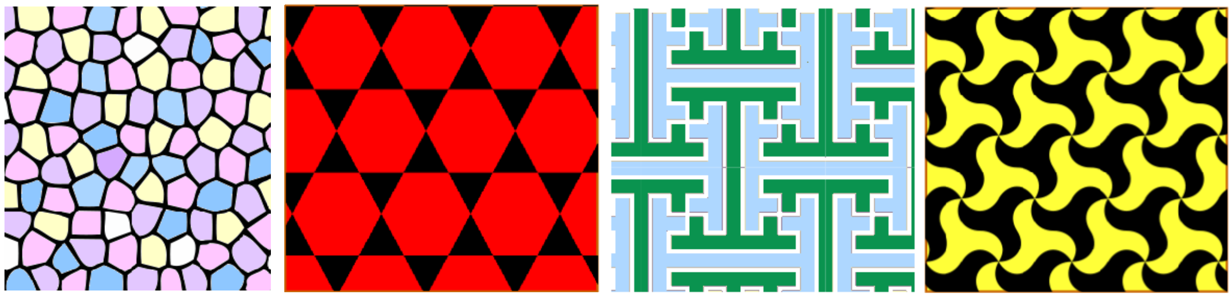
Find the missing angle.



$w = 30°$, $x = 45°$, $y = 45°$, $z = 45°$

Introduction to Tessellations

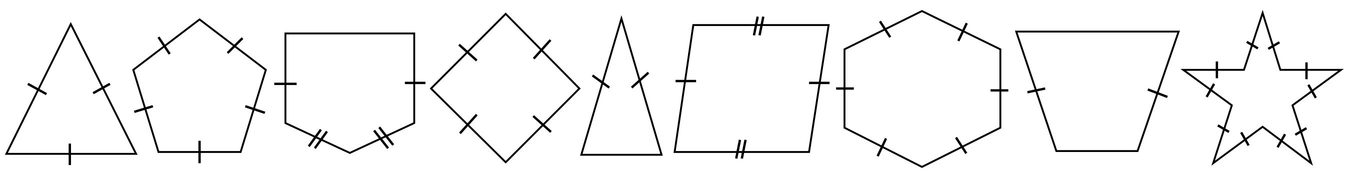
A **tessellation** is a collection of shapes that fit together with no gaps or overlaps. It forms a repeating pattern with the use of a specific number of shapes. In this lesson we will explore the different types of tessellations and what tessellations occur both in mathematics and in nature.



Angles in Geometric Shapes

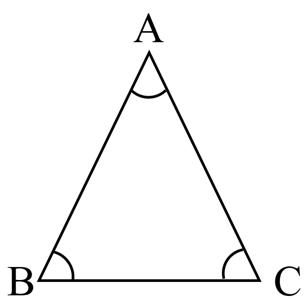
An **interior angle** is an angle inside of an enclosed shape.
A **polygon** is a shape with 3 or more straight sides enclosing it.
In order to be classified as a **regular polygon**, a shape must have all side lengths the same and all interior angles the same.

Exercise. Which of the following shapes are regular polygons?



1, 2, 4, and 7

Interior angles of regular polygons

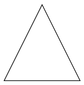
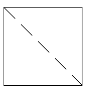
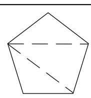
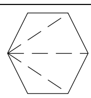
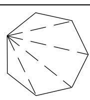
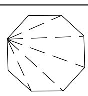


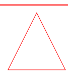
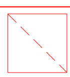
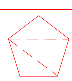
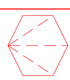
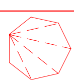
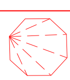
$\angle ABC + \angle BAC + \angle ACB = 180°$

As we have discussed in earlier lessons, the interior angles in *any* triangle add up to 180°.

The idea of the sum of interior angles can be generalized to other polygons. A way to come up with the interior angles of other polygons is to relate them back to triangles. If you can figure out the minimum number of triangles needed to create a polygon and you know the sum of a triangle's interior angles, then you just need to multiply the two numbers together. The sum of the interior angles of a triangle multiplied by the number of triangles you need to create the polygon will give you the sum of the interior angles of the polygon.
Once you know the sum of the interior angles, to find the size of one angle of a regular polygon you just need to divide by the number of angles, or vertices (where the angles occur), that are in the shape.

Exercise. Complete the following chart and determine the measure of the interior angles in each of the following regular polygons.

Shape	# of Triangles	Sum of Interior Angles	# of Vertices	Interior Angle
	1	$180°$	3	$\frac{180°}{3} = 60°$
	2	$180° \times 2 = 360°$	4	$\frac{360°}{4} = 90°$
				
				
				
				

Shape	# of Triangles	Sum of Interior Angles	# of Vertices	Interior Angle
	1	$180°$	3	$\frac{180°}{3} = 60°$
	2	$180° \times 2 = 360°$	4	$\frac{360°}{4} = 90°$
	3	$180° \times 3 = 540°$	5	$\frac{540°}{5} = 108°$
	4	$180° \times 4 = 720°$	6	$\frac{720°}{6} = 120°$
	5	$180° \times 5 = 900°$	7	$\frac{900°}{7} \approx 128.57°$
	6	$180° \times 6 = 1080°$	8	$\frac{1080°}{8} = 135°$

Formulas:

$\#$ of triangles $= \#$ of vertices -2

$$\text{Interior angle of a regular polygon} = \frac{180^\circ \times (\#\text{ of vertices } -2)}{\#\text{ of vertices}}$$

Exercise. What is the interior angle of an octadecagon (a polygon with 18 sides)?

$\#$ of vertices $= 18$

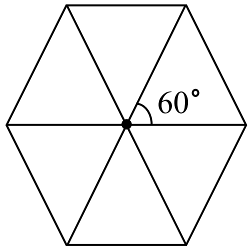
$$\begin{aligned}\text{Interior angle of an octadecagon} &= \frac{180^\circ \times (18-2)}{18} \\ &= \frac{180^\circ \times (16)}{18} \\ &= \frac{2880^\circ}{18} \\ &= 160^\circ\end{aligned}$$

Relationships Between Geometric Shapes

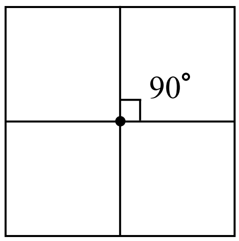
Regular Tessellations

A **regular tessellation** is made up of the same regular polygon.

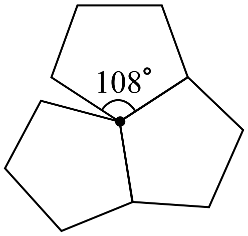
Which regular polygons can form regular tessellations?



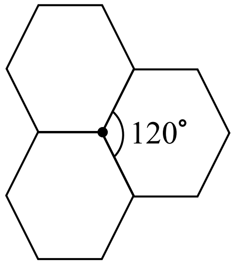
$60^\circ \times 6 = 360^\circ$



$90^\circ \times 4 = 360^\circ$



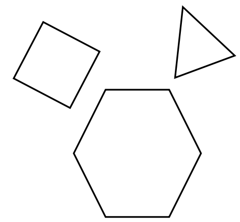
$108^\circ \times 3 = 324^\circ$



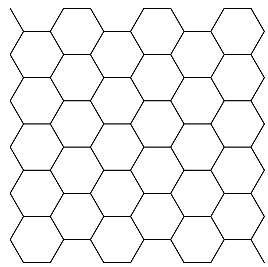
$120^\circ \times 3 = 360^\circ$

A regular polygon can form a regular tessellation if the measure of one of its interior angles **divides** 360° **evenly.** This is necessary in order for the shapes to complete a full rotation (360°) around any given point where they come together in the tessellation. If the interior angle cannot divide 360° evenly, then there will be a gap where the shapes are not touching, or the shapes are overlapping. If either of these results occur then, by definition, the shapes do not form a tessellation.

Exercise. Can you create a tessellation using **one** of the following regular polygons? All of the side lengths are equal and you can use as many of each shape as you need.

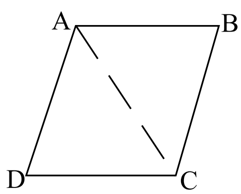


Answers will vary. One possibility is



Irregular Tessellations

An **irregular polygon** is any shape that does not have equal side lengths and equal interior angles, but does have all straight sides.

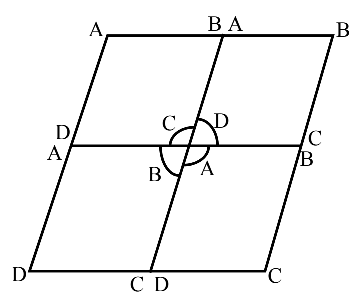


Just like a square, we can divide a parallelogram into 2 triangles. Thus, the interior angles add up to 360°
(i.e. $A + B + C + D = 360°$).

Irregular polygons have the same number of **sides** and **vertices** as their corresponding regular polygons, thus the formula for the sum of interior angles is unchanged. As a result, a four-sided irregular polygon has the same sum of interior angles as a square, and a six-sided irregular polygon has the same sum of interior angles as a regular hexagon.

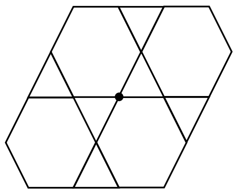
Can we create an irregular tessellation with the parallelogram from above?

Yes, since $A + B + C + D = 360°$!



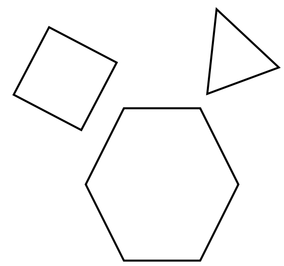
The same rule as when creating regular tessellations applies when creating irregular tessellations: the sum of all the interior angles at any given point must equal 360°. This is still necessary to ensure that there are no gaps or overlapping shapes.

Semi-Regular Tessellations



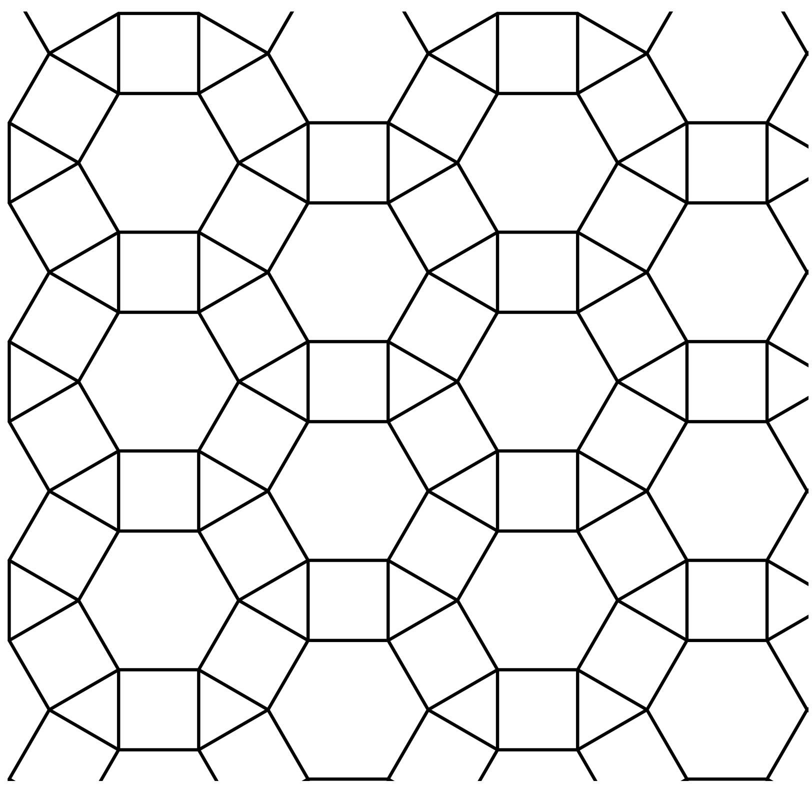
A **semi-regular tessellation** is a tessellation that is formed by multiple regular polygons. In order to be considered a tessellation the arrangement at every vertex within the pattern must be the same.

As you can see in the example of a semi-regular tessellation, every point within the design has two triangles and two hexagons in the same arrangement, the corresponding shapes opposite one another.



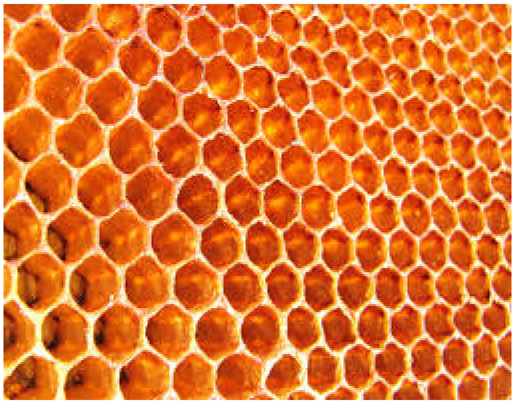
Exercise. Can you create a tessellation using **all** of the following regular polygons? All of the side lengths are equal and you can use as many of each shape as you need.

Answers will vary. One possibility is



Tessellations in Nature

Though tessellations may seem like a relatively obscure concept, tessellations appear rather commonly in nature. Examples of tessellations can be seen in a variety of areas including plants, animals skins, and various rock formations.

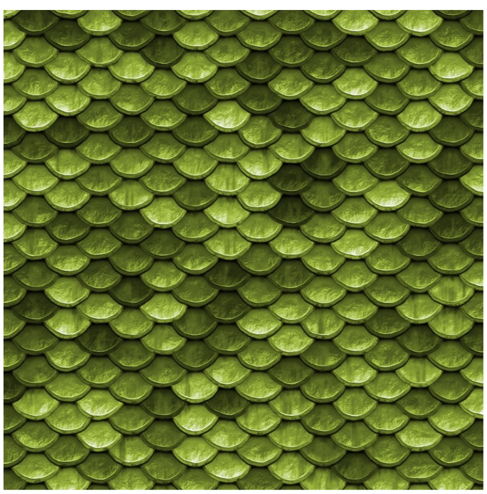


Honeycomb

Honeycomb is an example of a regular tessellation that occurs in nature. The structure of a honeycomb consists of entirely regular hexagon-shaped wax cells created by honey bees.

Giant's Causeway

This area on the coast of Northern Ireland is known for its columns of hexagonal rock. There are about 40,000 columns in this area, most with six sides, but some with slightly more or less. These columns start on land and extend into the sea. The hexagonal rock formations cover this area fitting together in a repeating pattern. This means that the whole area is one big rock tessellation.



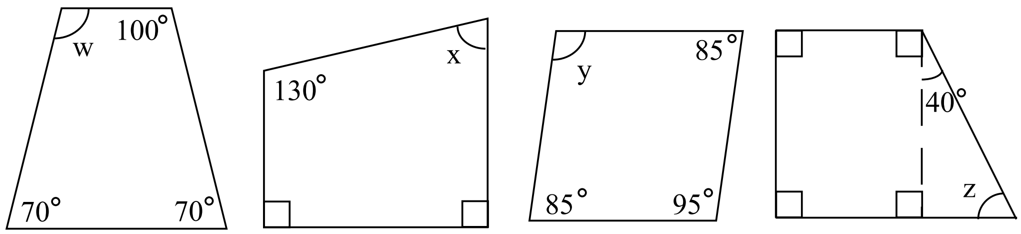
Fish Scales

Though they are not a regular geometric shape, fish scales are also an example of a naturally occurring tessellation. The repeated patterns occur in many different types of fish with many different sizes and shapes of scales.

Exercise. Can you think of any examples of tessellations (regular or irregular) that you have seen in your life?

Problem Set

1. Find the missing angle.



$w = 360° - 100° - 70° - 70° = 120°$
$x = 360° - 130° - 90° - 90° = 50°$
$y = 360° - 95° - 85° - 85° = 95°$
$z = 360° - 90° - 90° - 90° - 40° = 50°$

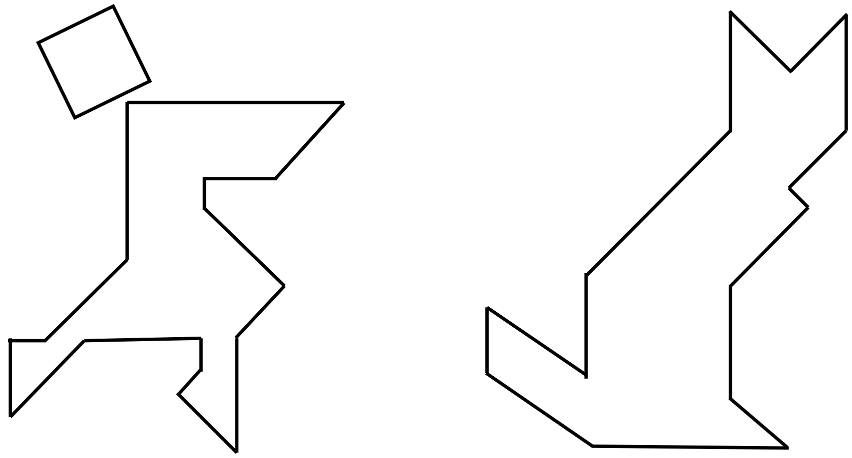
2. If you had a regular polygon with 50 sides (called a pentacontagon), what would the sum of all of the interior angles be? What is the measure of each angle?
# of vertices = 50

$$\begin{aligned}\text{Sum of Interior Angles} &= 180° \times (50 - 2)\\ &= 180° \times (48)\\ &= 8640°\end{aligned}$$

$$\begin{aligned}\text{Interior Angle} &= \frac{8640°}{50}\\ &= 172.8°\end{aligned}$$

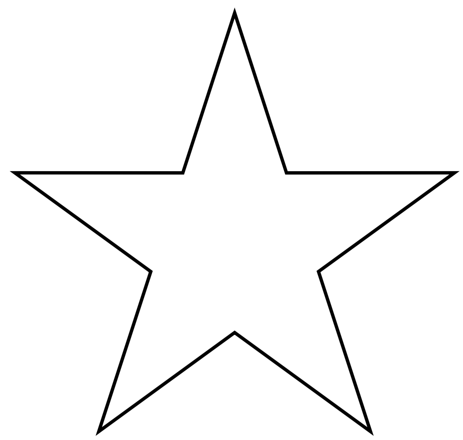
3. Is it possible to create a regular tessellation with a circle? Why or why not? No; circles cannot fit together without gaps or overlaps

4. Divide the following shapes into smaller polygons using straight lines. Classify each of your smaller polygons as a regular or irregular polygon.



Answers will vary

5. What do the interior angles in a star add up to? (Hint: Try dividing it into familiar shapes.)

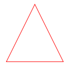
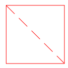
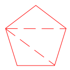
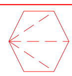
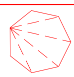
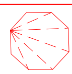


$1440°$

Consider the star as 5 triangles and a pentagon. Considering the sum of interior angles of these shapes, the interior angles of a star $= 5 \times 180° + 1 \times 540° = 1440°$.

6. Claire is trying to draw a regular polygon with an interior angle less than $60°$. Is this possible? (If you don't know the answer, try drawing it out.)

No; observing the table below, we can see that the interior angle decreases as the number of sides decreases. Since an interior angle of $60°$ corresponds to 3 sides, to have a polygon with an interior angle less than $60°$ would require a polygon of less than 3 sides. A polygon with less than 3 sides is not possible.

shape	# of Triangles	Sum of Interior Angles	# of Vertices	Interior Angle
	1	$180°$	3	$\frac{180°}{3} = 60°$
	2	$180° \times 2 = 360°$	4	$\frac{360°}{4} = 90°$
	3	$180° \times 3 = 540°$	5	$\frac{540°}{5} = 108°$
	4	$180° \times 4 = 720°$	6	$\frac{720°}{6} = 120°$
	5	$180° \times 5 = 900°$	7	$\frac{900°}{7} \approx 128.57°$
	6	$180° \times 6 = 1080°$	8	$\frac{1080°}{8} = 135°$

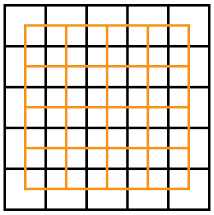
7. Can a regular polygon have an interior angle greater than 180°?
   No; 180° is a straight line.

CHALLENGE

8. We found that the interior angles of regular polygons must divide into 360° evenly in order to create regular tessellations. What is the minimum number of shapes you need to create a regular tessellation using regular polygons? (Hint: Think about your answer to the previous question.)
   3, this is the smallest number of shapes needed to have the angles at a specific point sum to 360°. This is because in order to be 2 shapes the angles would have to be 180°, which, as is expressed in the previous question, is not possible.

9. There are only a specific number of regular tessellations that can be created,
   (a) How many are there?
   (b) What are they?

   (a) 3
   (b) Equilateral triangles, squares, and regular hexagons.
   These are the only shapes whose interior angles evenly divide 360°.

10. 

Dual tessellations are drawn by identifying the **centre** of each shape in a tessellation and then connecting by **lines** the centres of shapes that **share an edge** with each other. For example, on left is a square tessellation (in black) and its dual which is again a square tessellation. Tessellations will not always have the same tessellation as a dual. Draw the dual tessellations on top of the following three tessellations: